$|g \in G\rangle$

$G = \mathbb{Z}_2 \times \mathbb{Z}_2$
$= \{e, x, y, z\}$
$= \{e\otimes e, x\otimes e, x\otimes x, e\otimes x\}$

span$\{e, x\}$ is a qubit.

3.5 Spin structures & fermions

a sign for every loop in spacetime M

parallel transport of vectors for each path $\gamma \in M$ $\longrightarrow$ an el't of $SO(D)$

(M is a D-dim'l oriented mfld)

parallel transp of spinors for each path $\gamma \in M$ $\longrightarrow$ an el't of

$$\begin{array}{c} \mathbb{Z}_2 \to Spin(D) \\ \downarrow \\ SO(D) \end{array}$$

e.g.:

$(Spin(3) = SU(2))$.

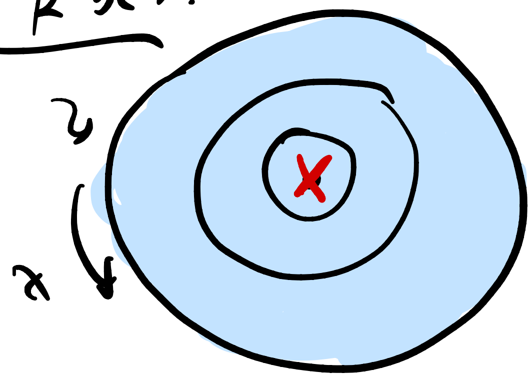
eg: $M = S^1$. has two spin structures:

$$\psi(x+L) = \pm \psi(x)$$

$+$ is Ramond

$-$ is Neveu-Schwarz

w/ R bcs:



$z = e^{ix}$ coord on disk

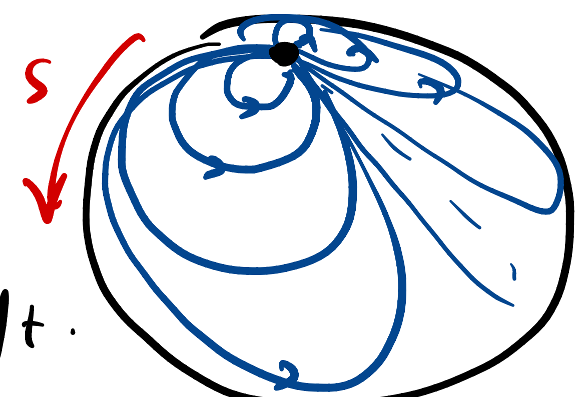
$$\psi(z e^{2\pi i}) = +\psi(z).$$

on $M = T^D = (S^1)^D$ there are 2^D spin structures.

obstruction to $\exists$ of spin structure: $S^2 < M$

$S^2 \longrightarrow$ family of loops $\gamma(t,s)$

$$\begin{cases} \gamma(t,0) = \gamma(t,1) = N \quad \forall t. \\ \gamma(0,s) = \gamma(1,s) = N \quad \forall t. \end{cases}$$



Parallel transport vectors along $\gamma(\cdot, s) \longrightarrow$ an el't of $SO(D)$ for each s.

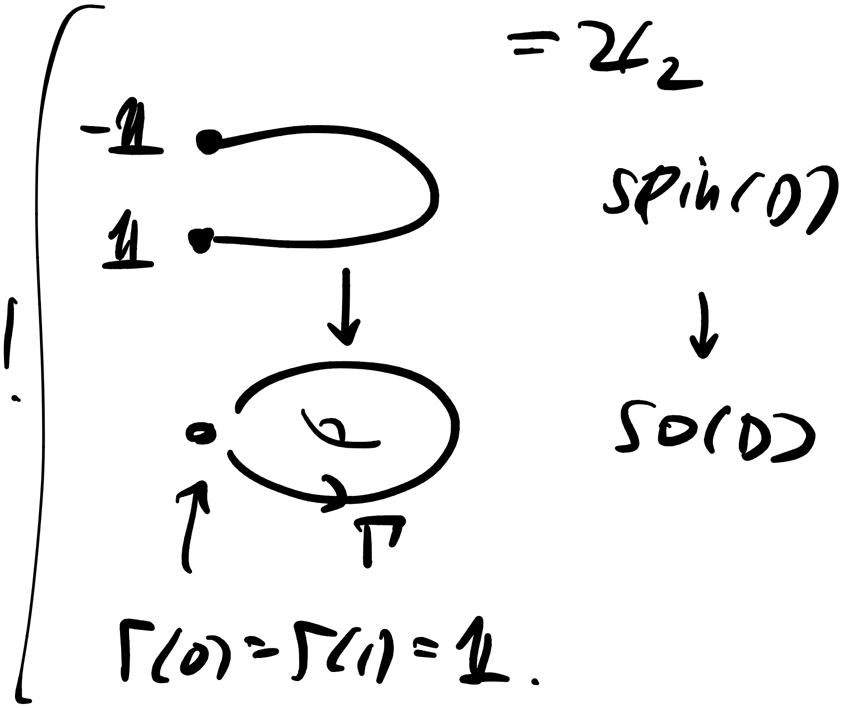
$\gamma(t,s) \rightsquigarrow$ a loop in $SO(D)$

$\Gamma \in \pi_1(SO(D)) = \mathbb{Z}_2$

If $[\Gamma] \in \pi_1(SO(D))$ is nontrivial

a spinor on M must be singular at N!

$[\Gamma]$ is a property of TM.



$\Gamma \neq 1$ is a $\mathbb{Z}_2$ monopole on S^2.

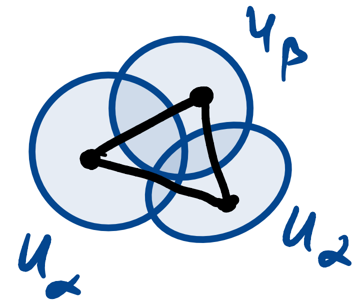
<u>eg</u>: $M = \mathbb{CP}^2 = SU(3)/U(2) = G/H$

<u>A sol'n</u>: if ψ is charged. couple to a bg. magnetic monopole field satisfying <u>wrong</u> Dirac quantization: $2eg = n + \frac{1}{2}$, $n \in \mathbb{Z}$.

("spin_c structure".)

Stiefel-Whitney classes:

Take a good cover of M.



$\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ are all balls.

transitions f'-s of TM

$$g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to O(D)$$

(If M is oriented $g_{\alpha\beta} \in SO(D)$).

$\longrightarrow$ associate a sign

$$(w_1)_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to \mathbb{Z}_2 = O(D)/SO(D)$$

$$\left((w_1)_{\alpha\beta} = \det(g_{\alpha\beta}). \right)$$

Claim: $(\delta w)_{\alpha\beta\gamma} = w_{\alpha\beta} w_{\beta\gamma} w_{\gamma\alpha} = 1.$ $\mathbb{Z}_2$-valued cocycle.
on $\mathcal{U}_{\alpha\beta\gamma}$

$[w_1] \in H^1(M, \mathbb{Z}_2)$

If $w_1 = \delta s$ ie $(w_1)_{\alpha\beta} = s_\alpha s_\beta$

ie $[w_1] = 0$ in $H^1(M, \mathbb{Z}_2)$

Then $s_\alpha : U_\alpha \to \mathbb{Z}_2$ specifies an orientation of TM.

$w_1(TM) = [w_1]$ is the 1st Stiefel Whitney class of M.

Suppose $w_1 = 0$. choose an orientation of M.

$$\underbrace{g_{\alpha\beta} : U_{\alpha\beta} \to SO(D)}.$$

A spin structure is

$$\underline{\underline{\hat{g}_{\alpha\beta}}} : U_{\alpha\beta} \longrightarrow Spin(D).$$

<u>Recall</u>: $g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = (\delta g)_{\alpha\beta\gamma} = 1$ on $U_{\alpha\beta\gamma}$.

$$\Rightarrow W_{\alpha\beta\gamma} = \hat{g}_{\alpha\beta}\, \hat{g}_{\beta\gamma}\, \hat{g}_{\gamma\alpha} : U_{\alpha\beta\gamma} \to \mathbb{Z}_2 .$$

Claim: $(\delta w)_{\alpha\beta\gamma\delta} = 1.$

$\Longrightarrow \quad [w] \equiv w_2(TM) \in H^2(M, \mathbb{Z}_2)$

2d Stiefel-Whitney class.

if $0 = w_2 \longrightarrow w_2 = \delta f$

$f_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{Z}_2$

$\Longrightarrow \quad \hat{g}_{\alpha\beta} = g_{\alpha\beta} f_{\alpha\beta}.$ is a spin structure.

$w_2 \neq 0$ is an obstruction to $\exists$ a spin structure on M.

Confession: $A_{CS} = e^{i\frac{k}{4\pi}\int A \wedge dA}$

is invt under $A \to A + \omega$ $\quad \left(\begin{array}{c} d\omega = 0 \\ \omega \neq d\lambda \end{array}\right)$

Requires k to be even!

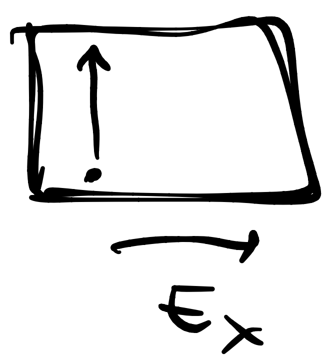
$k=1$ describes IQHE of fermions.

eg: space: T^2. $\begin{cases} x = x + L_x \\ y = y + L_y \end{cases}$

thread 2π flux through x dir. $\Rightarrow A \to A + dx$

$$A_x = \frac{2\pi t}{L_x} \qquad \oint_x A = 2\pi t \qquad t \in [0,1]$$

$\to E_x$. $\xrightarrow{\sigma^{xy} \neq 0}$ pumps a single unit of charge around the y direction



APBC on electrons $\Rightarrow$ $\underline{A \to -A}$

3.6 Characteristic classes & classifying spaces

A G bundle on M $\longleftrightarrow$ a map $\underline{\gamma} : M \to \underline{BG}$

pull back operation

$$\gamma^* : H^n(BG, A) \longrightarrow H^n(M, A)$$

Image of γ^* = char classes.

<u>classifying space</u>: $\exists$ a universal G-bundle EG.

$$\begin{array}{ccc} G & \to & EG \\ & & \downarrow \\ & & BG \end{array}$$

- EG is <u>contractible</u>.
- G acts freely on EG.

$BG = EG/G$.

eg: $U(1) \to S^{2\infty+1}$
$$\begin{array}{c} S^{2\infty+1} \\ \downarrow \\ \mathbb{CP}^\infty = BU(1) \end{array}$$

$\mathbb{Z}_2 \to S^\infty$
$$\begin{array}{c} S^{\infty} \\ \downarrow \\ \mathbb{RP}^\infty = B\mathbb{Z}_2 \end{array}$$

$SU(2) \to S^{4\infty+1}$
$$\begin{array}{c} S^{4\infty+1} \\ \downarrow \\ \mathbb{HP}^\infty = BSU(2). \end{array}$$

Application: Def of Z_{DW}

data: $\nu \in \underline{H^D(G, U(1))}$ ^ discrete group G

claim: $\underset{\text{group cohomology}}{H^D(G, U(1))} = \underset{\text{ordinary cohomology of } BG.}{H^D(BG, \underline{\underline{U(1)}})}$

Triangulate M. specify a gauge field config by $\gamma : T \to BG$. for each D-simplex $T \subset M$.

$$W[\gamma] = \prod_T \nu^{s(T)}(\gamma(T))$$

a discrete rep of the characteristic class $\gamma^*(\nu) \in H^D(M, U(1))$.

3.7 Classification of SPTs.

Recall stacking: given $A, B \longrightarrow A + B$

$$\mathcal{M}^d_G \equiv \text{def classes of } \{(\mathcal{H}, \rho, H)\}$$

Hilbert sp, Rep of G, Hamiltonian

$$SPT^d_G \equiv \text{invertible elts of } \mathcal{M}^d_G.$$

vs:

$SRE^d_G \equiv$ G-symmetric non-SSB systems that can be be deformed to the trivial state if we break G.

who is SPT/SRE ?

for fermions
- In $D=0+1$ an odd # of Majorana modes.
- In $D=1+1$ Kitaev chain
- In $D=2+1$ $p+ip$ SC.

for bosons :
- in $D=2+1$ E_8 state.

$U(1)^8$ CS thy w $K=K_{E_8}$.

$|\det K|=1$

- in $D=4+1$ $S[B,C]=\int \frac{B\wedge dC}{2\pi}$.

3.8 Global anomaly inflow

$\text{SPTs}^G_D \longleftrightarrow$ anomalies of G in $D-1$ dims.

edge thy $\xrightarrow{\cdots}$ extra dim'l thy.

eg: $D=3+1$ TI $\rightsquigarrow$ $G = U(1)\rtimes \mathbb{Z}_2^T$

($\exists$ A symmetric mass term $\rightarrow$ no anomaly.)

path integral on X Z | Dirac ferm.

$$Z_X = \det D \qquad D = i\gamma^\mu D_\mu$$

$$= \prod_i \lambda_i \qquad D = D^\dagger$$

$$\lambda_i \in \mathbb{R}.$$

w/ 2 Dirac ferm

$$Z_X^2 = \prod \lambda_i^2 \geq 0 \qquad \left(Z^*(X) = Z(\bar{X}) \right)$$

$$\text{sign } Z_X = (-1)^{\#\text{ of negative evals of } D}.$$

Define the sign of

$Z_X = Z_X[g, A]$ by finding a path from $g_0 A_0$.

choose $Z_X[g_0, A_0] > 0$.

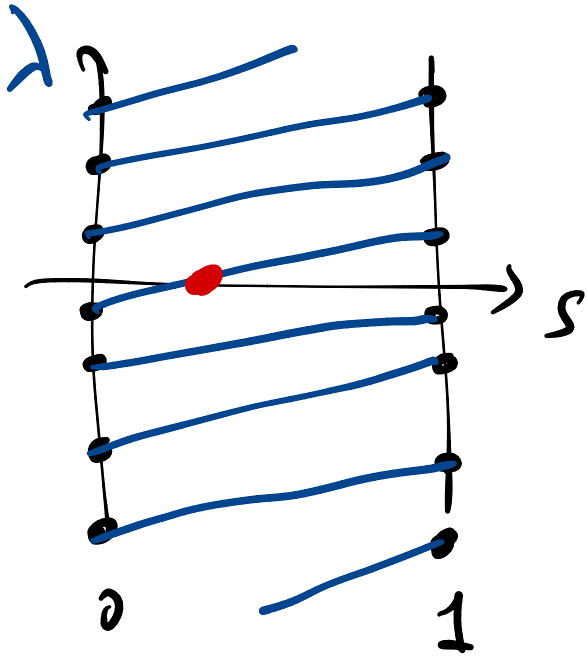
Let ϕ be some gauge transf. $(g_0, A_0) \to (g_0^\phi, A_0^\phi)$.

$$\begin{cases} A_s = (1-s)A_0 + s A_0^{\phi} \\ g_s = (1-s)\, g_0 + s\, g_0^{\phi} \end{cases} \qquad s\in[0,1] \qquad (\, g_s \geq 0.\,)$$

$$Z(g_0, A_0) = (-1)^{\Delta}\, Z(g_0^{\phi}, A_0^{\phi})$$

Δ = # of evals crossing zero

= index of a 3+1 D dirac operator on Y_ϕ = mapping torus



spectral flow.

$$Y_\phi = I \times X \big/ {(0,x) \sim \phi(1,x)}$$

(if ϕ only acts on A this $Y_\phi = S^1 \times X$.)

$\Rightarrow Z_X(g,A)$ is a section of some bundle (w/ structure group $\mathbb{Z}_2$) over the space of BG fields

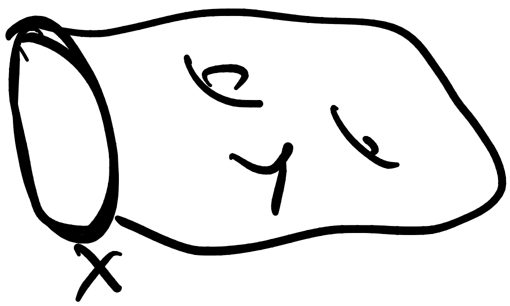
Discrete anomaly inflow:

anomaly in D-1 dims $\leftrightarrow$ a D-dim'l gapped thy "anomaly theory"

Z_X is a section of a bundle $\longmapsto$ $Z_{Y,\partial Y = X}$ is too.

In this example it's just $S_Y = \int_Y d^4x\, \bar{\Psi}(i\cancel{D} - M)\Psi$ w/ $M<0$.

$$Z_{bulk} = \int D\Psi\, e^{iY} = M^{\mathcal{U}}$$



has sign $(-1)^{\mathcal{U}}$ if $M<0$.

$e^{2\pi i \eta_Y}$ = partition fn of the SPT on Y, $\partial Y = X$.

eg: "local anomalies" of free fermions (triangle diagrams)

$Y = \partial Z \qquad Z = X \times D$

$\uparrow$ 2-disk.

APS index theorem $\Rightarrow$

$$\mathbb{Z} \ni \mathrm{Ind}(D_Z) = \eta_Y + \int_Z I_{D+1}$$

$$I = \left(\hat{A}(R) \wedge \mathrm{tr}\, e^{F/2\pi}\right)$$

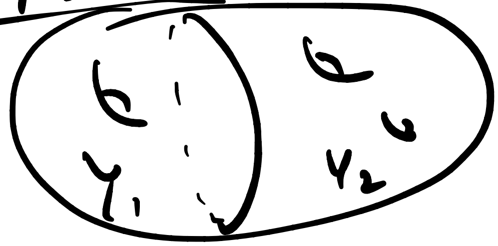
$$\Rightarrow e^{2\pi i \eta_Y} = e^{2\pi i \int_Z I_{D+1}}$$

δ(this) reproduces local anomalies by WZ descent.

global anomaly $\equiv$ anomaly in an ele't not connected to $\mathbb{1}$.

$[Y, G]$

Gluing properties:

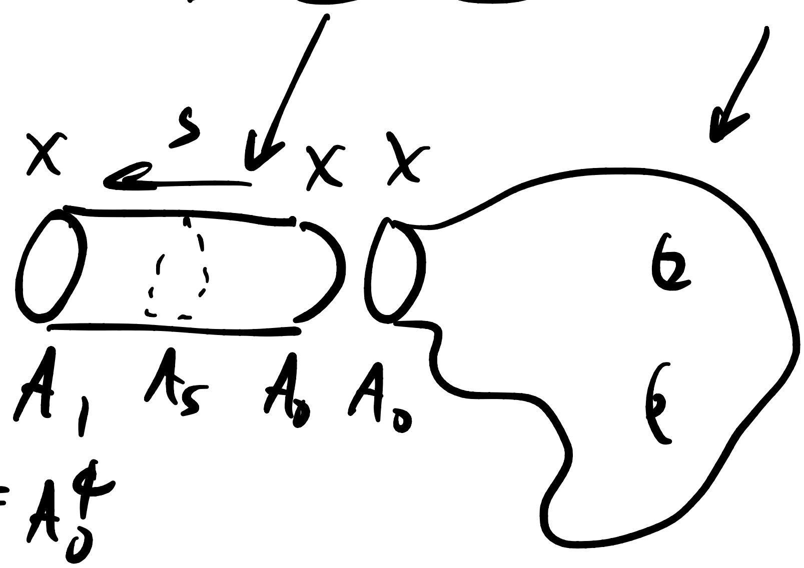


$Y = Y_1 \cup Y_2$

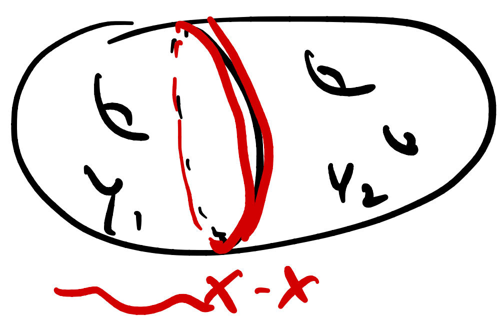
$\partial Y_1 = -\partial Y_2$

$$e^{2\pi i \eta_Y} = e^{2\pi i \eta_{Y_1}} e^{2\pi i \eta_{Y_2}} .$$

$$Z[A_0^\phi] = \underbrace{e^{2\pi i \eta_{X\times I}}} Z[A_0]$$



More general anomalies:



If $\underbrace{e^{2\pi i \eta(Y_1 \cup Y_2)} \neq 1}$

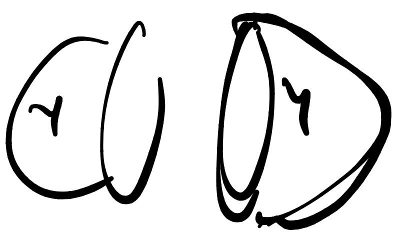
then Z_X is ambiguous!

If local anomalies vanish η_Y is a <u>BORDISM inv't.</u>

$\underbrace{Y_1 \overset{\text{bordant to}}{\simeq} Y_2}_{D \text{ manifold}}$ if $\exists$ $D+1$ mfld Z w/ $\partial Z = Y_1 - Y_2$

$$\Rightarrow \eta_{Y_1} = \eta_{Y_2}.$$

SPTs $\longleftrightarrow$ $U(1)$-valued bordism inv'ts satisfying · gluing rule · unitarity



$= \langle \Phi_Y | \Phi_Y \rangle = Z_{Y \cup \bar{Y}} > 0.$

$\Omega_D^{spin}(W) \equiv$ equivalence classes of D-dim'l spin manifolds w/ a map to W. mod $\simeq$

(W: manifold)

G-bundles on Y $/\sim$ $\leftrightarrow$ $[Y, BG]$.

$$e^{2\pi i \underline{\underline{\eta}}} : \Omega_D^{spin}(BG) \longrightarrow U(1)$$

a group homomorphism.

More generally $S_Y^\alpha[A, g)$

- local fn'l of bg fields
  gauge inv't
- $\left| e^{2\pi i S_Y^\alpha} \right| = 1$.
- $S_{\bar{Y}}^\alpha = - S_Y^\alpha$
- gluing.
- $S_Y^{\bar{\alpha}} = S_{\bar{Y}}^\alpha$

• No symmetry. $G = \{e\}$.

$\rightarrow$ Y is oriented

$$S^\alpha = \int_Y \Sigma \left(\begin{array}{c} \text{products of } P_n \\ \text{and } w_n \text{ 's} \end{array} \right)$$

— Bordism invariants

— determine a bordism class.

no G.

$$e^{2\pi i S_Y^\alpha} : \Omega_{D,SO}(pt) \rightarrow U(1)$$

oriented.

$$\mathrm{Hom}(\Omega_{D,SO}(pt) \to U(1)) = \Omega^D_{SO}(pt, U(1))$$

"Co-bordism group".

$$bSPT^D_{G=\{e\}} = \Omega^D_{SO}(pt, U(1)) \Big/ \Omega^D_{SO}(pt, \mathbb{R})$$

free part

• Time reversal sym $G = \mathbb{Z}_2^T$.

Y can be unoriented.

$\Rightarrow \alpha = \bar{\alpha} \quad 2 S_Y^\alpha = 0 \mod 1$

$S_Y^\alpha = 0, \frac{1}{2} \mod 1$

$$bSPT^D_{G=\mathbb{Z}_2^T} = \Omega_0^D(pt, U(1)).$$

$$\bigoplus_D \Omega_{0D}(pt) \overset{\text{Thom}}{=} \mathbb{Z}_2[\{x_j\}]$$

$j \neq 2^i - 1$

$j = 2, 4, 5, 6, 8 \ldots$

D	1	2	3	4	5
$\Omega_{D0}(pt)$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$

<u>$D = 1+1$</u>. $\int_Y w_1^2 = \int_Y w_2 =$ euler character of Y (mod 2).

$Z_Y = e^{2\pi i \frac{1}{2} \int w_2} = (-1)^{\chi(Y)} = \pm 1$ if Y is oriented or not.

this is the Haldane phase!

- $\underline{D = 2+1}$. $\quad w_1 w_2 = w_3 = w_1^3 = 0$

  for all closed 3-mflds.

  every $Y = \partial Z$.

  no $\mathbb{Z}_2^T$ SPTs.

- $\underline{D = 3+1}$. $\quad \left\{ \begin{array}{l} w_3 w_1 = w_2 w_1^2 = 0 \\ w_4 + w_2^2 + w_1^4 = 0 \end{array} \right.$

  leave 2 generators.

  $\frac{1}{2} w_1^4$ and $\underset{\sim}{w_2^2}$. $= 0$ if Y is spin

  w/ a neutral fermion Y must be Spin.

  $\longrightarrow \quad w_2^2 = 0.$

  This is the "efmf thy"

  whose edge is all-ferm TC.

$w_1^4 \rightsquigarrow$ group cohomology SPT

$$H^4(\mathbb{Z}_2^T, U(1)) = \mathbb{Z}_2.$$

- $D = 4+1$ $\int \frac{1}{2} w_2 w_3$

vs $H^4(\mathbb{Z}_2^T, U(1)) = 0.$

$$\longleftarrow \quad \int \frac{B \wedge dC}{2\pi}.$$

Bordism is an example of <u>generalized cohomology</u>

$$\mathrm{bSPT}^{D}_{G} = \Omega^{D}_{\underline{\underline{SO}}}(BG, U(1)) \Big/ \text{free part}$$

unitary

$$\int \mathrm{tr}\, F \wedge F$$

$$\Omega_{SO, D}(BG) = \mathbb{Z} \oplus \mathbb{Z}_n \oplus \mathbb{R} \cdots$$

free part