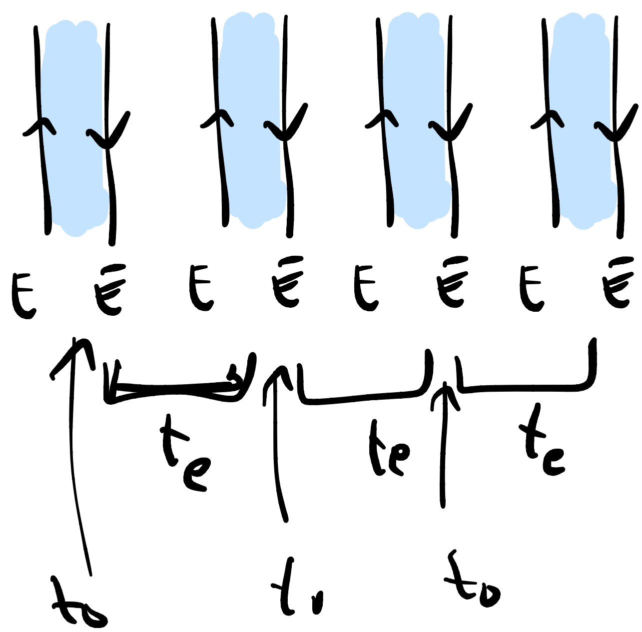
Announcements:

- lecture next Monday usual time & place.
- please fill out evaluation!

Coupled layer construction:



$t_e \gg t_0$

$\longrightarrow$ SPT

- **IQHE**. $D=1+1$ edge $\rightarrow$ $D=2+1$ SPT.

Fermions: $\underbrace{E+\bar{E}}$ is a nonchiral boson.

Could gap out $\sum_i t_0 \cos(\phi_L^i + \phi_R^i)$

instead: $\sum_{i=1}^{N-1} t_e \cos(\phi_R^i + \phi_L^{i+1})$

OR: $S = \int d\tau\, dx\, \mathcal{L}$

$$\mathcal{L} = \sum_j c_j^\dagger (i\partial_t - i s_j \partial_x) c_j - t_j c_{j+1}^\dagger c_j + h.c.$$

$s_j = (-1)^j$

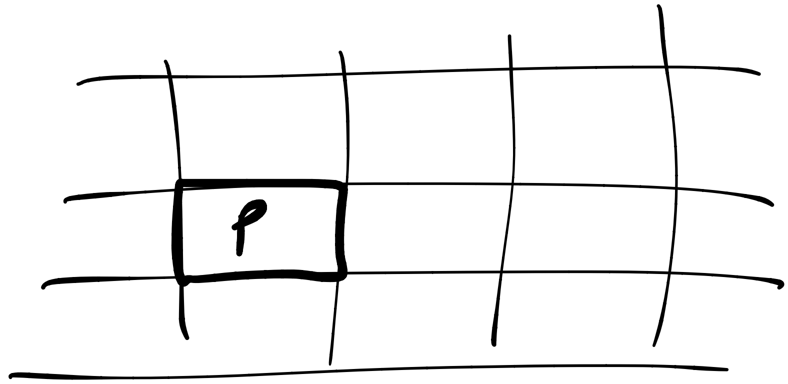
$$t_j = \begin{cases} t_e \\ t_o \end{cases}$$

$t_e < t_o : C = 0$

$t_e > t_o : C = 1$

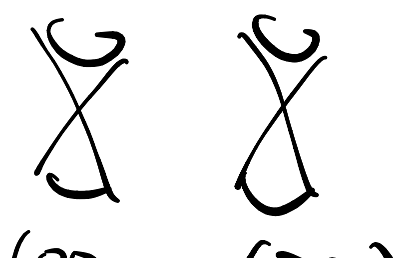
when $t_e = t_o$

$$\prod_{\ell \in \partial p} t_\ell = -|t^4|$$

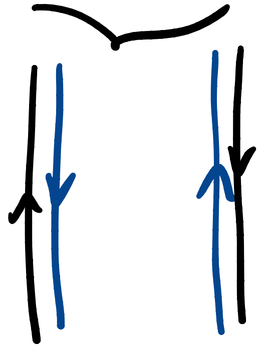


π-flux square lattice.

$\longrightarrow$



IQH Bosons:

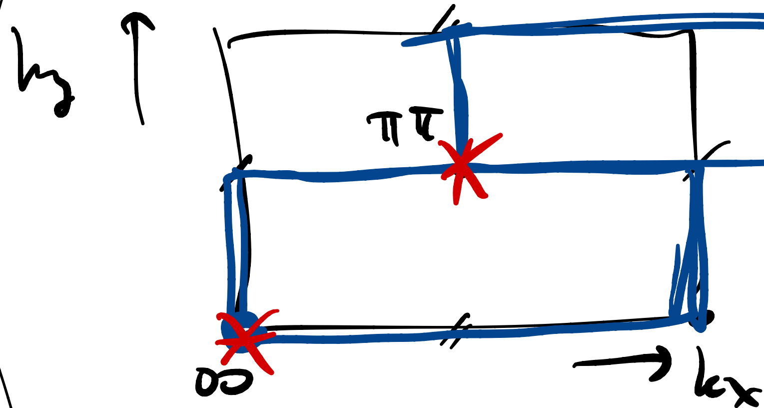


• $D=3+1$ TI $\rightsquigarrow$ $G = U(1) \rtimes \mathbb{Z}_2^T$

$E =$ $\qquad E\bar{E} = 2$ dirac cones

$$H_0 = \sum_i \oint d^2k \; c_i^\dagger(k)\, h_0(k)\, c_i(k)$$

$$h_0(k) = \sin k_x \sigma_y - \sin k_y \sigma_x$$

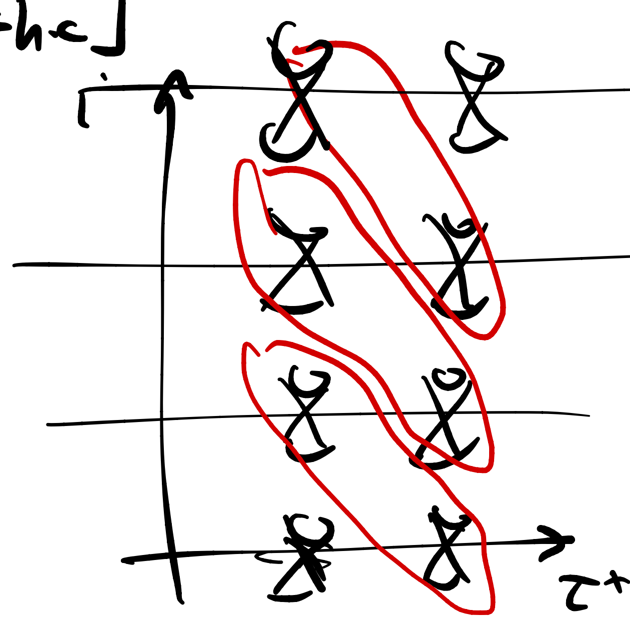


$\sim k_x \qquad \sim k_y$

$$= \sum_i \oint_{\square} d^2k \; c^\dagger_{ik}\, \tau^x \left(\sin k_x \sigma^y - \sin k_y \sigma^x\right) c^\dagger_{ik}$$

valley index

$$H_e = \sum_{i=1}^{N-1} \left[\oint d^2k \; c_i^\dagger(k)\, h_+\, c_{i+1}(k) + h.c.\right]$$



$$h_+ = \tau_z - i\tau_y .$$

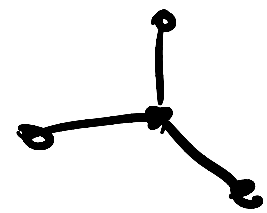
Wan hodge $\sim \pm \tilde{k} \times \vec{\sigma}$

• $G = \mathbb{Z}_2^T$ Boson SPT in $D=3+1$

E = edge theory: all-fermion toric code. ($D=2+1$)

Construction: $U(1)^4$ CS theory

$$\text{w/ } K = K_{SO(8)} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & +1 & & \\ -1 & & +1 & \\ -1 & & & +1 \end{pmatrix}$$

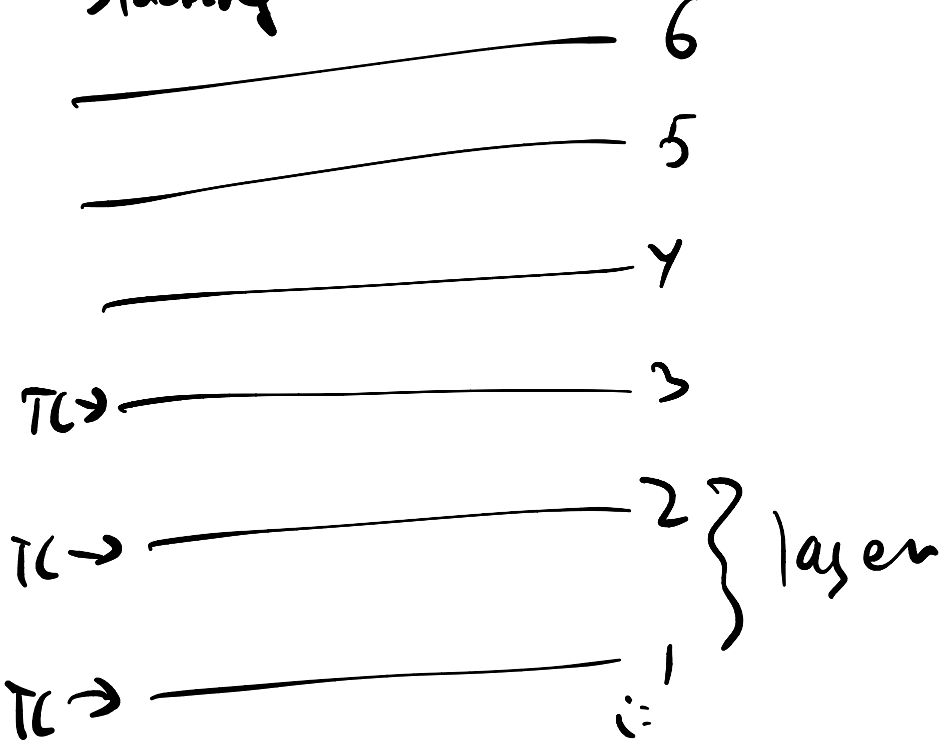


catch: Not T-inv't !!

$c_- = 4.$

$E + \bar{E} = TC + \overline{TC}$

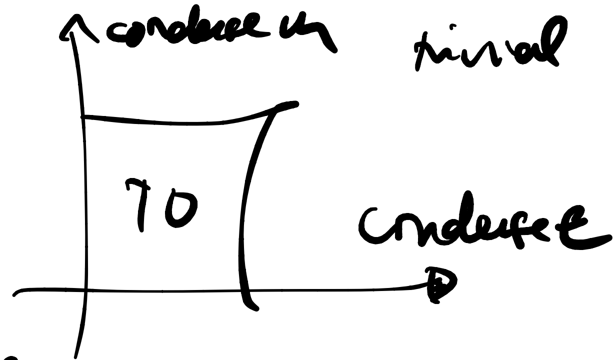
↑ stacking



list anyons in each

		TC^2	$APTC^2$
1	1	B	B
1	e	B	F
1	m	B	F
1	ε	F	F
e	1	B	F
e	e	B	B
e	m	B	B
e	ε	F	B
	⋮	⋮	⋮
		6 F & 10 B	6 F & 10 B.

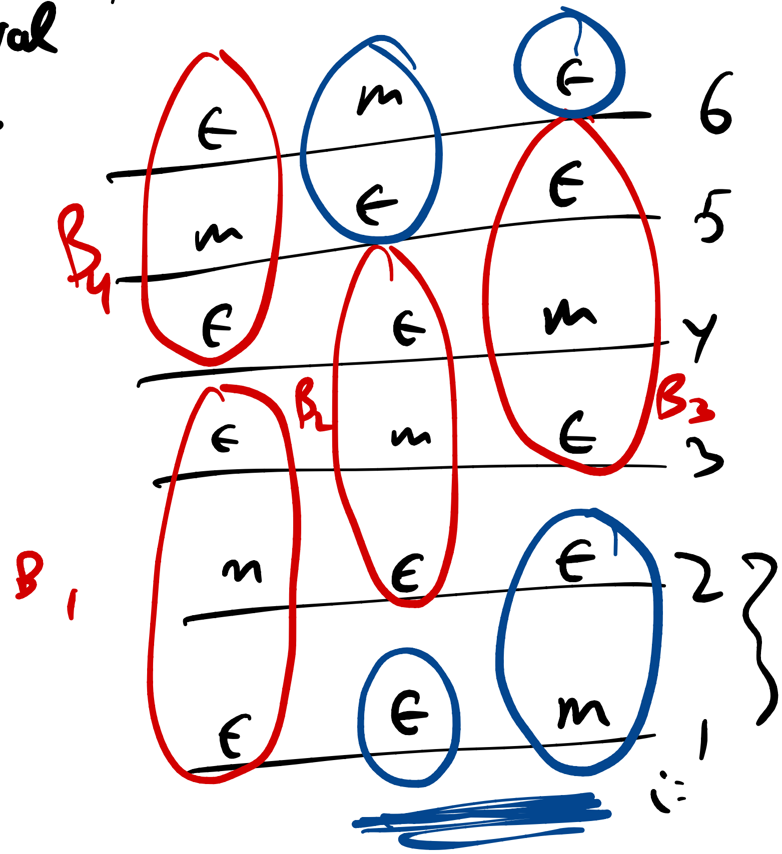
$B_i \equiv \epsilon_i m_{i+1} \epsilon_{i+2}$.



- is a boson.
- B_i & B_j are mutual bosons.

→ can condense B.

$$\Delta H = -\sum_{i=1}^{N-2} B_i$$



if $\langle B \rangle \neq 0$

all anyons in bulk are destroyed!

$$\left\{ \begin{array}{ll} \epsilon_1 , & m_1 \epsilon_2 \\ \epsilon_N & \epsilon_{N-1} m_N \end{array} \right.$$

have trivial braiding w/ B's

→ survive

<u>ALL FERMIONS!</u>

Decorated Defects construction :

paramagnet = ordered phase + condense defects

$\left\langle \begin{matrix} \text{creation op} \\ \text{for defects} \end{matrix} \right\rangle \neq 0.$

SPT =
interesting paramagnet = ... defects carry some decoration.

3.4 Wavef'ns for SPTs :

an EFT for SPTs of bosons in 3+1d.

bosons b^I w symmetry current j^I_μ , $\partial^\mu j^I_\mu = 0.$

solve continuity eqn : $j^I_\mu = \epsilon_{\mu\nu\rho\sigma}\partial^\nu B^{I\,\rho\sigma}/2\pi$

($B^I \to B^I + d\Lambda^I$ preserves j^I)

want a gapped phase.

$$S[B^I, a^I, A^I] = \int \left(\sum_I \frac{1}{2\pi} B^I \wedge da^I + \vartheta K_{IJ} \frac{da^I \wedge da^J}{4\pi^2} \right) + j^{I\mu} A_{I\mu}.$$

$$0 = \frac{\delta S}{\delta a} = dB = 0.$$

$\longrightarrow$ gap.

Notes:

- $\frac{k}{2\pi} B^I \wedge da^I \qquad k \in \mathbb{Z}$

  $k > 1 \longrightarrow \mathbb{Z}_k$ gauge thy.

  $k = 1$ no TO (SPT)/

- $T \Rightarrow \vartheta \in 0, \pi.$

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eom: $0 = \frac{\delta S}{\delta B} \propto da^I + dA^I \longrightarrow$ field line of a = vortex line of the bosons.

$\boxed{K = \sigma^x}$

$$\log \int DaDB\, e^{iS} = \frac{2\vartheta}{8\pi^2} \int dA \wedge dA \qquad (A = A^1 - A^2).$$

$$\begin{cases} \vartheta = \pi & \to & \Theta = 2\pi & \text{is nontrivial.} \\ \vartheta = 0 & \to & \Theta = 0 & \text{is trivial.} \end{cases}$$

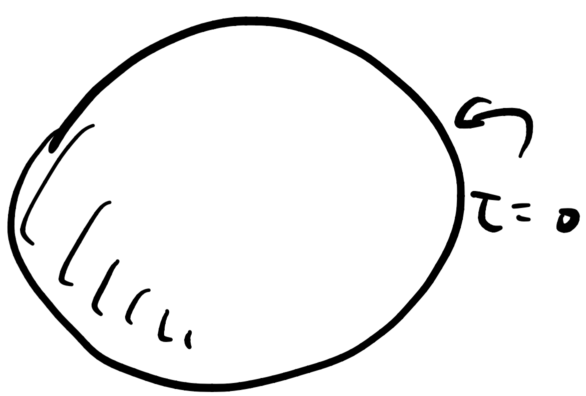
Slicing the path integral:

on a closed mfld $Z(\vartheta + 2\pi) = Z(\vartheta)$.

w/ a bdy ϑ matters.

eg: bdy in space $\to$ edge modes.

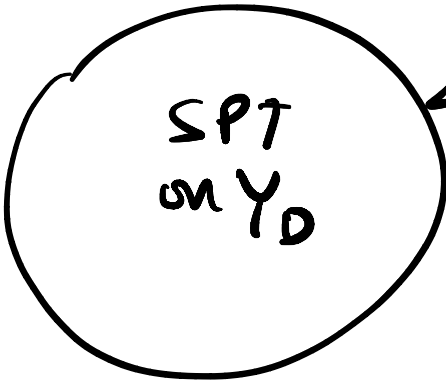
bdy in time.



radius of ball

$$Z_{Ball}[bcs] = \langle bcs | \underbrace{e^{-HT} | \text{arbitrary} \rangle}_{\text{groundstate.}}$$

= g.s. wave f'l.



$X = \partial Y$
$= \{t = 0\}$

$$\underbrace{\Psi\left[\text{bdy vars}_{\vec a, \vec B}\right]} = \underbrace{\Psi_0[\vec a, \vec B]}_{\text{trivial wavefn}}\ \underbrace{e^{i\vartheta \int_X \frac{a^I \wedge f^J K^{IJ}}{8\pi^2}}}_{\vartheta \cdot \text{ linking \# of } 2\pi \text{ flux lines} = \text{vortex lines of } b^1 \,\&\, b^2}$$

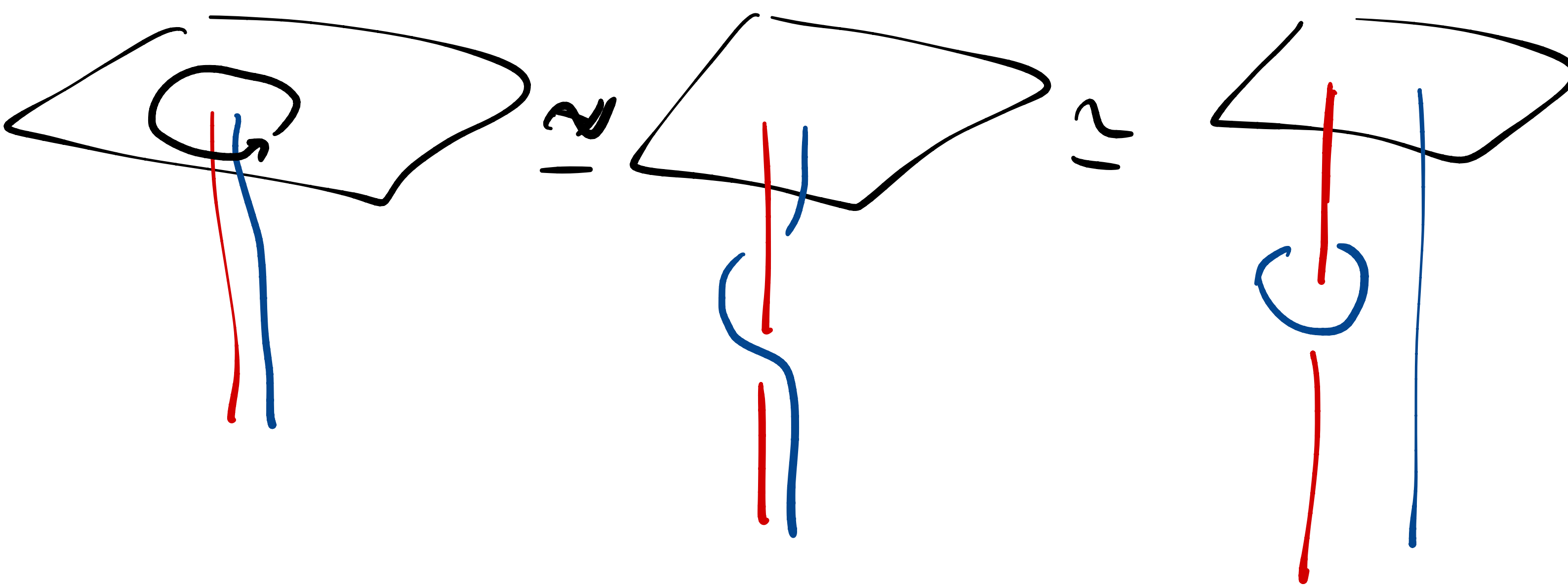
$\vartheta = 0$

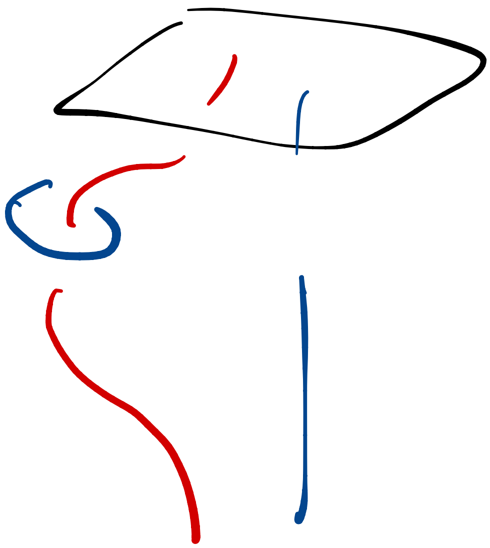
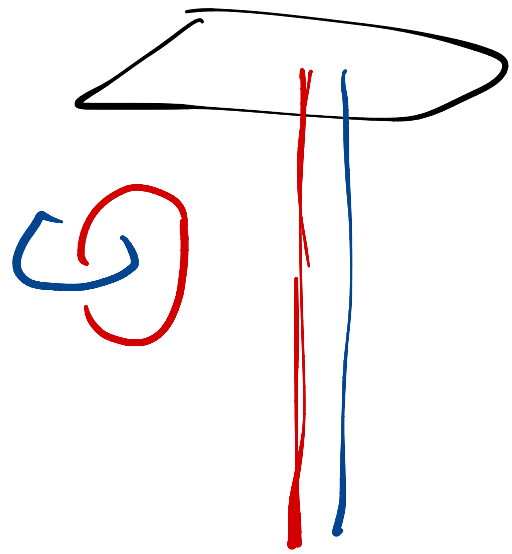
$K = \sigma^x$

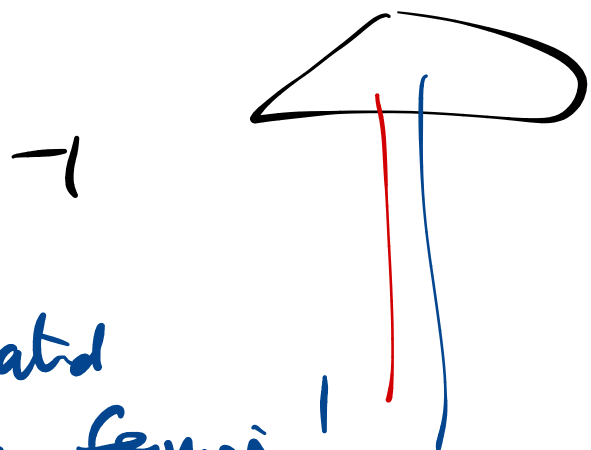
$= (-1)$

Imagine we break $U(1)_1 \times U(1)_2 \to U(1)$

vortex lines collimate.



$\simeq$  $\simeq$ 

$= -1$ 

→ end of the collimated vortex lines is a <u>fermion</u>!

Same strategy works for NLSMs = [maps: ϕ: spacetime → M.]

w/ Θ-terms.

in the <u>strong-coupling</u> limit.

Note: $\underline{\underline{e^{i\,CS(a)}}} = \Psi[a]$

Solves Schrod eqn for YM thy!

catch: not norm'ble as a wavefn

for $a(x) = \sum_k e^{ikx}\, \underline{\underline{\hat{a}_k}} + h.c.$

Group Cohomology SPTs:

Divide spacetime into a simplicial complex

dofs $= g_i \in G$ on the 0-cells.

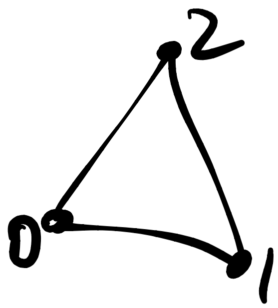
$$Z = \sum_{\{g\}} e^{-S[\{g\}]}.$$

- G symmetric
- $e^{-S[\{g\}]} = 1$ on a closed mfld.
- subdivision inv't, $\left(|e^{-S}|^2 = 1 \right)$
  (RG fixed pt.)

essential ingredient: group cocycle ν_D

$$\nu_D : G^{D+1} \longrightarrow U(1)$$

A D-simplex is specified by D+1 vertices

eg. 

$= [012]$.

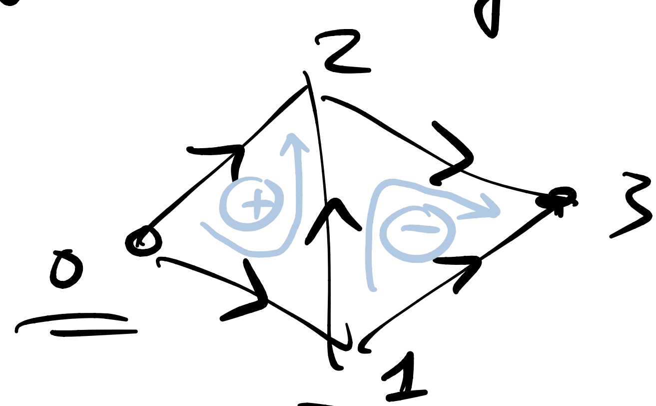
$$Z = \frac{1}{|G|^{N_v}} \sum_{\{g\}} \prod_{\substack{\text{simplices} \\ [i_0 \cdots i_D]}} \nu_D^{s_{i_0 \cdots i_D}}(g_{i_0} \cdots g_{i_D})$$

$s_{i_0 \cdots i_D} = \pm 1$ according to orientation.

To orient: put a branching structure.

= an arrow for each edge w/ NO loops

# of arrows going in = # of vertex 0



$s_{012} = +1$

$s_{123} = -1$.

Def of cocycle:

$\left[\begin{array}{l} s(g) = +1 \text{ if } g \text{ is unitary} \\ -1 \text{ if } g \text{ is antiunitary} \end{array}\right.$

① G-invt

$$\nu_D(g_0, g_1, \dots, g_D)^{s(g)} = \nu_D(gg_0, gg_1, \dots, gg_D)$$

② cocycle condition:

$$(\delta\nu) = 1$$

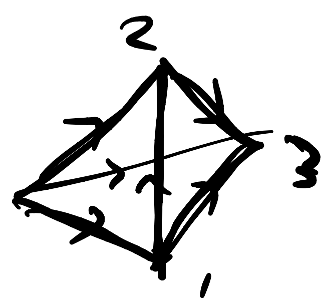
$$\left\{\begin{array}{l} \delta : D\text{-}\cancel{\text{cycle}}^{\text{chains}} \to D+1 \text{ cocycle} \\ \underline{\underline{\delta^2 = 1}} \end{array}\right.$$

$$(\delta\nu)(g_0 \dots g_{D+1}) = \prod_{i=0}^{D+1} \nu_D^{(-1)^i}(g_0 \dots g_{i-1}, g_{i+1}, \dots g_{D+1})$$

D=2.

eg:

$$\delta\nu(0123) = \frac{\nu(123)\,\nu(013)}{\nu(023)\,\nu(012)} \overset{!}{=} 1.$$

$e^{-S}\Big($  $\Big) = 1.$

$\nu_D \to \nu_D \, (\delta \mu_{D-1})$ would work as well.

$\mu_{D-1} : G^D \to U(1)$ a $D-1$ cochain.

But μ's all cancel on a closed mfld.

models of this form $\longleftrightarrow$ $H^D(G, U(1))$

"group cohomology group"

$$\equiv \frac{\text{(closed) cocycle } \nu}{\text{exact cochain } \delta\mu}$$

cocycle $\equiv$ closed cochain.

Why is $\mathbb{Z}_r$ natural?

① couple to BG G lattice gauge fields

$\longrightarrow$ Dijkgraaf-Witten gauge thy.

discrete G gauge thy "twisted" by a D-cocycle ν_D of G eg: $H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$

trivial ν : ordinary Ising model $\xrightarrow[\mathbb{Z}_2]{\text{gauge}}$ toric code

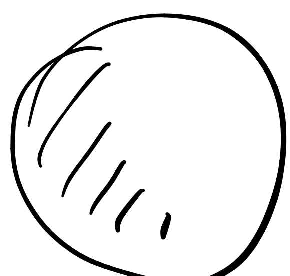
$$\Psi(\text{loops } c) = 1.$$

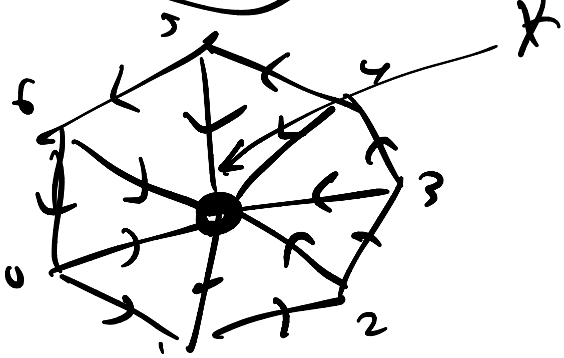
$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

nontrivial ν : Levin-Gu SPT $\longrightarrow$ double semion model

$$\Psi(\text{loops } c) = (-1)^{\#\text{of loops}}$$

$$K = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}.$$

② Construct $\Psi_\nu(\{h\}) =$  $\leftarrow \{h\}$



"minimal diagram"
just one bulk vertex!
*

$$\Psi_v(\{h\}) = \cancel{\frac{1}{|G|}\sum_{g^*}} \underbrace{\prod_{\substack{\text{simplices}\\ i_1\ldots i_{D-1}*}} \nu_D^{s_{i_1\ldots i_{D-1}*}}(h_{i_1}\ldots h_{i_{D-1}}, g^*)}_{\text{ind of } g^*}$$

$$= \underbrace{\prod_{\substack{\text{simplices}\\ i_1\ldots i_{D-1}*}} \nu_D^{s_{i_1\ldots i_{D-1}*}}(h_{i_1}\ldots h_{i_{D-1}}, g^*)}_{\text{ind of } g^*}$$

$$\mathcal{H} = \bigotimes_{\substack{\text{bdy}\\ \text{values } i}} \operatorname{span}\{|g_i\rangle\}$$

$$|1\rangle_i = \sum_{g\in G} |g\rangle_i \qquad |\mathbb{1}\rangle = \bigotimes_i |1\rangle_i$$

$$H_0 = -\sum_i |1\rangle\langle 1|_i \qquad \left(-\sum_i X_i\right)$$

$$\Psi_0(\{h\}) = \langle \{h\} | \mathbb{1} \rangle = 1 .$$

$$U_{g^*} \equiv \sum_{\{h\}} \prod_{\substack{\text{simplices} \\ i_1 \ldots i_{D-1} *}} \nu_D^{s\ldots}(h_{i_1} \ldots h_{i_{D-1}}, g^*) \, |\{h\}\rangle\langle\{h\}|$$

- unitary
- local
- finite depth
- not a product of G symmetric unitaries.

$$H \equiv - U_{g^*} H_0 U_{g^*}^\dagger$$ is G-symmetric

has gs $U_{g^*} |\mathbb{1}\rangle = \underline{\underline{|\Psi_\nu\rangle}}$.

eg: Haldane chain

$$\nu \in H^2(\mathbb{Z}_2^T, U(1))$$

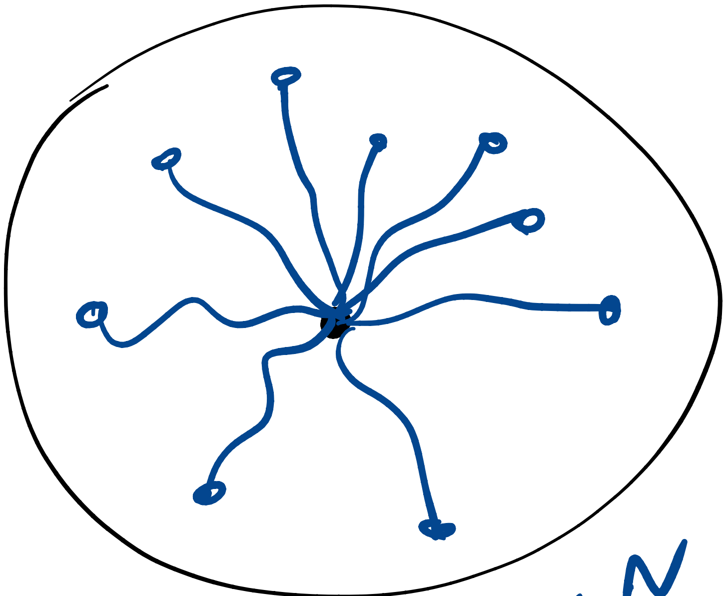
$$\mathbb{Z}_2^T = \{e, t\}$$

$H^2(G, U(1)) \equiv$ projective reps.

$$\left(U(t)^2 = -\mathbb{1}. \right)$$

eg: $D = 2+1$ $\quad H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$

$\longrightarrow$ Levin Gu SPT.



Ψ(positions x_i of N Wilson lines in S^2)

$$= \left\langle \prod_i^N e^{i\varphi(x_i)} \times e^{-iQ\varphi(\infty)} \right\rangle$$

$$\mathcal{L} \ni \bar{\Psi} \gamma^{[\mu}\gamma^{\nu]} \psi B_{\mu\nu} + \bar{\Psi} \gamma^{\mu} \psi a_{\mu}$$