Symmetric gapped (TO) surface of a TI in 3+1 d.

Dirac surface phase

$$S[\eta] = \int d^{2+1}x \, \bar{\eta} \, i \gamma^\mu \partial_\mu \eta$$

= higgs phase of the gauge theory $\langle w \rangle \neq 0$.

U(1) gauge theory of a_μ in some other phase

r_w

$$\eta = \chi w^n$$

χ: Dirac fermion; w: scalar

	χ	ϕ	w
A	1	2	0
a	n	$2n$	-1

Condense w
break $U(1)_a \times U(1)_A$

$\longrightarrow U(1)_A$.

$$\eta = \chi \langle w^n \rangle \propto \chi.$$

$$L_\phi = \bar{\phi} \chi_a i \epsilon^{ab} \chi_b + h.c.$$

Fu-Kane SC term w/ $\underline{\Delta = \bar{\phi}}$

what if $\langle \phi \rangle \neq 0$?

$$U(1)_a \times U(1)_A \xrightarrow{\langle \phi \rangle} \widetilde{U(1)}$$ generated by $\tilde{q} = q_A - q_a/n$.

(ϕ is neutral under $\tilde{q}$.)

$$|D\phi|^2 = \ldots \quad (2A + \underline{2na})^2$$

Higgs:

eom for $a \Rightarrow a = -2A/2n$

(no mass for A.)

effective charge:

	χ	ϕ	w
A	0	0	$-1/n$

$\tilde{q} = q_A - q_a/n$

moreover:

actually

$$U(1)_A \times U(1)_a \longrightarrow \tilde{U}(1) \times \underline{\mathbb{Z}_{2n}}$$

under $U(1)_a$: $\phi \to (e^{i\alpha})^{2n}\phi$

if $2n\alpha = 2\pi$ is invariant.

$\to$ a $\mathbb{Z}_{2n}$ gauge theory.

Ignoring fermions : vortices of ϕ

$\int \frac{f}{2\pi} = \frac{1}{2n}$ are m particles of the $\mathbb{Z}_{2n}$ g.t.

a general excitn is (k, v)

k: e charge, v: ν charge

$$W^x_{kv} W^y_{k'v'} = W^y_{k'v'} W^x_{kv} \, e^{\frac{2\pi i (kv' - k'v)}{2n}}.$$

Dirac eqn for χ in a vortex of ϕ :

same as Fu-Kane sc.

$\Rightarrow$ majorana ZM!

on an actual quasiparticle!

$\longrightarrow$ N.A. statistics

~ Moore Read

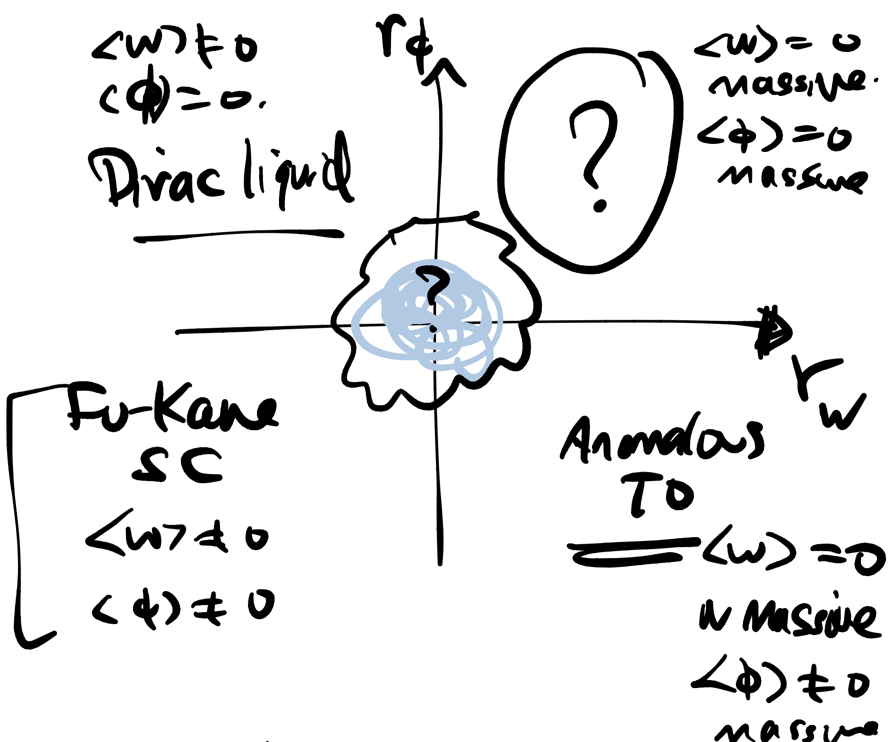
but T invt!

charge-ν vortex has ν majorana zmes
γ_i

But $\Delta H = i\gamma_i\gamma_j$ gaps out an even #.

$$\Rightarrow \begin{cases} \text{odd } \nu \text{ vortices have a MZM} \\ \text{even } \nu \text{ '' have none} \end{cases}$$

$$\Delta L = u_\phi |\phi|^4 + r_\phi |\phi|^2 + u_w |w|^4 + r_w |w|^2$$



$$\eta \xrightarrow{T} \gamma^0 \eta(-t, x)$$

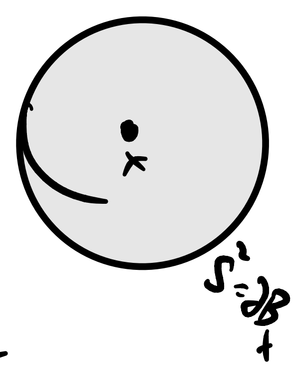
What can n be?

Gauge-inv't operators:

$\bar{\chi} \cdot\cdot \chi$ $\quad \bar{\Phi}\phi \ldots$ also monopoles.

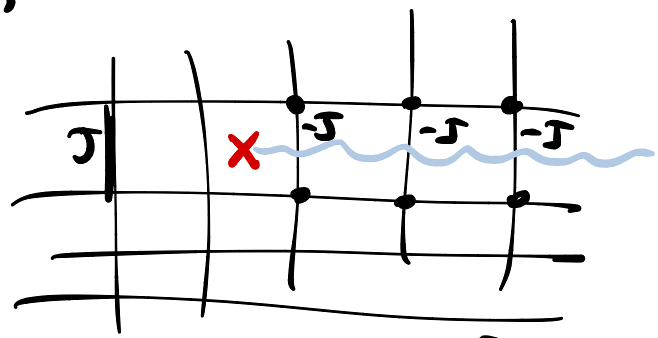
Monopole operator $\quad e^{i\sigma(x)}$

$$\langle e^{i\sigma(x)} \ldots \rangle = Z^{-1} \int_{\oint f = 2\pi} [Da]\, e^{-S} \ldots \qquad f = da$$



Disorder op in the Ising model:

$$\langle \mu(x) \cdots \rangle \equiv \sum_{\{s\}} e^{-\tilde{H}_\mu} \cdots$$



what are $U(1)_A$ charge & spin of $\underbrace{e^{i\sigma(x)}}$?

$=$ quantum #s of the groundstate(s) on S^2 w/ $\oint_{S^2} f = 2\pi$.

[state-op correspondence of CFT.]

χ has charge n under a

$\xrightarrow{\text{Index thm}}$ n complex zm operators $\chi_{i=1..n}$.

$$\chi(\theta) = \sum_i \hat{\chi}_i \underbrace{\psi_i(\theta)}_{\text{mode}} + \cdots \xrightarrow{\text{ETCR}} \begin{cases} \{\chi_i, \chi_j^\dagger\} = \delta_{ij} \\ \{\chi_i, \chi_j\} = 0 \end{cases}$$

$$\chi_i |\Downarrow\rangle = 0 \qquad \chi_i^\dagger |\Downarrow\rangle \cdots\cdots$$

$$\cdots \quad \chi_i^\dagger |\Uparrow\rangle = 0 \qquad 2^n \text{ states.}$$

$$|\Uparrow\rangle = \chi_1^\dagger \cdots \chi_n^\dagger |\Downarrow\rangle$$

each $\chi^\dagger$ adds $\Delta q = (\Delta q_A, \Delta q_a) = -(1, n)$.

and $\frac{1}{2}$ integer spin.

$$q_\Uparrow = -q_\Downarrow = q_\Downarrow + n\Delta q.$$

$$\Rightarrow q_\Downarrow = \underline{\left(\frac{n}{2}, +\frac{n^2}{2}\right)}.$$

$\chi_{i_1}^\dagger \cdots \chi_{i_{n/2}}^\dagger |\Downarrow\rangle$ have charge $(0, 0)$.

$$\left(\frac{n}{2}, +\frac{n^2}{2}\right) + \frac{n}{2}(-1, -n) = (0, 0).$$

<u>If n is odd</u>: if $|\Downarrow\rangle$ has $\frac{1}{2}$-integer spin

$|\Uparrow\rangle$ has integer spin

$\times$

Assume $n = 2s$ even.

$\Rightarrow$ $\underbrace{\chi_{i_1}^{+} \cdots \chi_{i_{n/2=s}}^{+} |\Downarrow\rangle}$ are neutral.

$|\Downarrow\rangle$ has spin 0 $\longrightarrow$ has spin $\frac{s}{2}$ mod 1

In a system made of electrons $\leftarrow$

all spin-$\frac{1}{2}$ objects carry odd electric charge!

"spin charge rel'n".

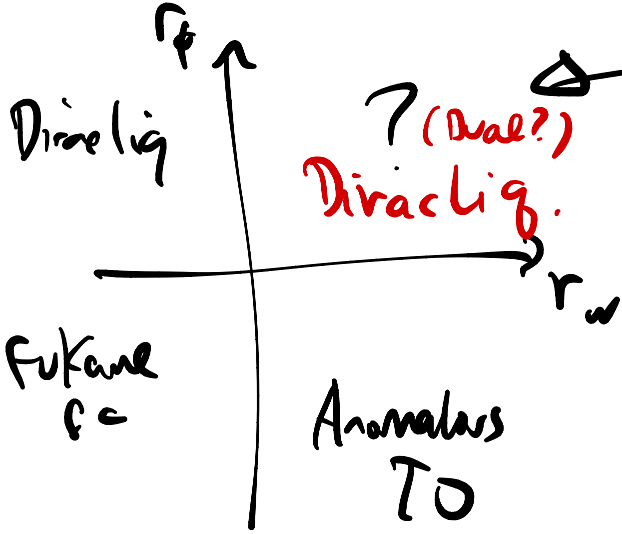
odds violates this rule

$\longrightarrow$ take s even

$\Longrightarrow$ $\boxed{n = 4, 8 \ldots}$

minimal case is $\mathbb{Z}_8$ gauge theory!

$2n = 8$



U(1) g.t. w/ one Dirac ferm χ in D=2+1.

$$N_f = 1 < N_f^{\text{critical}}$$

$\overset{?}{\Longrightarrow}$ confines.

gauge invt ops: $e^{i\sigma(x)}$ ...

$$\eta = \chi w^n .$$

3.3 Coupled-layer construction

Start with a "Layer"

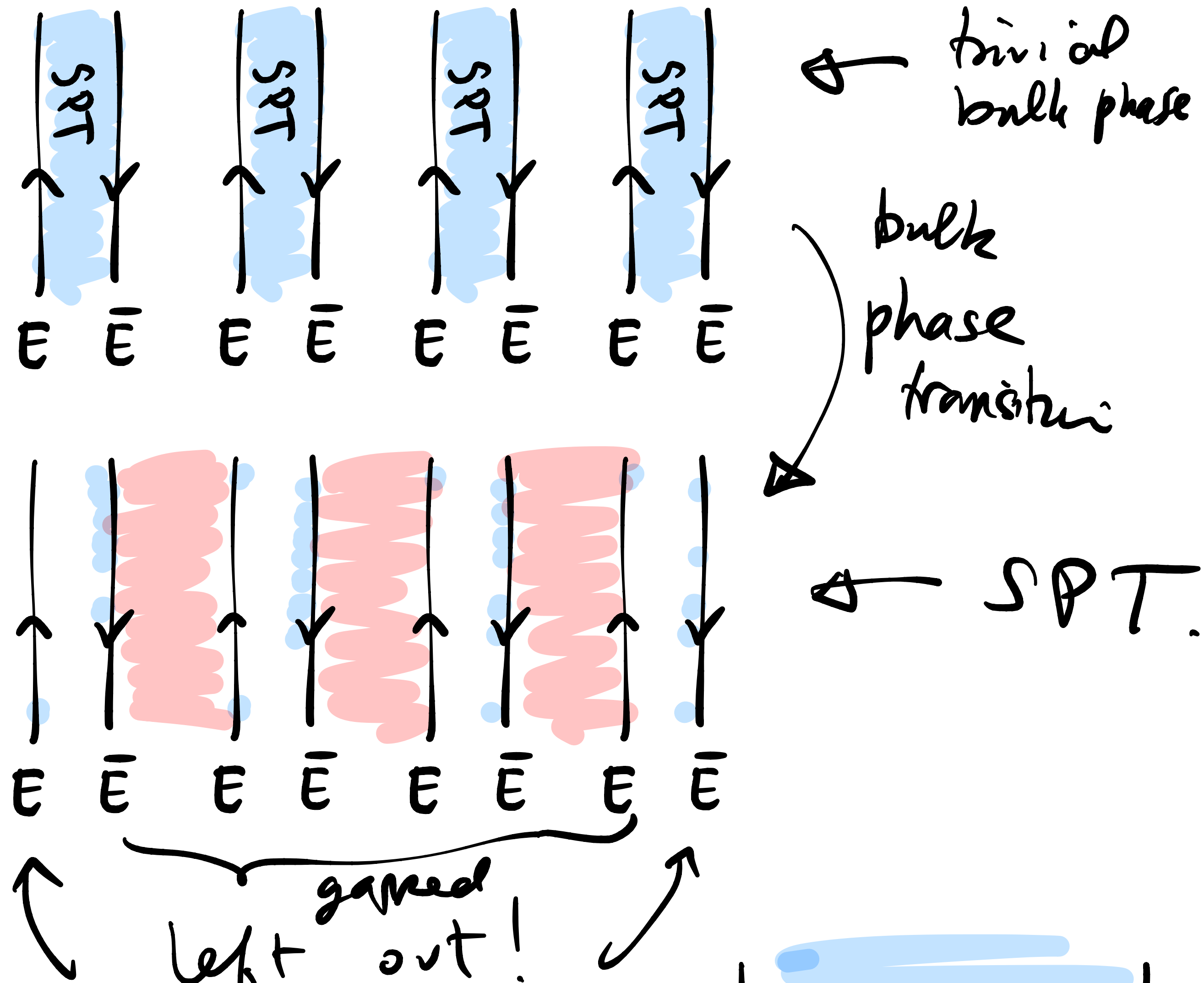
d space dims

E $\bar{E}$

opposite anomalies

no anomaly.

has a local, symmetric d-dim'l H.



EXAMPLES

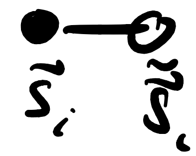
• <u>AKLT/Haldane</u> :

edge thy : spin $1/2$, a projective rep of G.

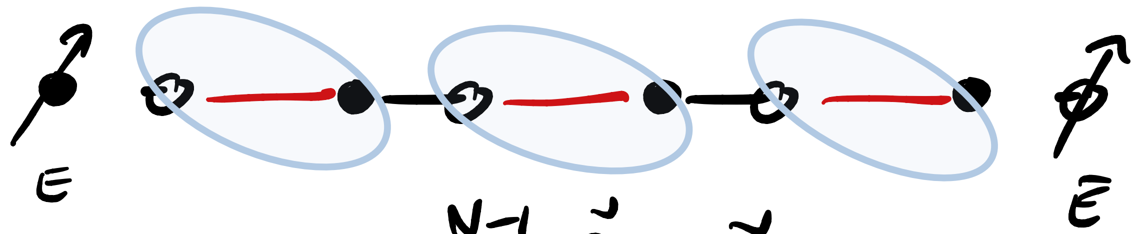
<u>$G = SO(3)$.</u> A <u>PAIR</u> of spin $1/2$ is a linear rep of $SO(3)$.

LAYER  $\leftarrow \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$.

$$H_0 = t_0 \sum_{i=1}^{N} \vec{S}_i \cdot \vec{\tilde{S}}_i$$



$t_0 \gg t_e$



$t_e \gg t_0$

$$H' = t_e \sum_{i=1}^{N-1} \tilde{\vec{S}}_i \cdot \vec{S}_{i+1}$$

→ edges of the chain transform in projective rep. of G.

SPTs in D=1+1 are labelled by a projective rep $\omega \in H^2(G, U(1))$.

$$U_{g_1} U_{g_2} = \omega(g_1, g_2)\, U_{g_1 g_2}$$

OR: $G = \mathbb{Z}_2^T$

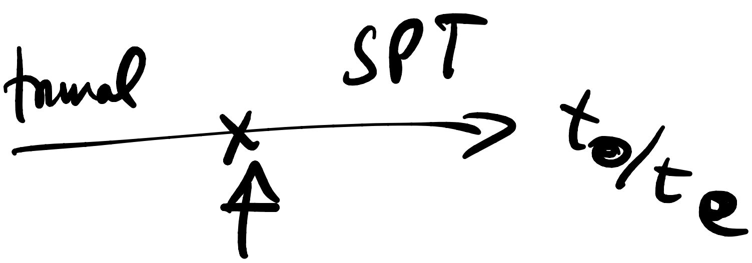
on a spin $\frac{1}{2}$ $T^2 = -1$. (projective rep)

Kramers' doublet.

on a pair of spin $\frac{1}{2}$ s $T^2 = \mathbb{1}$.

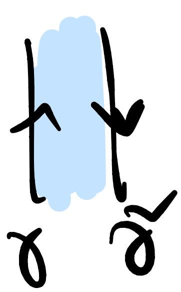
$$H_0 = -(X\tilde{X} + Z\tilde{Z})$$

when $t_o = t_o$ → Heisenberg spin $\frac{1}{2}$ chain.
a CFT.



- Kitaev chain

LAYER = Rep of one cx ferm $\{c, c^\dagger\} = 1$



$$c = \frac{\gamma + i\tilde{\gamma}}{\sqrt{2}}$$

$\{\gamma, \tilde{\gamma}\} = 0$

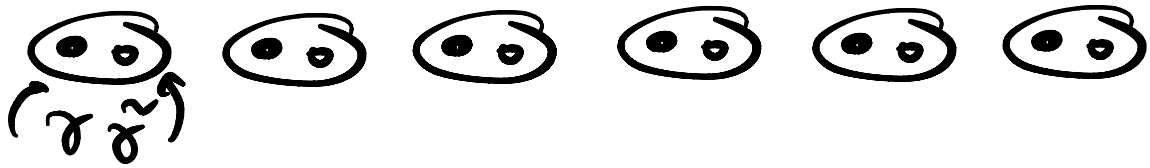
$\gamma^2 = 1 = \tilde{\gamma}^2$

majorana modes.

$G = \mathbb{Z}_2^T$ or **Nothing**

$c \to c,\ i \to -i$

$\Rightarrow \gamma \to \gamma,\ \tilde{\gamma} \to -\tilde{\gamma}.$

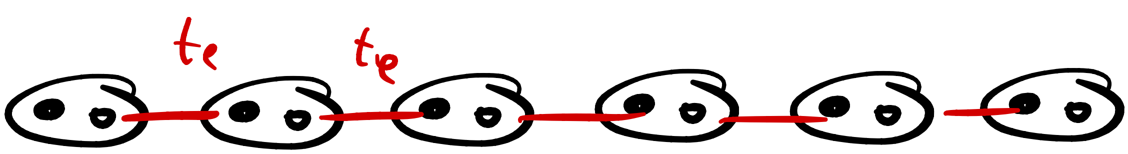


$$H_0 = t_0 \sum_i c^\dagger_i c_i$$

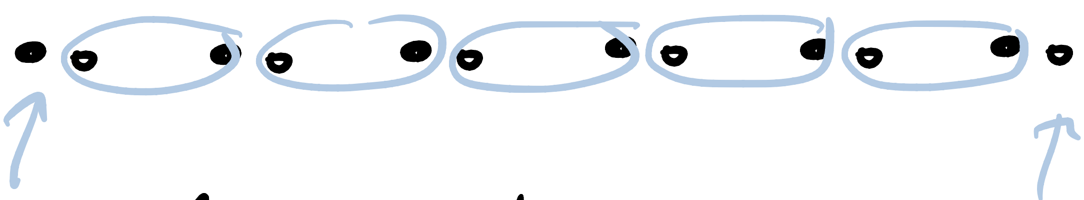
$$= t_0 \sum_i i \gamma_i \tilde{\gamma}_i$$

$$H_e = t_e \sum_{i=1}^{N-1} i \tilde{\gamma}_i \gamma_{i+1}$$

$t_e \ll t_0$:



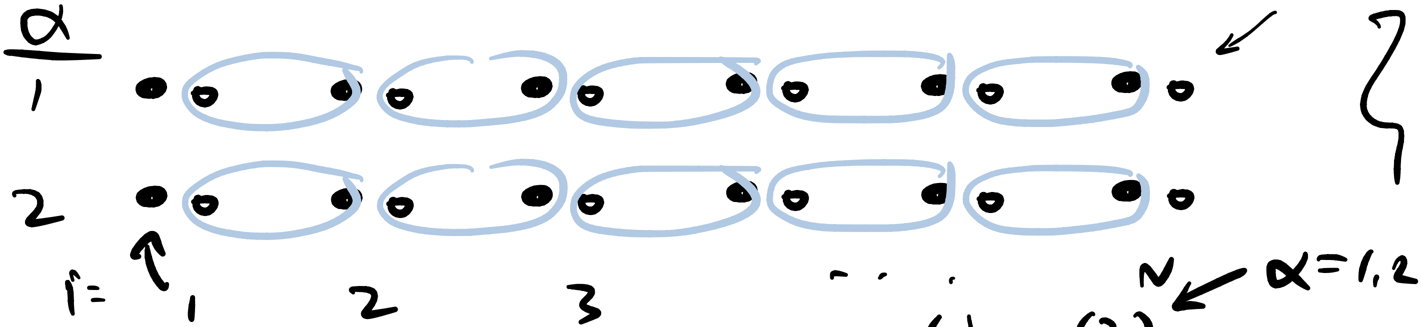
$t_e \gg t_0$:



$\underline{t_e = t_0}$: gapless Majorana chain

$$L = \bar{\psi} \gamma^\mu \partial_\mu \psi$$ w/ $\psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}$ real. $c_L = c_R = 1/2$.

(nonchiral)



w/ $G = \{e\}$ $\qquad \underline{\Delta H} = i\,\gamma^{(1)}_{i=1}\,\gamma^{(2)}_{i=1}$

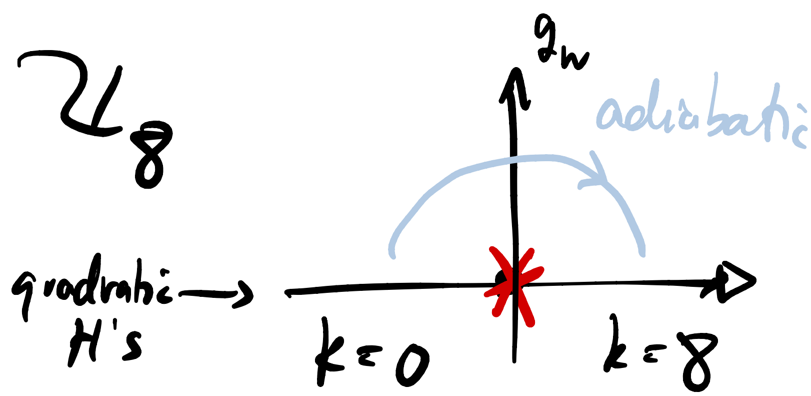
$$+\, i\,\tilde{\gamma}^{(1)}_{i=N}\,\tilde{\gamma}^{(2)}_{i=N}\,. \longrightarrow \text{trivial}$$

$\Longrightarrow$ mod 2 classification.

if $\underline{G = \mathbb{Z}_2^T}$: $\underline{\Delta H \text{ is not allowed}}$.

$\overset{?}{\Longrightarrow}$ integer classification ?

$$\mathrm{FSPT}^{D=1+1}_{\mathbb{Z}_2^T} = \mathbb{Z}_8$$



ingredient : made from 8 majoranas

$$W = \underline{\gamma_1\gamma_2\gamma_3\gamma_4} + \dots$$

- $W = W^\dagger$
- T even
- W a unique lowest eval.

- Integer QH states $D=2+1$.

Bogoliubov transf.: $\gamma \; \tilde{\gamma} \quad \gamma \; \tilde{\gamma}$

$$H = \sum_i i \gamma_i \gamma_{i+1} t_i \qquad t_i = \begin{cases} t_e \\ t_o \end{cases}$$

$$= \sum \Big(\# c^\dagger_i c_i + \# c_i c_{i+1} + h.c. \Big)$$

$$= \oint dk \left(c^\dagger_k c_k f(k) + \underline{c_k c_{-k}} g(k) + h.c. \right)$$

$$d_k \equiv u_k c_k + v_k c^\dagger_k$$

$$\{ d_k, d^\dagger_{k'} \} = \delta_{kk'} \quad \{ d_k, d_{k'} \} = 0 \ldots$$

$$\Rightarrow |u_k|^2 + |v_k|^2 = 1.$$

Choose u, v s.t. $H = \int d^2k \, d^\dagger_k d_k \underline{\varepsilon_k}$

$$\Rightarrow E(k) = \pm \varepsilon_k$$