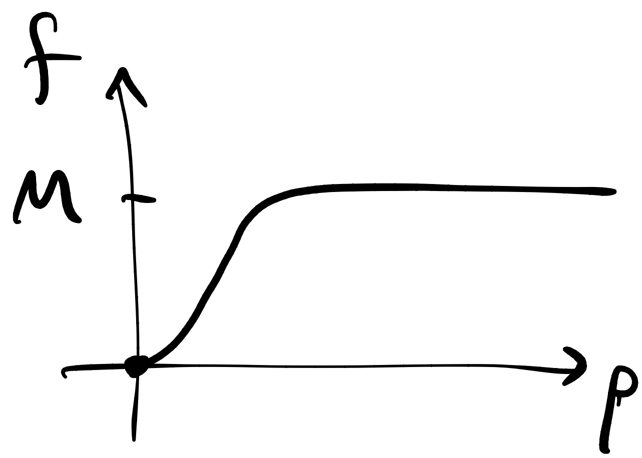
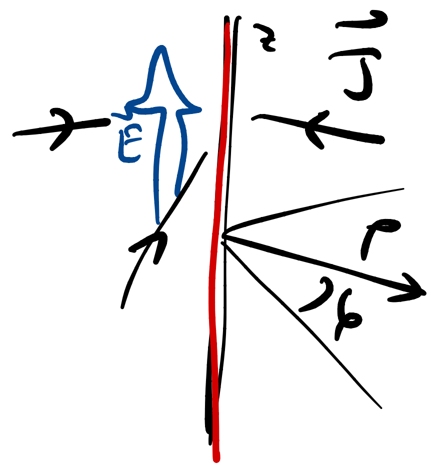
Day of fermion zeromodes on vortices:

Anomaly inflow, continued:

$$S[\Psi, A, \Phi] = \int d^{3+1}x \quad \bar{\Psi}\Big[(i\gamma^\mu D_\mu) + (\Phi_1 + i\gamma_5 \Phi_2)\Big]\Psi$$

eg: $\quad \Phi = \Phi_1 + i\Phi_2 = e^{i\varphi} f(\rho)$





Resolution of charge conservation:

vortex line traps a complex, charged chiral fermion z.m.!

$$0 = \frac{\delta S}{\delta \bar{\Psi}(x)}\gamma^0 \Bigg|_{A=0} \Rightarrow \Big(i\gamma^a \partial_a + i\gamma^2(\cos\varphi + i\gamma^{23}\sin\varphi)\partial_\rho\Big)\Psi_{\mp}$$

$$= f(\rho)e^{\pm i\varphi}\Psi_\pm .$$

$\boxed{\gamma^5 \Psi_\pm = \pm \Psi_\pm}$ $\quad$ along string $a = 0, 1$

along string is $a=0,1$

transverse to string : $2,3$

$$\begin{cases} \gamma^{01} \equiv -\gamma^0\gamma^1 \\ \gamma^{23} \equiv i\gamma^2\gamma^3 \end{cases}$$

note: $\cos\varphi + i\gamma^{23}\sin\varphi = e^{i\gamma^{23}\varphi}$

$$(\gamma^{23})^2 = 1.$$

Look for zero mode : $(\gamma^a \partial_a)(\,) = 0$

Ansatz: ind of φ

$$\boxed{\psi_+ = -i\gamma^2\psi_- \qquad \psi_- = \eta(x^a)e^{\alpha(\rho)}}$$

$$i\gamma^2 e^{i\gamma^{23}\varphi}(\partial_\rho\alpha)\eta = -f(\rho)e^{+i\varphi}\, i\partial^2\eta$$

$$\Longleftrightarrow \quad \partial_\rho\alpha = -f(\rho)$$

$$e^\alpha = e^{-\int_0^\rho d\rho' f(\rho')}$$ is exp'lly localized at $\rho=0$.

claim: ZM is guaranteed by an index theorem due to Callias.

$$\boxed{i\gamma^a \partial_a \eta = 0 \qquad \gamma^{01}\eta = -\eta}$$

η is a 1+1 d charged chiral spinor field! localized to $x_\perp = 0$.

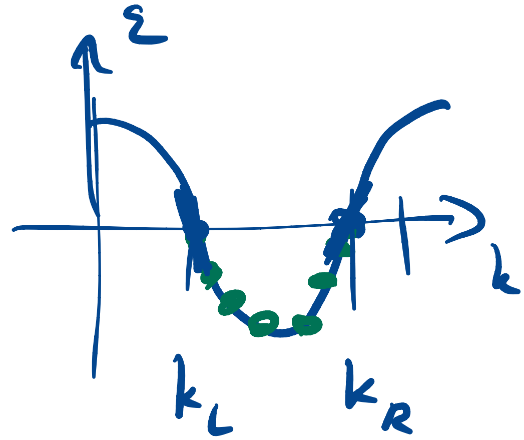
$$\delta_\lambda \left(S_{bulk} + S_{strip}[\eta, t] \right)$$

$$= \int_\Sigma \lambda \underbrace{\partial^\mu J_\mu}_{\text{var of Bulk action}} + \int_\Sigma \lambda \partial^\mu J_\mu$$

2D chiral anomaly

$$= \int_\Sigma \lambda \left(\frac{-\epsilon_{ab} F^{ab}}{4\pi} \right) + \int_\Sigma \lambda \frac{\epsilon_{ab} F^{ab}}{4\pi} = 0.$$

[Chiral Anomaly in $D = 1+1$
from flux-threading.



$$H = -\sum_i c_i^\dagger c_{i+1} + h.c$$

$$\rightsquigarrow S = \int \bar{\Psi} i \not{\partial} \Psi$$

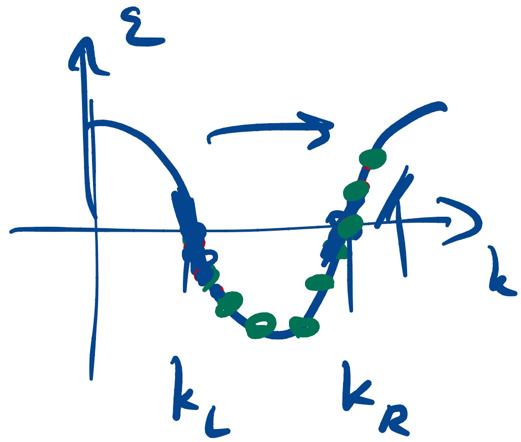
$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$c_x = e^{ik_L x} \psi_L + e^{ik_R x} \psi_R .$$

Thread flux: $\oint_{C_x} A = \oint A(t) \to E_x$.

$\partial_t P = e E_x$ on each particle.

$$\Delta p = e \int dt \, E_x(t)$$



$$\Delta Q_A = \Delta(N_L - N_R)$$

$$= 2\,\frac{\Delta p}{2\pi/L} = \frac{L}{\pi}\Delta p = \frac{L}{\pi} e \int dt\, E_x(t)$$

$$= \frac{e}{2\pi}\int \epsilon_{ab} F^{ab}$$

$\Rightarrow$ anomaly for a $\underline{\underline{\text{chiral}}}$ ferm

is $\frac{e}{4\pi}\epsilon_{ab}F^{ab}$.

Wess-Zumino descent (I_k is a k-form.)

$$I_{D+2} = d\, I^{(0)}_{D+1}\,, \quad \delta I_{D+1} = d\, I^{(1)}_D$$

$\underline{\text{eg}}$: $D=2$. $\quad I_4 = \frac{F \wedge F}{8\pi^2} \quad I_3^{(0)} = \frac{A \wedge F}{8\pi^2}$

$$I_2^{(1)} = \frac{\lambda}{2\pi} F/4\pi$$

$$S_\theta[A] = \int_{M_4} \theta\, I_4 \overset{\text{IBP}}{=} -\int_{M_4} d\theta \wedge I_3^{(0)} \qquad (\partial M_4 = 0)$$

$$\delta_\lambda S_\theta = \delta\left(-\int_{M_4} d\theta \wedge I_3^{(0)}\right)$$

$$= -\int_{M_4} d\theta \wedge \delta I_3^{(0)}$$

$$= -\int_{M_4} d\theta \wedge d I_2^{(1)}$$

$$\overset{\text{IBP}}{=} +\int_{M_4} d^2\theta \wedge I_2^{(1)}$$

<u>vortex</u>: $d^2\theta = 2\pi \cdot (\text{density of vortices})$

$$\oint_{\substack{\text{around}\\ \text{string}}} d\theta = \Delta\theta = 2\pi$$

$$d^2\theta = 2\pi\, \delta_2(\Sigma)$$

string worldsheet

$$\Rightarrow \delta_\lambda S_\theta = 2\pi \int_\Sigma I_2^{(1)} .$$

If spacetime is curved

$$I_4 = \frac{F \wedge F}{8\pi^2} - \frac{\operatorname{tr} R \wedge R}{48 \cdot (2\pi)^2}$$

$$\longrightarrow I_3^{(0)}, \quad I_2 \quad \ldots$$

$I_D = D$-form part of

$$\underbrace{I \equiv \hat{A}(TM) \wedge \operatorname{tr} e^{F/2\pi}}$$

$$\hat{A}(TM) = 1 - \frac{\operatorname{tr} R \wedge R}{48 (2\pi)^2} + \ldots$$

- Vortex in a $p+ip$ superconductor in $D = 2+1$ has a majorana fm ! [Read & Green] (Not T-inv't)

- edge of TI w/ an s-wave SC on top.

(is T-inv't)

$$S[\eta,\Delta] = \int d^{2+1}x \, \Big(\bar{\eta}\, i\gamma^\mu D_\mu \eta + \Delta\, \eta_a \underbrace{i\epsilon_{ab}}_{=\sigma^2} \eta_b + \Delta^* \eta^*_a i\epsilon_{ab} \eta^*_b \Big]$$

BG field.

$a,b = \uparrow\downarrow$ 2+1d spinor indices.

$$\Delta = e^{i\varphi} f(\rho) \qquad \text{set } A = 0$$

$$0 = \frac{\delta S}{\delta \bar{\eta}} \gamma^0 = \left(i\gamma^0 \partial_0 + i\gamma^i \partial_i \right)\eta - \Delta^* \sigma^2 \eta^*$$

$\partial_0 \to 0$ look for zms.

assume η ind. of t

choose: $\left[\begin{array}{l} \gamma^1 = \sigma^2 = i\epsilon \\ \gamma^2 = \sigma^3 \\ \gamma^0 = i\sigma^1 \end{array}\right]$

$$i\gamma^1 (e^{i\gamma^{12}\varphi}) \partial_\rho \eta = \Delta^* \gamma^1 \eta^*$$

$$\gamma^{12} \equiv i\gamma^1\gamma^2$$

Demand: $\gamma^{12}\eta_\pm = \pm \eta_\pm$

$$e^{\pm i\varphi} i\partial_\rho \eta_\pm = f(\rho) e^{-i\varphi} \eta_\pm^*$$

$\Rightarrow \eta_+ = 0.$ to match $e^{-i\varphi}$.

$$\boxed{i\partial_\rho \eta_- = f(\rho)\eta_-^*}$$

NOT $\mathbb{C}$-LINEAR!

$$\eta_- = e^{i\alpha}\eta_0, \ \alpha \in \mathbb{R}/2\pi\mathbb{Z}$$

$\Rightarrow$ phase of η_- is fixed!

$$e^{i(\frac{\pi}{2}+2\alpha)} \partial_\rho \eta_0 = f(\rho)\eta_0$$

WANT:

$$-\partial_\rho \eta_0 = f(\rho)\eta_0$$

$$\Rightarrow \eta_0 = e^{-\int_0^\rho d\rho' f(\rho')}$$

NORMABLE.

$$\Rightarrow \frac{\pi}{2} + 2\alpha \overset{!}{=} \pi$$

$$\Rightarrow \alpha = \pi/4.$$

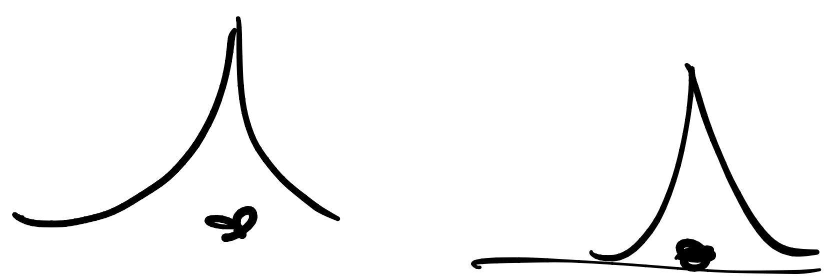
$$\eta(x^\mu) = \sum_k \left(\hat{a}_k \underbrace{u_k^+(x)}_{\sim e^{\pm ikx} \text{ for } \rho\to\infty} + \hat{b}_k^\dagger \underline{u_k^-(x)} \right) + \sum_{A=1}^{N} e^{i\nu}\, \underline{\eta_0^A(\rho)}\, \hat{\gamma}_A$$

$$\underline{\hat{\gamma} = \hat{\gamma}^\dagger}.$$

$$\xrightarrow{\text{CCRs}} \quad \{\hat{a}_k, \hat{a}_{k'}^\dagger\} = i\delta_{k-k'} \quad \ldots$$

$$\Longrightarrow \quad \underline{\hat{\gamma}^2 = 1}.$$

Suppose we have N well-separated vortices:



CCR $\Longrightarrow$

$$\{\gamma^A, \gamma^B\} = \delta^{AB}$$

$$A, B = 1 .. N.$$

For N zero modes, the Rep. of the algebra has $2^{[N/2]}$ states $\overset{N\gg 1}{\sim} \sqrt{2}^N$

→ unitary reps of the braid group on N el'ts.

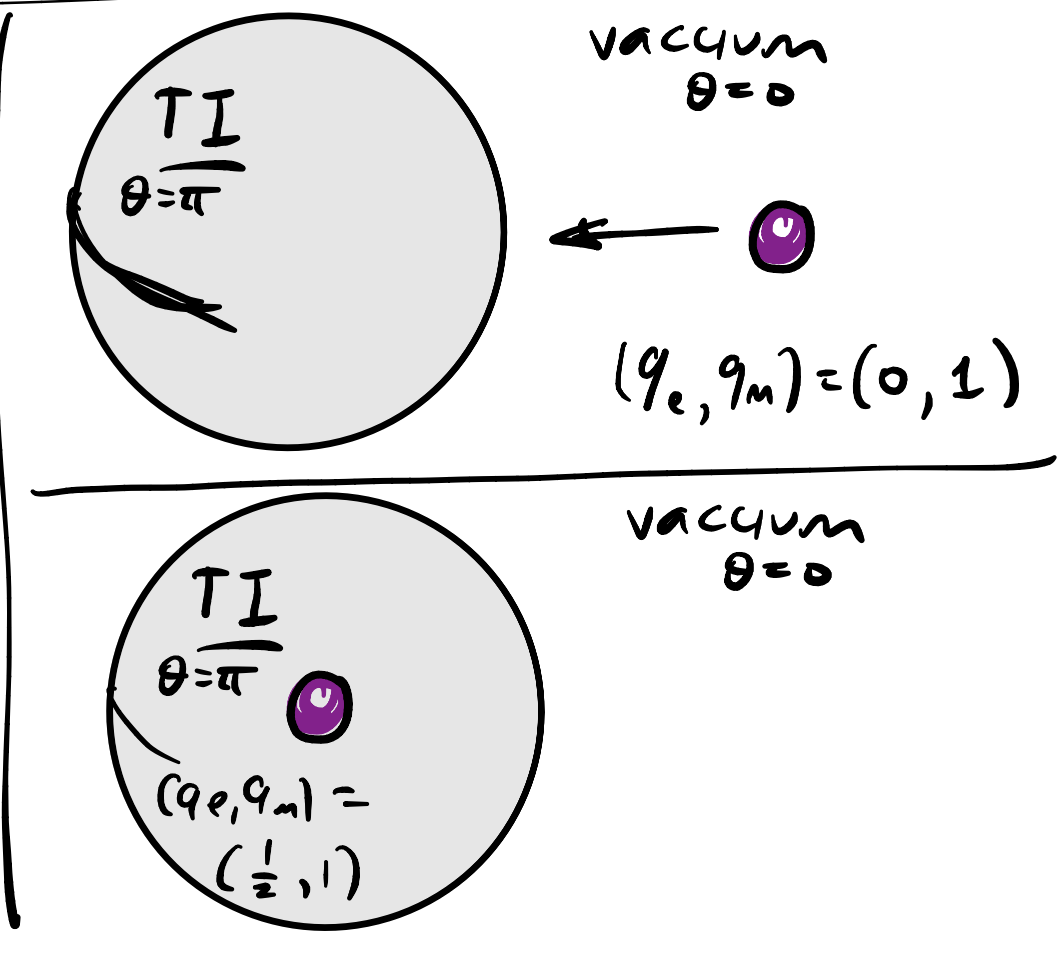
→ vortices have non-abelian statistics.

↑

extrinsic defects ⇒ not N-A TO.

vs {Moore-Read ← (NOT T-inv't) & (T.O.) gapped, symmetric edge of TI. ← (T-inv't.)} Ising TO.

Witten effect :



$$\rho = \frac{\theta}{2\pi} \nabla \cdot B$$

⇒ stuck to surface is charge $e/2$!

- Fu Kane SC. $\Delta\eta\eta$ Breaks $U(1)$ sym

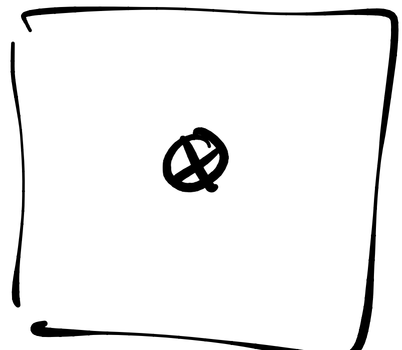
can absorb any charge.

- stick a magnet on top Breaks T: $\pm m\bar{\eta}\eta$

$$\left\{\begin{array}{l} -\text{gapped} \\ -\text{has } \sigma^{xy} = \pm\frac{e^2}{h}\frac{1}{2} = \frac{e^2}{h}\nu \end{array}\right.$$

threading 2π flux

has charge $e\nu$.



surface

- Gapless edge. $S[\eta] = \int_{S^2\times\mathbb{R}} i\bar{\eta}\cancel{D}\eta$

on S^2 w/ magnetic monopole inside.

w/ $\oint_{S^2} F = 2\pi$.

Atiyah-Singer index thm:

$$n_+ - n_- = \int_X I = \int_X \hat{A}\wedge \mathrm{tr}\, e^{F/2\pi} \overset{X=S^2}{=} \underbrace{\int_{S^2} F/2\pi}_{\in\mathbb{Z}} = 1.$$

# of zms of $D = i\gamma^\mu D_\mu$ on X

w/ $\gamma^5\eta_\pm = \pm\eta_\pm$.

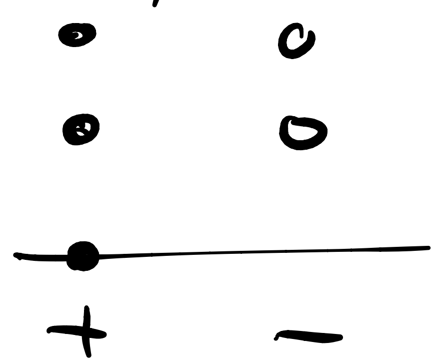
why is $n_+ - n_- = \mathrm{tr}\,\gamma^5$ topological?

$$H = \{D, D^\dagger\} \quad \text{is supersymmetric}$$

nonzero evals of H come in pairs one of each chirality

$$\{\gamma^5, D\} = 0 = \{\gamma^5, D^\dagger\}$$

$$H|\phi\rangle = E|\phi\rangle, \quad H(D|\phi\rangle) = E(D|\phi\rangle)$$



$\exists$ a complex 1 fcn η_0 on S^2.

$$\eta(x) = \eta_0\, c + \text{nonzeromodes}$$

$$\xrightarrow{\text{CCRs}} \{c, c^\dagger\} = 1.$$

$$c|\Downarrow\rangle = 0 \qquad |\Uparrow\rangle = c^\dagger|\Downarrow\rangle.$$

$$(c^\dagger|\Uparrow\rangle = 0.)$$

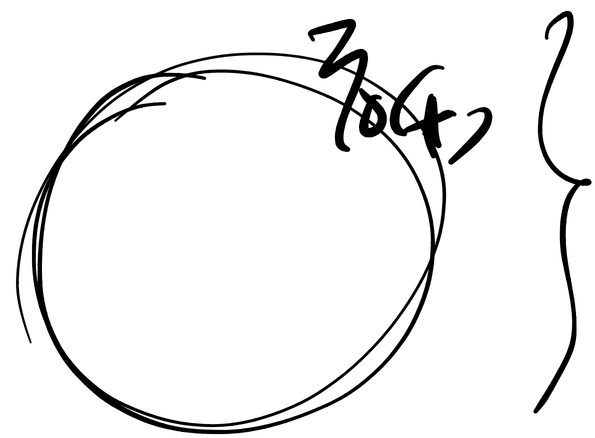
$$q(\Uparrow) = q(\Downarrow) + e$$

$$C\,|\Uparrow\rangle = e^{i\varphi}\,|\Downarrow\rangle$$

$$\Rightarrow\ q(\Uparrow) = -\,q(\Downarrow)$$

$$\Rightarrow \begin{cases} q(\Downarrow) = -e/2 \\ q(\Uparrow) = +e/2 . \end{cases}$$

charge spreads over surface.



- <u>Gapped, symmetric surface.</u>

charge from monopole is localized
$\Rightarrow$ must be carried by a quasiparticle
w/ $q = \pm e/2$.
$\Rightarrow$ fractionalization, T.O.

Parton Construction of gapped symmetric surface:

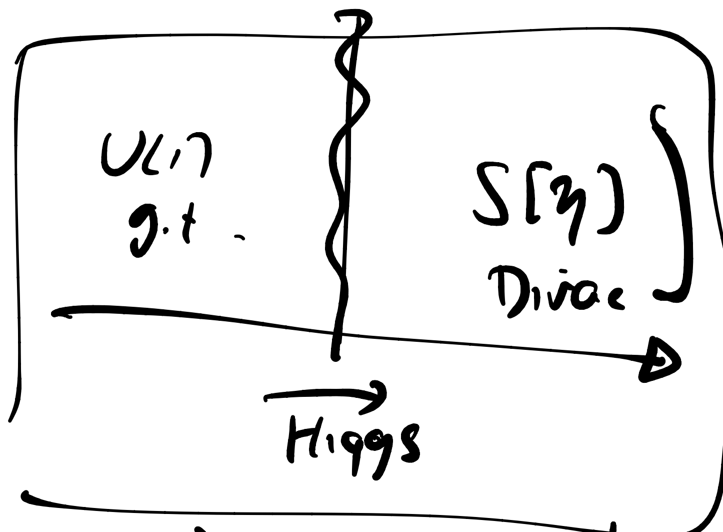
$$S[\eta, A] = \int d^{2+1}x \, \bar{\eta} i \gamma^\mu D_\mu^A \eta$$

Regard as the Higgs phase of a $\underset{\text{new}}{\wedge}$ $U(1)$ g.t.

Anomaly is a property of $\mathcal{H}$ and the rep. of G on $\mathcal{H}$

NOT of H.

or phase.



$\Rightarrow$ same anomaly.

Dirac fermion

scalar.

$$\eta = \chi w^n$$

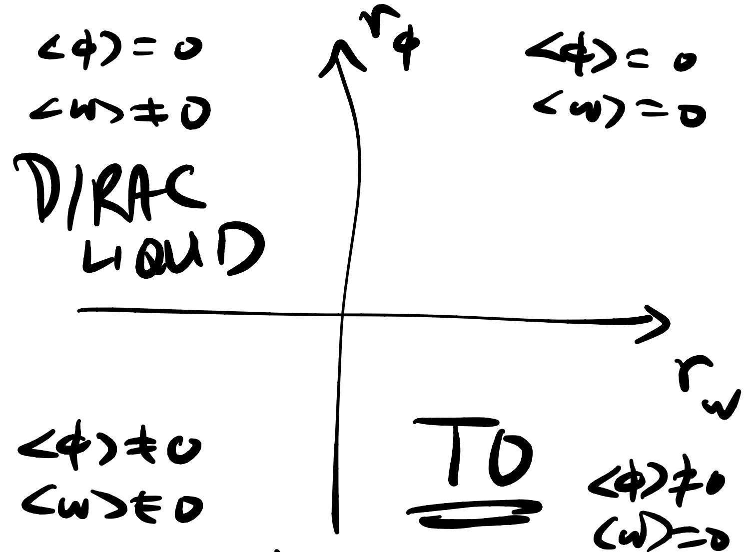
	χ	ϕ	w	η
A	1	2	0	1
a	n	$2n$	-1	0

new dynamical $U(1)$ g.f.

Condense w : $\langle w\rangle = 1$, assumption: ϕ is massive $\langle\phi\rangle=0$

$$|Dw|^2 = a^2|\langle w\rangle|^2 \longrightarrow \text{mass for } a$$

$$\eta = \chi w^{u} = \chi . \longrightarrow \underline{S[\eta]}$$



$$U = u_\phi|\phi|^4 + r_\phi|\phi|^2 + u_w|w|^4 + r_w|w|^2$$