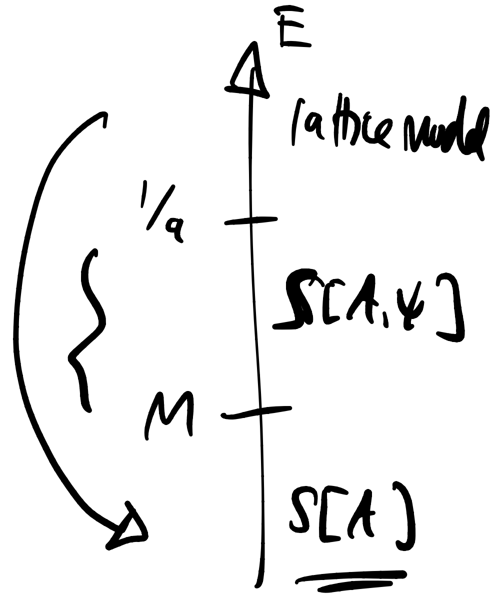
3+1 D TI ~ $G = \underbrace{U(1)}_{A} \rtimes \mathbb{Z}_2^T$

[free fermion lattice model = discretize massive Dirac fermion.



$$\theta = -\frac{1}{4\pi}\oint_{BZ} d^3k \, \epsilon_{ijk}\left(A_i \partial_j A_k - i\frac{2}{3} A_i A_j A_k\right)$$

$S(A) \ni \theta \int \frac{F \wedge F}{8\pi^2}$

$$\vec{A}_i = -i \langle \psi_k | \frac{\partial}{\partial k_i} | \psi_k \rangle$$

1-particle state

$\in 0, \pi$

$$\left(\theta = \frac{1}{4\pi^2}\frac{\partial P}{\partial B}\right) \qquad (\Leftarrow T \text{ invariance})$$

shortcomings:
① assumes transl inv.
② assumes free fermions

solution: [Niu Thouless Wu]

analogous case in 2d.

$$\sigma^{xy} = \frac{e^2}{h}\oint_{BZ} d^2k \, \mathcal{F} \qquad \text{[TKNN]}$$

Reinterpret k as a flux:

many body state $\Psi(\vec{x}_1 \dots \vec{x}_N)$ put on T^d w/ $x^i \equiv x^i + L^i$

choose bcs: "$\Psi(x_1 \dots \underbrace{x_\alpha + L_i}, \dots x_N)$" $= \underline{e^{i\Theta^i}} \Psi(x)$

$$\equiv \underline{e^{i \Pi_\alpha^i L_i \hat{i}}} \Psi(x) \leftarrow \text{gauge inv't.}$$

$$\Pi_\alpha^i \equiv -i \frac{\partial}{\partial x_\alpha^i} + A_i(x^\alpha)$$

can remove Θ from the bc by

$$\Phi(x) \equiv \underline{e^{-i \sum_i \frac{\Theta^i}{L^i} \sum_{\alpha=1}^N x_\alpha^i}} \Psi(x)$$

$\Rightarrow$ $\Phi(x_1 \dots x_\alpha + L_i, \dots x_N) = \Phi(x)$.

acting on Φ, $-i \frac{\partial}{\partial x^i} \rightsquigarrow -i \frac{\partial}{\partial x^i} + \Theta^i$

$\rightarrow$ Θ^i appears wherever k^i would appear.

If $H = \underline{\sum_i f(\Pi^i)} + \underline{V(x)} \xrightarrow[\text{analysis}]{\text{prev.}} \theta = \begin{pmatrix} \dots \\ \dots \\ k \to \Theta \end{pmatrix}$

3.2 Anomaly Inflow & zeromodes on defects

$$S[\Psi, A, \Phi] = \int d^4x \Big[\bar{\Psi}\, i\gamma^\mu \underset{\uparrow}{D_\mu} \Psi$$

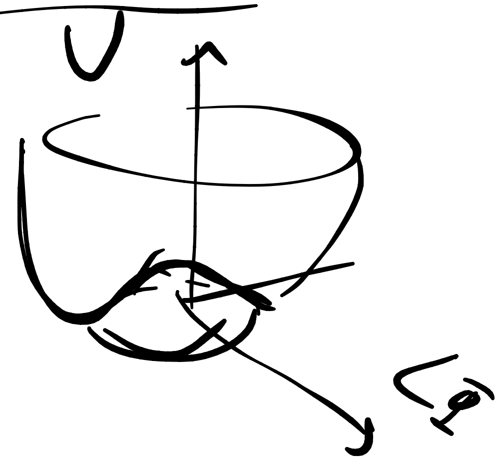
$$\equiv \partial_\mu - i A_\mu$$

$$- \bar{\Psi}(\Phi_1 + i\gamma^5 \Phi_2)\Psi$$

$$- \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \partial_\mu \Phi^* \partial^\mu \Phi - V(|\Phi|^2) \Big]$$

$$\Phi \equiv \Phi_1 + i\Phi_2$$

$$\langle |\Phi| \rangle = M > 0$$



S is inv't under

$$U(1)_A : \begin{cases} \Phi \to e^{i\alpha}\Phi \\ \Psi \to e^{-i\gamma^5 \alpha/2}\Psi \end{cases}$$

$\{\gamma^5, \gamma^\mu\} = 0$

$\gamma^5 \psi_\pm = \pm \psi_\pm$

one L & R.

chiral transformation.

Axial (chiral) anomaly:

$$\delta_\alpha S_{eff} = \int \alpha \frac{F \wedge F}{8\pi^2} = \int \alpha\, \partial_\mu j_A^\mu .$$

i.e. $\partial_\mu j_A^\mu = \dfrac{F \wedge F}{8\pi^2}$ (in flat space)

If Φ were dynamical, $U(1)_A$ is SSB.

$\Phi = M e^{i\theta}$ $\quad\theta$ is a goldstone "axion".

The coupling of θ to Ψ

$\bar{\Psi} M e^{i\theta\gamma^5}\Psi$ can be removed

by $\Psi \to e^{i\theta\gamma^5/2}\Psi$ (a chiral rotation)

$\Longrightarrow$

$$S_{eff}[A,\theta] = \int d^4x \left[\frac{\theta}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}\partial_\mu\theta\,\partial^\mu\theta\right]$$

$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$

Return to TI: $\Phi = \pm M \in \mathbb{R}$. $\Leftarrow \mathbb{Z}_2^T$

$\Longrightarrow$ $M>0$ has $\theta = 0$, $M<0$ has $\theta = \pi$.

$M>0$ is vac. $\qquad M<0$ is TI.

$$0 = \frac{\delta S_{\text{axion-electrodynamics}}}{\delta A_\nu(x)} = \partial^\mu F_{\mu\nu} - J_\nu$$

$$\underline{\underline{J_\nu}} = \frac{e^2}{16\pi^2}\partial^\mu\left(\theta \tilde{F}_{\mu\nu}\right)$$

Witten effect

monopole carries e charge $e\theta/2\pi$

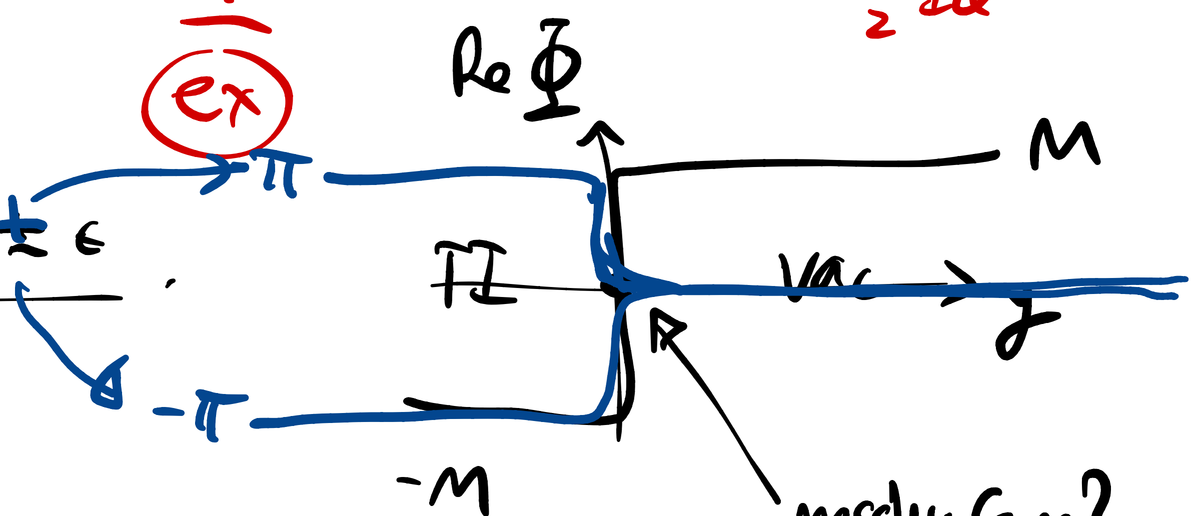
$$\begin{cases} \rho \sim \theta\, \vec{\nabla}\cdot\vec{B} + \underline{\underline{\vec{B}\cdot\vec{\nabla}\theta}} \\ \vec{J} \sim \dot{\theta}\vec{B} + \underline{\underline{\vec{\nabla}\theta}} \times \vec{E} \end{cases}$$

surface charge $\underline{-e\theta/2\pi}$.

surface hall response $\frac{1}{2}IQ$

? ex

Let $\mathrm{Im}\,\Phi = \pm\epsilon$



massless ferms?

θ is ill-def'd.

$$S_\theta = \frac{1}{8\pi^2}\int \theta F \wedge F \overset{\text{IBP}}{=} -\int \frac{d\theta}{\pi} \frac{A \wedge F}{8\pi}$$

<u>dw:</u> $d\theta = \Delta\theta\, \delta(y)\, dy \longrightarrow = \frac{\Delta\theta}{\pi}\frac{1}{2}\int_{y=0} \frac{A\wedge F}{8\pi}$

$\underline{\underline{\Delta\theta = \pm\pi}} \longrightarrow$ surface Hall response $= \pm\frac{1}{2}\frac{e^2}{h}$.

1) $S_{eff}[A]$ should be gauge inv't!

2) what produces this Hall conductivity?

$$0 = \frac{\delta S}{\delta\bar\psi}\gamma^0 = \underline{\underline{\left[i\gamma^a\partial_a + i\gamma^y\partial_y + g\Phi(y)\right]\psi}}$$

$\underline{a=0,1,2} \leftarrow$ along the wall

(set $A_\mu = 0$)

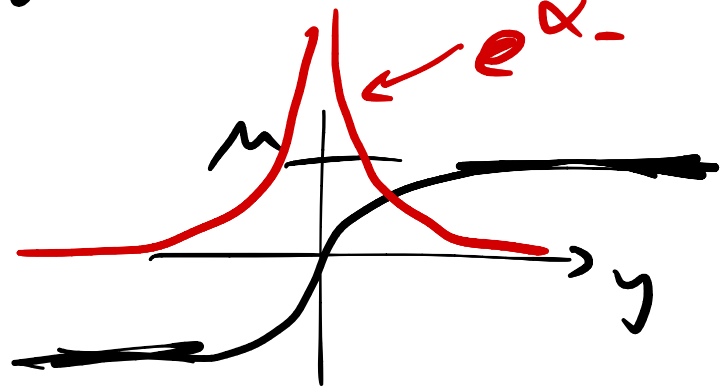
ansatz: $\psi = \eta_\pm(x^a)\, e^{\alpha(y)}$

choose $\underline{\gamma^y\eta_\pm = \pm i\eta_\pm}$.

$$\Rightarrow \mp \partial_y \alpha + g\Phi(y) = 0$$

$$\Rightarrow \alpha_{\pm}(y) = \pm \int_0^y dy' \, g \Phi(y')$$

$\Rightarrow e^{\alpha_-(y)}$ is norm'ble



$$\Rightarrow \boxed{\gamma^a \partial_a \eta_- = 0}$$

~~w/ $\gamma^y \eta_- = -i\eta_-$~~

Dirac eqn for a single Dirac cone in $D=2+1$.

$$S_{eff}[\eta] = \int_{y=0} d^{2+1}x \; \bar{\eta} \, i \not{D} \, \eta$$

~~No way to~~ quadratic term can give a mass w/ $U(1) \times \mathbb{Z}_2^T$.

$-m\bar{\eta}\eta$ breaks $\mathbb{Z}_2^T$

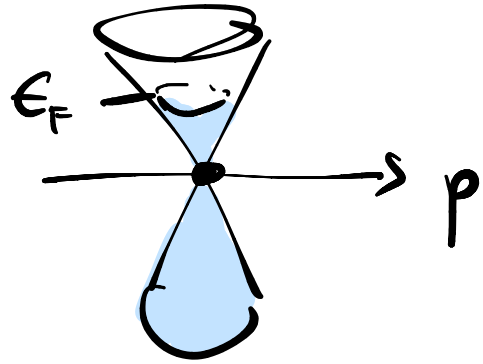
$-\Delta \eta_a i \varepsilon_{ab} \eta_b + h.c.$ breaks $U(1)$

"Fu Kane supercond."

$$-\mu\eta^\dagger\eta$$ (breaks Lorentz, preserves $U(1)\times \mathbb{Z}_2^T$)

$$= \mu\bar{\eta}\gamma^0\eta$$ $(\bar{\eta}\eta \equiv \eta^\dagger\gamma^0\eta)$

$\rightarrow$ fermi surface!



each of these MUST BE INTERESTING!

$$\pm m\bar{\eta}\eta \longrightarrow \sigma^{xy} = \pm\frac{1}{2}\frac{e^2}{h}.$$

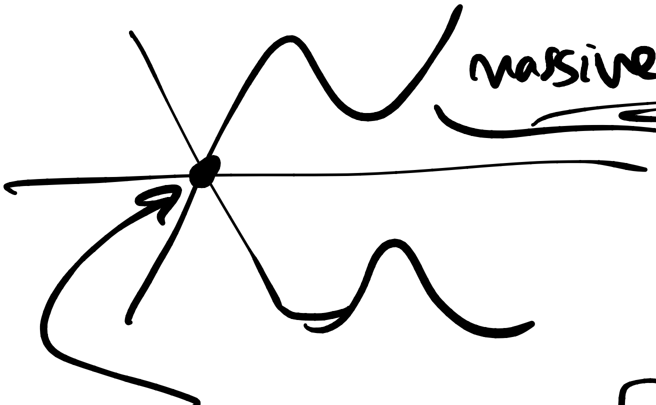
$\rightarrow$ even a single massive Dirac ferm can't arise locally in $D=2+1$

$$\not{D} \equiv \gamma^\mu D_\mu.$$

2+1 d
2×2 real γ's.

OR: Berry curvature near gap:

A Dirac cone emits Berry flux
can't live on a compact BZ.



massive Dirac fermion

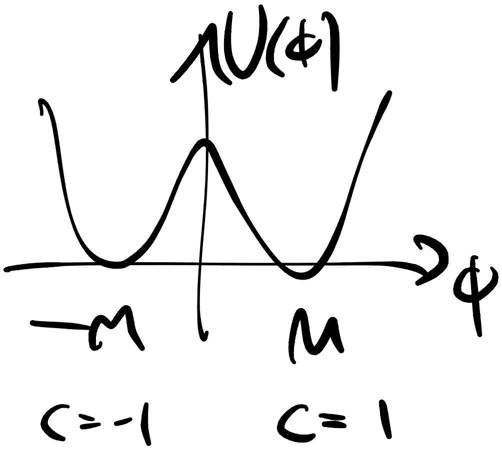
$$S_{eff}[\eta, A] = \int d^3x \left[i\bar{\eta}\not{D}\eta + \frac{A\wedge dA}{8\pi} \right].$$

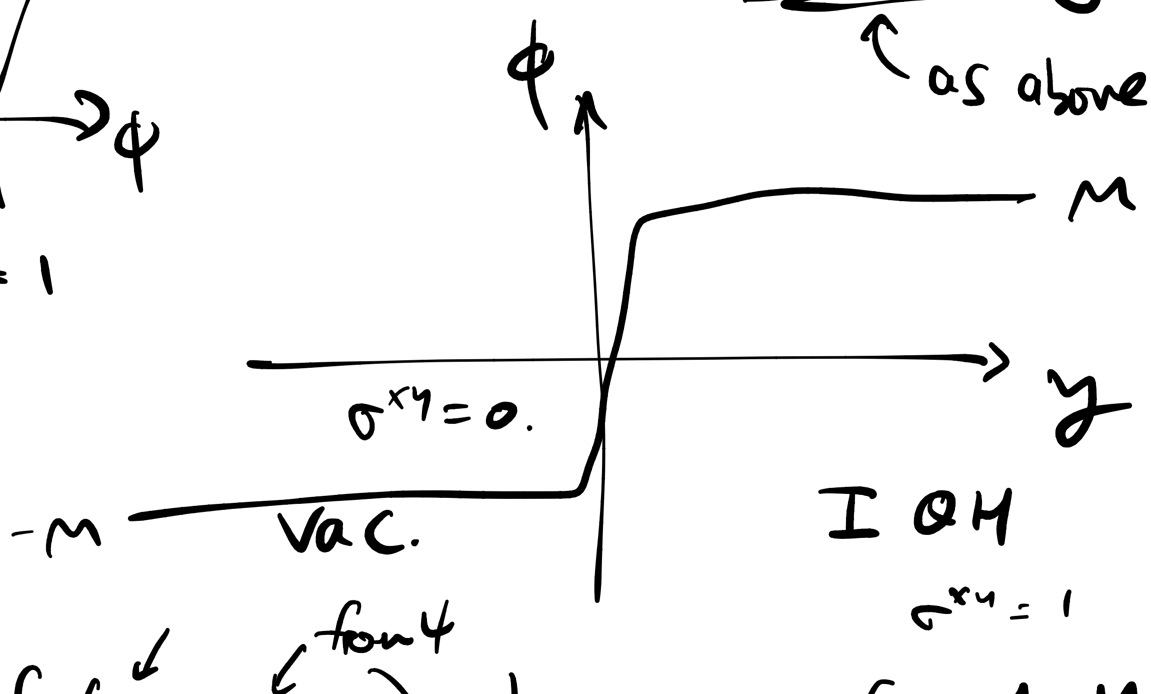
Anomaly inflow for edge states of IQHE:

$$S[\psi, A, \phi] = \int d^{2+1}x \left[\bar{\psi} i \not{D}_A \psi - g\phi\bar{\psi}\psi + \underline{\int \frac{A\wedge dA}{8\pi}} \right]$$

real scalar

as above





from ψ

$$S_{eff}[A] \approx \int \left(1 + \mathrm{sign}(y)\right) \frac{1}{8\pi} A\wedge dA = \int_{y>0} \frac{A\wedge dA}{4\pi}$$

$A \to A + d\lambda$

$$\delta_\lambda S_{\text{eff}}[A] = \int_{y>0} \frac{d\lambda \wedge dA}{4\cancel{2}\pi} \overset{\text{IBP}}{=} -\int_{y=0} \lambda \frac{\epsilon_{ab}F^{ab}}{4\cancel{2}\pi}$$

$$0 = \frac{\delta S}{\delta \bar\psi}\gamma^0 = \left(i\gamma^a\partial_a + i\gamma^y\partial_y - g\phi(y)\right)\psi$$

$\underline{a=0,1}$ along wall.

ansatz: $\psi = \eta_\pm(x^a)\, e^{\alpha(y)}$

$$\gamma^{01}\eta_\pm = \pm\eta_\pm \qquad \gamma^{01} = \gamma^0\gamma^1$$

chirality along wall.

$$\rightarrow \boxed{\underbrace{i\gamma^a D_a \eta_+ = 0 \qquad \gamma^{01}\eta_+ = \eta_+}_{\text{is exp'lly localized}}} \quad \underline{\text{chiral}}$$

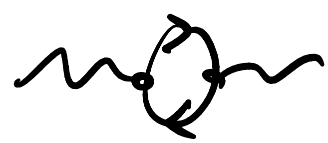
charged $\underline{\text{chiral}}$ fermion on the wall!

$U(1)$ of A acts the $D=1+1$ chiral symmetry!

$$\delta_\lambda S_{2d}[A] = \int \lambda\, \partial_\mu \hat{j}^\mu$$

$$= \int \lambda\, \frac{\epsilon_{ab} F^{ab}}{4\pi}.$$

2d axial anomaly from just a right-handed fermion.



$$\delta_\lambda S_{full} = \delta_\lambda S_{bulk} + \delta_\lambda S_{2d} = 0.$$

Anomaly Inflow in codim 2.

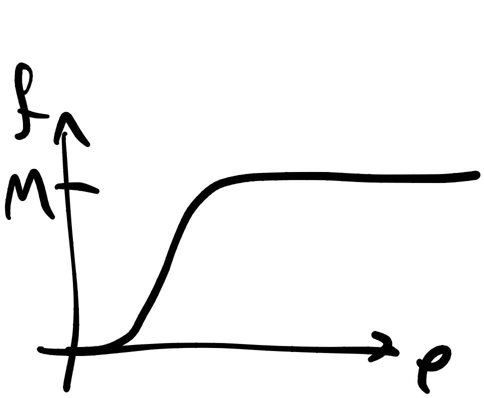
D=3+1 axion EM.

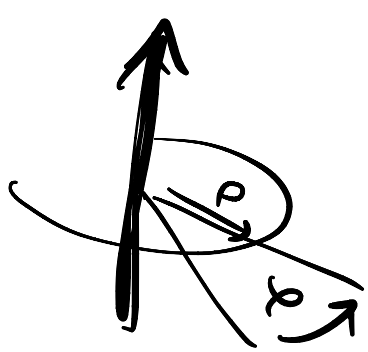
$\pi_1(V = S^1) = \mathbb{Z}$.

$\Rightarrow$ codim 2 defects. vortex string of Φ

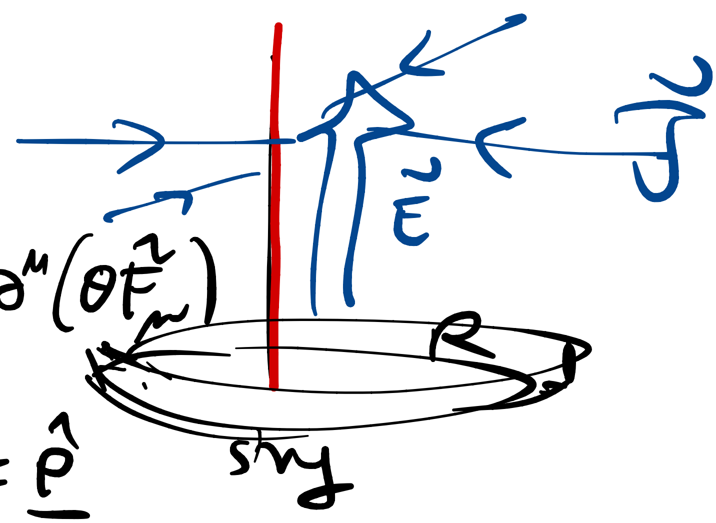
$$\Phi = f(\rho)e^{i\varphi}$$

ie: $\theta = \varphi$.





apply $\vec{E} = E\hat{z}$



$$\partial^\mu F_{\mu\nu} = J_\nu = \frac{e\theta}{\cancel{4\pi}}\partial^\mu(\theta \tilde{F}_{\mu\nu})$$

$$\Longrightarrow \vec{J} = -\frac{e}{2\pi^2} E \frac{\hat{\rho}}{\rho}$$

$\vec{\nabla}\cdot\vec{J} = 0$ for $\rho > 0$.

But $$\int_R \vec{\nabla}\cdot\vec{J} = \oint_{\partial R} d\vec{n}\cdot\vec{J}$$

$$= \int_0^{2\pi} d\varphi\, \rho J_\rho = -\frac{eE}{\pi}$$

$$\Longrightarrow \frac{dQ}{dt} = -\int_R \vec{\nabla}\cdot\vec{J} = \frac{eE}{\pi}.$$

i.e. $\partial^\mu J_\mu = \delta^2(x_\perp)\frac{1}{2\pi}\epsilon^{ab}F_{ab}$ $\boxed{?!}$

to be continued!