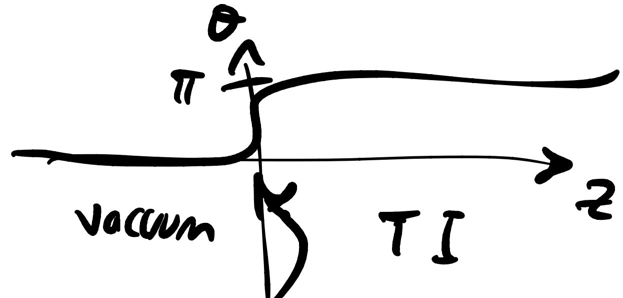
Recap: $D = 3+1$ $\quad \underline{\underline{G = U(1) \rtimes \mathbb{Z}_2^T}}$

$= \mathcal{T}_{i \to -i}$

$$\longrightarrow S_{\text{eff}}[A] \;\underline{\underline{=}}\; \dots + \theta \underbrace{\frac{e^2}{8\pi^2}\int_X F\wedge F}_{\in \mathbb{Z} \text{ (if } X \text{ is spin)}} \qquad F = dA$$

$$\mathbb{Z}_2^T : \begin{cases} \underline{\theta \to -\theta.} \\ F\wedge F \propto E\cdot B \to -F\wedge F. \end{cases} \qquad \Rightarrow \underline{\underline{Z_X(\theta + 2\pi) = Z_X(\theta)}}$$

$\Rightarrow \theta = 0, \pi$ are $\mathcal{T}$-inv't.



a gapped $U(1)$-symmetric interface between $\theta = 0, \pi$

has $\sigma^{xy} = \frac{e^2}{h}\left(\frac{1}{2} + n\right)$

can add IQHE.

"<u>magneto-electric response</u>": $\quad \frac{dP}{dB} = \frac{dM}{dE} = \theta/4\pi^2.$

$\vec{P} \equiv$ density of el. dipoles

$D = 1+1 \rightsquigarrow G = U(1) \times \mathbb{Z}_2$
parable

$\uparrow$ $\partial_\mu \to -\partial_\mu$.

$$\underbrace{S_\theta[A] = \int_{X_2} \frac{\theta}{2\pi} F}$$ $F = dA$

$\int F \in 2\pi\mathbb{Z} \to Z(\theta + 2\pi) = Z(\theta)$

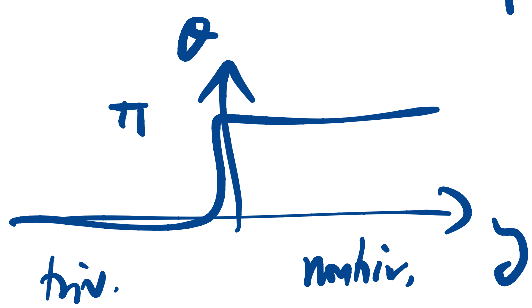
$\theta = 0, \pi$ are $\mathbb{Z}_2$ -inv.

$\left(\int F = \int F_{01} = \int E. \right)$

Coupling of an electric dipole is $\vec{d} \cdot \vec{E}$

$\Rightarrow P = \frac{\theta}{2\pi}$ is the dipole density
$\equiv$ polarization.

$$\boxed{d\theta = \pi\, \delta(y)}$$



$$j^\mu(x) = \frac{\delta S_{eff}}{\delta A_\mu(x)} = \frac{\epsilon^{\mu\nu}}{2\pi} \partial_\nu \theta + \ldots$$

$$\Rightarrow \rho = \delta(y) \times \frac{\Delta\theta}{2\pi} = \delta(y) \cdot \frac{1}{2}.$$

microscopically what is θ?

assume: – transl. sym
– free fermions

claim: $\theta = \oint dk \sum_{n,\text{occ.}} A_{nn}(k)$

Berry connection.

$$A_{nn}(k) \equiv \langle \psi_n^{(k)} | i \nabla_k | \psi_n^{(k)} \rangle$$

pf: $\frac{d\vec{P}}{dt} = \vec{J}$. vary some param λ in H to produce P from $P=0$.

$$|\delta \psi_{nk}\rangle \overset{\text{pert th}}{=} -i\hbar \dot{\lambda} \sum_{m \neq n} \frac{\langle \psi_{mk} | \partial_\lambda \psi_{nk} \rangle}{E_{nk} - E_{mk}} |\psi_{mk}\rangle$$

$$\vec{J}_n = \frac{d\vec{P}_n}{dt} = \left\langle \frac{e\vec{p}}{m} \right\rangle_n = \frac{i\hbar e}{m} \dot{\lambda} \sum_{m \neq n} \oint_{BZ} d^d k \; \frac{\langle \psi_{nk} | \vec{p} | \psi_{mk} \rangle \langle \psi_{mk} | \partial_\lambda \psi_{nk} \rangle}{E_{nk} - E_{mk}}$$

$+ h.c.$

Bloch: $\psi_{nk}(r) \equiv \langle r | \psi_{nk} \rangle$

$$= e^{ik\cdot r} \underbrace{u_{nk}(r)}_{u_{nk}(r+a) = u_{nk}(r).}$$

$$\underline{H_k |u_{nk}\rangle = E_{nk} |u_{nk}\rangle}$$

expand to 1st order in δk.

$$H_k = \frac{(p+\hbar k)^2}{2m} + V$$

(Feynman–Hellman)

$$\Longrightarrow \vec{p} = \frac{m}{\hbar} \vec{\nabla}_k H_k - \vec{k}$$

$$\Rightarrow \frac{d\vec{p}}{d\lambda} = ie \sum_{n,occ} \oint_{BZ} d^dk \left\langle \vec{\nabla}_k u_{nk} \middle| \partial_\lambda u_{nk} \right\rangle$$

$$+ h.c.$$

$$\Rightarrow \vec{P} = \int_0^1 d\lambda \frac{d\vec{p}}{d\lambda} = e \,\mathrm{Im}\left(-i \sum_{n,occ} \oint_{BZ} d^dk\, \vec{A}_{nn}(k) \right)$$

$$u_{nk}(r) \to g(k) u_{nk}(r) \Longrightarrow A^{(1)} \to A^{(1)} + g^{-1} \partial_k g.$$

$$g(k) = e^{imak}, m \in \mathbb{Z} \Rightarrow \oint_{BZ} A \to \oint_{BZ} A + 2\pi m.$$

$\longrightarrow \quad \theta \to \theta + 2\pi m,$

$\rho \to \rho + 1$

goldstones for translation

Compare: $S_\nu[\theta, A] \overset{1+1d}{=} \int \frac{\nu}{2\pi} A \wedge d\theta.$

gauge inv $\Rightarrow \nu \in \mathbb{Z}.$

$\overset{IBP}{=} -\frac{\nu}{2\pi} \int \theta F$ same!

$\frac{\theta}{2\pi}$ = polarization.

$$S_\nu[\theta^I, A] = \frac{\nu}{(2\pi)^d} \int A \wedge d\theta^1 \wedge \dots \wedge d\theta^d$$

$\theta^I \to \theta^I + 2\pi p^I$

$$\xrightarrow{IBP} \sum_{i=1}^{d} (-1)^{d+1} \frac{\nu}{(2\pi)^{d-1}} \int p^i \, d\theta^1 \wedge \dots dA \wedge d\theta^{i+1} \wedge \dots d\theta^d$$

BG $\mathbb{Z}^d$ gauge fields $\quad z^i$ 1-form

$$\begin{cases} \oint_{C_i} z^i = L_i \\ dz^i = \text{density of dislocation w/ Burgers' vector } i. \end{cases}$$

$$z^i = \frac{d\theta^i}{2\pi} \quad !$$

$$\int_i d\theta^i = L_i$$

$d^2\theta^i$ = density of dislocation.

$$\int_{\substack{\text{Disc} \\ D}} d^2\theta^i = \int_{\partial D} d\theta^i = \text{winding } \# \text{ of } \theta^i.$$

quantized charge.

Thouless pump: $D = 1+1$. w/ $G = U(1)$ (gap)

suppose we can vary θ adiabatically..

$$\theta \to \theta + 2\pi n.$$

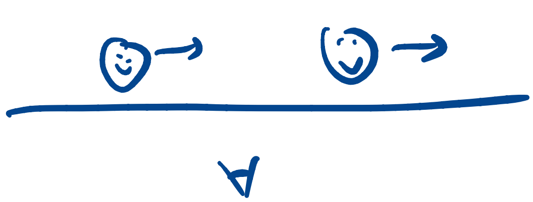
during time interval T.

$$j^\mu = \frac{\nu}{2\pi} \epsilon^{\mu\nu} \partial_\nu \theta$$

charge gets pumped past some location

$$\Delta Q = \int_0^T dt\, j^x(t) = \int_0^T dt \frac{\partial \theta}{\partial t} \frac{\nu}{2\pi}$$

$$= \frac{\nu}{2\pi} \oint d\theta = \nu n \in \mathbb{Z}.$$



eg:

thread flux.

pumped charge σ^{xy}.

$$\frac{\theta}{4\pi^2} = \frac{\partial P}{\partial B} = \frac{\partial M}{\partial E}$$

$M =$ density of

Polarization = translation quantum #s of magn. flux

$$1+1\,d: \quad S_\nu = \nu \int p F = \nu p$$

insert 2π flux

$F = 2\pi\, \delta^2(x)$

(event)

always codim 2

$$\oint_{\substack{2 \text{ transv}\\ \text{dirs}}} F \overset{!}{=} 2\pi$$

$2+1\,d$: flux is a particle.

$$S_\nu = 2\pi \int_{\text{worldline}} dt \left(p^x \partial_t \theta^y - p^y \partial_t \theta^x\right)$$

$$\Pi_y(x) \equiv \frac{\delta S}{\delta \partial_t \theta^y(x)} = \ldots + 2\pi p^x$$

$$T_y \equiv e^{i a \Pi_y} = T_y^{(p=0)} e^{2\pi i p^x}.$$

$(a=1)$

$\Rightarrow$ 2π flux particle carries lattice momentum

$$\vec{K} = 2\pi(-p^y, p^x) .$$

3+1 d : $S_v = 2\pi \int_{\substack{1+1 \\ \text{worldsheet}}} \epsilon^{ijk} p^i \underline{\partial_t \theta^j} \partial_x \theta^k$

flux is a string.

$$\pi_{\theta^j} = \ldots + 2\pi \epsilon^{ijk} p^i \partial_x \theta^k$$

$$\Rightarrow T_x T_y = T_y T_x e^{2\pi i p^z} .$$

claim : $\Rightarrow \Delta \vec{p} = \frac{\theta}{4\pi^2} \vec{B} .$

if $\theta \neq 0$: monopole carries electric charge

$$q = \theta/2\pi \qquad \text{[Witten effect]}$$

$$\Rightarrow T_x T_y = T_y T_x e^{2\pi i (p + \underset{\substack{\uparrow \\ \text{from } B}}{\Delta p})^z}$$

$$\Delta p^z = \frac{q B^z}{2\pi} = \frac{\theta}{4\pi^2} B^z$$

$$\Rightarrow \quad \theta = \frac{\partial P}{\partial B},$$

In 2+1d w/o GSD $\frac{\sigma^{xy}}{e^2/h} \in \mathbb{Z}$.

claim: $\theta/\pi \in \mathbb{Z}$ w/o fractionalization.

pf: on T^3. apply tiny $\vec{B} = B\hat{z}$

$$\int dx\,dy\; e B_z = 2\pi.$$

(minimal flux)

thread 2π flux

$$\oint_{C_z} A = \frac{2\pi t}{e} \qquad t \in [0,1]$$

$$\vec{E} = \hat{z}\,\frac{2\pi}{e L_z}$$

$$S_{eff}[A] = \begin{array}{c}\text{manifestly} \\ T \text{ symm-terms}\end{array} + S_\theta \qquad \boxed{\exists! \text{ g.s.!}}$$

groundstate to groundstate amplitude

$$Z = C e^{i S_\theta[E,B)} = C e^{i\theta} .$$

C: real constant

θ: calculation

T-symmetry $\Rightarrow$ T-reversed process has the same amplitude!

$$\oint A = \frac{2\pi}{e} \qquad \int B = -\frac{2\pi}{e}$$

$$\Rightarrow Z^T = Ce^{-i\theta} \overset{!}{=} Z = Ce^{i\theta}$$

$$\Rightarrow \theta = 0, \pi. \quad \text{mod } 2\pi.$$

Bosons: IQHE for bosons had $\sigma^{xy} = \underline{2} \frac{e^2}{h}$

$\theta \to \theta + 2\pi$ adds $\sigma^{xy} = \frac{e^2}{h}$ IQHE

on surface is not ok for bosons.

$$\underbrace{Z_{\text{bosons}}(\theta + 4\pi)} = \underbrace{Z_{\text{boson}}(\theta)}.$$

$\theta = 2\pi$ is nontrivial

(causes $\sigma^{xy} = \frac{2e^2}{2h}$ for bosons on surface.)

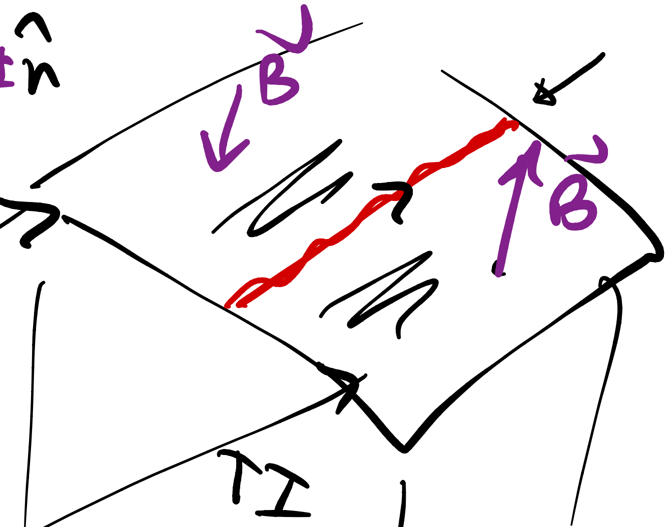
on a non-spin mfld X $\int_X \frac{F \wedge F}{8\pi^2} \in \frac{\mathbb{Z}}{2}$.

(for bosons
$\theta = \pi$ requires fractionalization!)

Using $S_{eff}[A]$ at surface requires a gapped U(1)-symmetric bdy.

eg: turn on m, $\vec{B}$ on bdy. (breaks T)

big $\vec{B} \propto \pm \hat{n}$

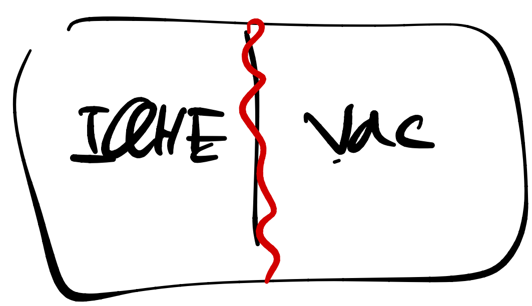


$$\sigma^{xy} = \frac{e^2}{h}\cdot\frac{1}{2}$$

$$\sigma^{xy} = -\frac{e^2}{h}\frac{1}{2}$$

wall carries a chiral edge mode

$$\Delta\sigma^{xy} = \frac{e^2}{h}\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) = \frac{e^2}{h}.$$

just like 

$\exists$ gapped & <u>symmetric</u> surface states.

w/ T.O. $\not\!\!\!\!-$ <u>anomalous</u>.

(T-symmetric Moore-Read state!)

<u>options for surface</u>: – gapless

– symmetry-breaking

– T.O.

(same as LSMOH alternative)

Systems w/ an LSMOH thm <u>are</u> anomalous but $\exists$ as lattice models w/o Bulk.

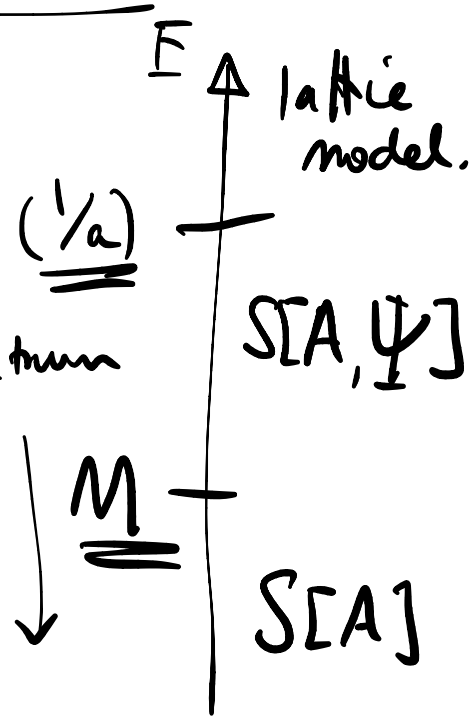
<u>How?</u>

<u>A</u>: symmetries are NOT on-site!
eg translation:

Free fermion TIs in D=3+1

$$S[A,\Psi] = \int d^3x\,dt \Big[\bar\Psi \, i\gamma^\mu D_\mu \Psi + \bar\Psi (m + i\tilde{m}\gamma^5)\Psi \Big]$$ continuum

$\partial_\mu - iA_\mu$



γ^μ 4×4. Ψ Dirac ferm.

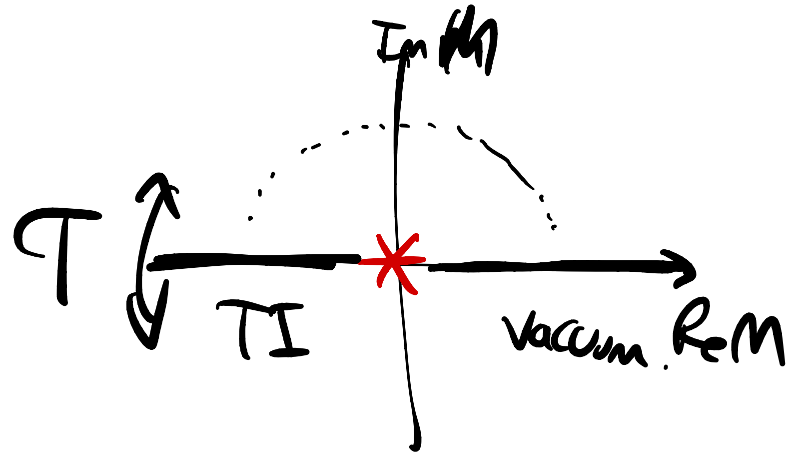
$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

U(1): $\Psi \to e^{i\alpha}\Psi$.

$$M = m + i\tilde{m}.$$

$$T: \quad i \to -i, \quad \psi(t,x) \mapsto \gamma^1\gamma^3\psi(-t,x)$$

$$M \to M^*.$$



$$\gamma^\mu \partial_\mu = \underline{\underline{\gamma^0 \partial_0}} + \gamma^i \partial_i$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \qquad \sigma^\mu = (1, \sigma^i)^\mu \qquad \bar\sigma^\mu = (1, -\sigma^i)^\mu$$

$$\begin{cases} (\gamma^2)^* \neq \gamma^2 . \\ (\gamma^{1,3,0})^* = \gamma^{1,3,0} . \end{cases}$$

$$\Psi = \begin{pmatrix} \text{particle} \uparrow \\ \text{particle} \downarrow \\ \text{antiparticle} \uparrow \\ \text{antiparticle} \downarrow \end{pmatrix} \qquad \gamma^1\gamma^3 = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$$

$$(\gamma^\mu)^* = \gamma^1\gamma^3 \; \gamma^\mu \; (\gamma^1\gamma^3)^{-1}$$

$$(-1)^\mu = \begin{cases} +1 & \mu = 0 \\ -1 & \mu = i \end{cases}$$