$\pi_k(G/H)$ continued:

$$\cdots \to \pi_k(H) \xrightarrow{i_*} \pi_k(G) \xrightarrow{j_*} \pi_k(G/H) \xrightarrow{\partial_*} \underline{\pi_{k-1}(H)} \to \cdots$$

$$\pi_k(G/H) \cong \pi_k(G,H)$$

α : e

$\underline{\partial_* : \alpha \to \alpha|_{\text{bottom face}}}$:

$(I^{k-1}, \partial I^{k-1}) \to (H, e)$.

- $\underline{\pi_2(G) = 0.}$

$\Rightarrow$

$$\pi_2(G) \to 0 \xrightarrow{j_*} \pi_2(G/H) \xrightarrow{\partial_*} \pi_1(H) \to \cancel{\pi_1(G)}^{\,0} \to \cdots$$

$\underline{\underline{\operatorname{Im} j_* = \ker \partial_* = 0.}}$ 1-1

$\underline{\underline{\text{IF}}}$ $\underline{\pi_1(G) = 0}$:

$$0 \to \pi_2(G/H) \xrightarrow{\partial_*} \pi_1(H) \to 0$$

1-1 onto

∂_* is an isomorphism

$$\boxed{\pi_2(G/H) \overset{\partial_*}{\cong} \pi_1(H).}$$

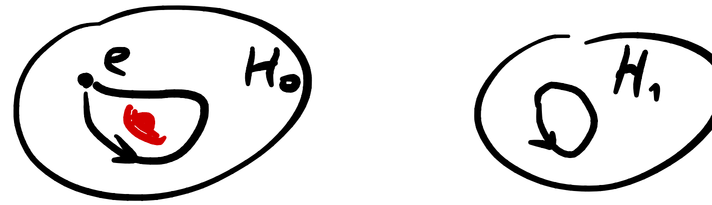
$V = G/H = \hat{G}/\hat{H}$

$\hat{G}$ = univ. cover

$\pi_1(\hat{G}) = 0.$

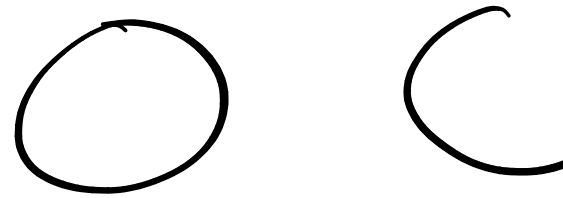
$$\pi_2(G/H) = \pi_1(H,e) = \pi_1(H_0, e).$$

H:



assume:

- $\pi_1(G) = 0$.



$$0 \to \pi_1(G/H) \xrightarrow{\partial_*} \pi_0(H) \xrightarrow{i_*} \pi_0(G) \to \cdots$$

$\uparrow$

$\pi_1(G)$

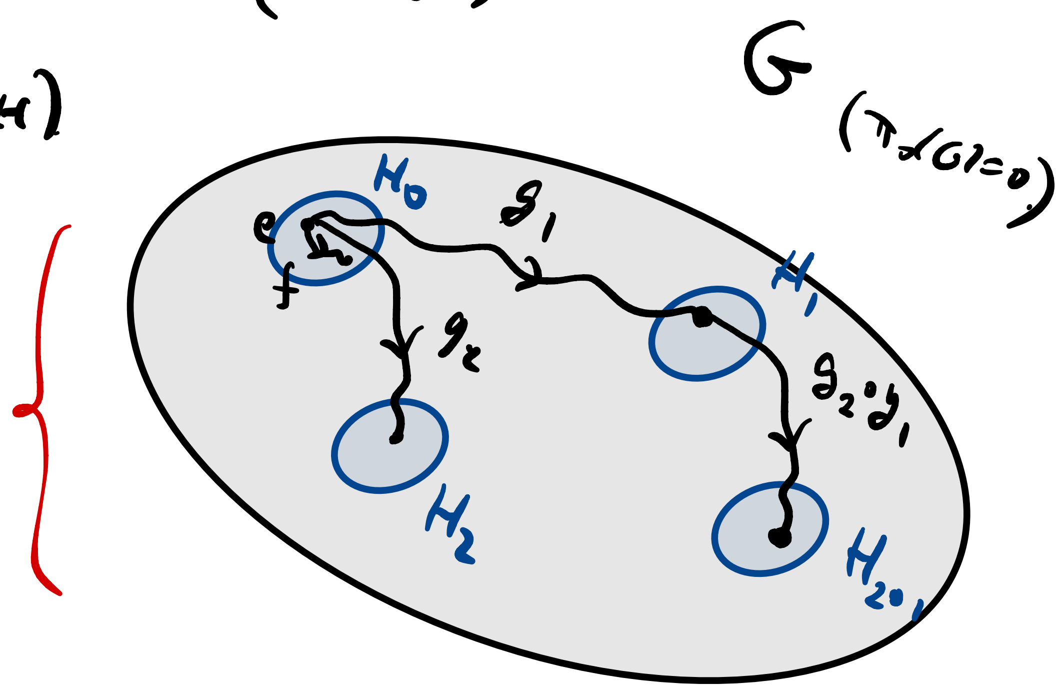
If: $\pi_0(G) = 0$ ie G is Connected

$$\pi_1(G/H) \cong \pi_0(H) \cong H/H_0.$$

$\uparrow$

(item 0.)

$[f] = 0 \in \pi_1(G/H)$



more generally: $\pi_1(G/H) \simeq \pi_0(H) / \pi_0(G)$.]

Loop automorphism of $\pi_1(G/H)$ on $\pi_{q-1}(G/H)$:

$\simeq \pi_0(H)$

$\simeq$ inner automorphism of H by conjugation

$h \longrightarrow h_i h h_i^{-1}$

$h_i H_0 \in H/H_0$ representative.

$h_i \rightsquigarrow g_i(t)$

$\begin{cases} g_i(0) = e \\ g_i(1) = h_i \end{cases}$

eg: $q = 3$. $\pi_2(G/H) \simeq \pi_1(H_0)$

$\underbrace{\text{a loop in } H_0.}_{\gamma(t)}$

$\gamma(t) \longrightarrow h_i \gamma(t) h_i^{-1}$. is the action of π_1 on π_2.

eg: Nemahic in 3d. $G = SU(2)$

take $\phi_0 = \hat{z}$

$$H = U(1) \ltimes \mathbb{Z}_2$$

rots about $\hat{z}$: $e^{i\frac{\theta}{2}Z}$

π rotation about $\hat{y}$. eg $i\hat{Y}$.

point defect (w/ basept):

$\pi_2(G/H) = \pi_1(H_0) \ni [u]$ is a loop in H_0.

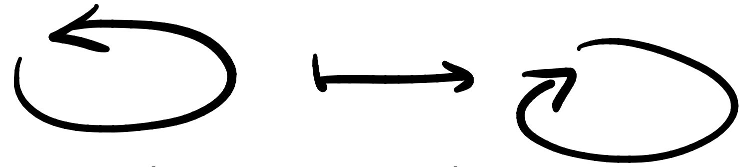
$$u(t) = e^{i\frac{\theta(t)}{2}Z}$$

winding # $n \in \mathbb{Z}$ $\quad \theta(t): S^1 \to S^1$

loop automorphism: $u(t) \longmapsto iY\, u(t)\, (iY)^{-1}$

$$= u(t)^{-1}$$
$$= u(-t).$$

$$\left[\begin{aligned} &YZY = -Z \\ \Rightarrow\ &Y e^{i\alpha Z} Y \\ &= e^{-i\alpha Z} \end{aligned}\right]$$



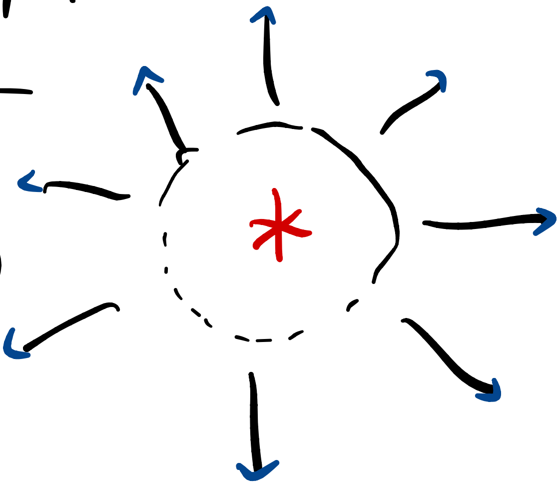
$n \longmapsto -n$.

what is the codim 3 defect :

$\hat{n} \propto \hat{r}$

"hedgehog"

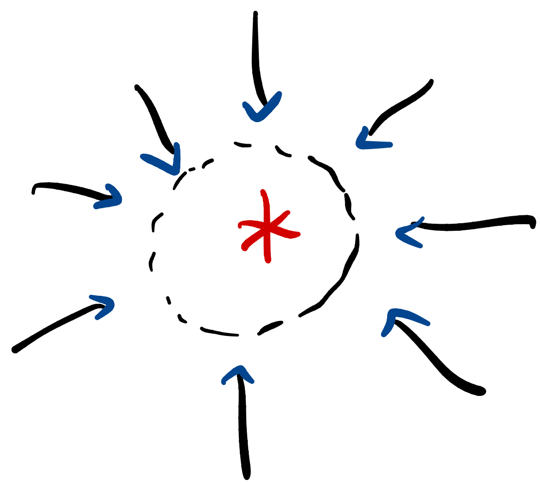
$n = 1$



$\hat{n}\big|_{S^2} : S^2 \to S^2$

$\simeq$

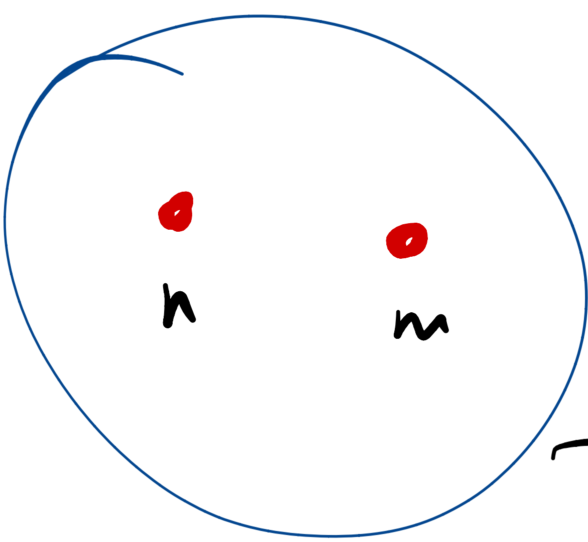
$n = -1$



$\Rightarrow$: nontrivial pt defects are labelled by $n > 0$

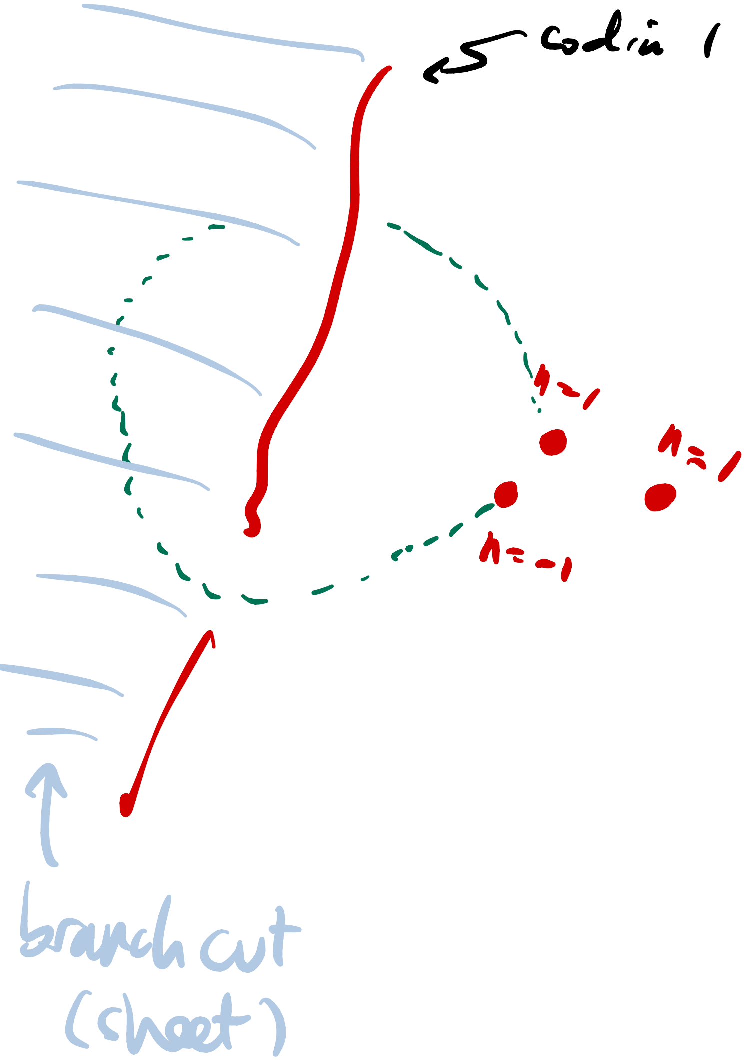
$n \in \mathbb{Z}_{>0}$.

not a group.

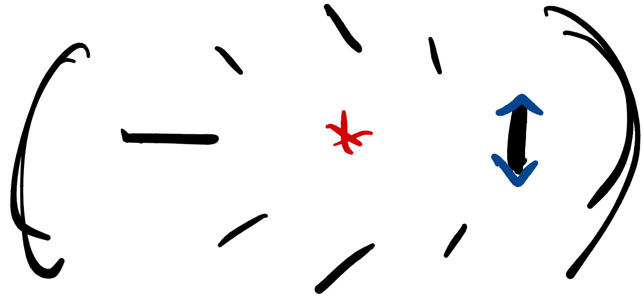


S^2

$\rightarrow$ either $n+m$ or $|n-m|$.



codim 1 defect : π disclination



Gauge Theory description of the nematic :

$$M_{ij} = n_i n_j - \dots$$

actual d.o.f.

"parton"

* $n(x) \to (-1)^{s_x} n(x)$ is a redundancy of description. "gauge invariance".

$F_{LG}(M)$ predicts 1st order trans.

$F_{LG}[\vec{n}, \sigma_{xy}]$ looks like ferromagnet.

* $n_x \to s_x n_x$, $\sigma_{xy} \to s_x \sigma_{xy} s_y$.
$(=\pm 1)$

$$H = -t \sum_{\langle xy \rangle} n_x^i \sigma_{xy} n_y^i - K \sum_{p} \prod_{\ell \in \partial p} \sigma_\ell + \ldots$$

plaquettes

[core energy of π disclination]

$t \gg K$: nematic phase :

$\langle n \rangle \neq 0$

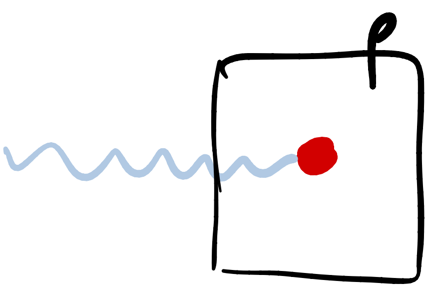
Higgs phase.

$K \gg t$: disordered, but σ deconfined.

(topological order)

Codim 2 defect : vortex line w/ holonomy (-1). $\hat{n} \to -\hat{n}$.

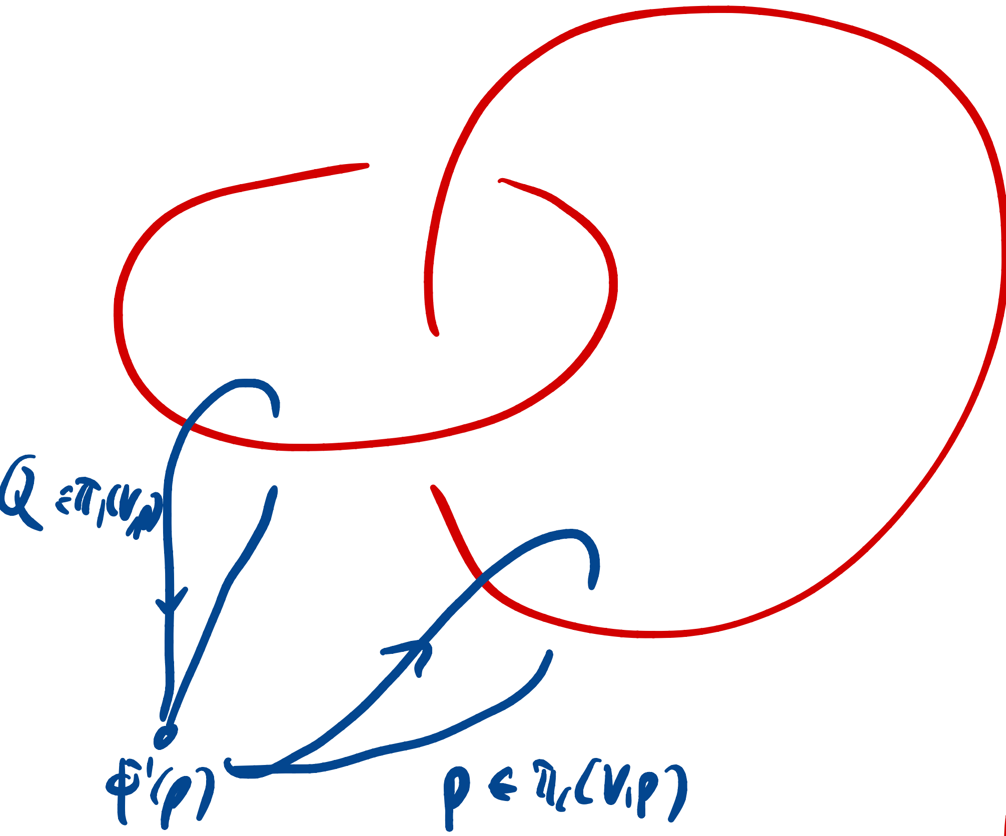
$\equiv \pi$ disclination.



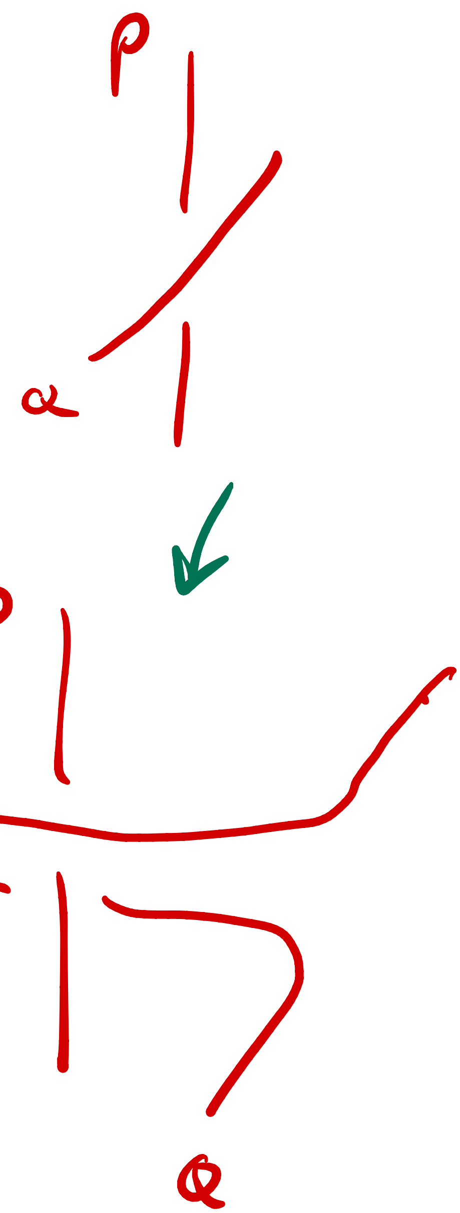
$\prod_{\ell \in \partial p} \sigma = -1$ costs energy $2K$.

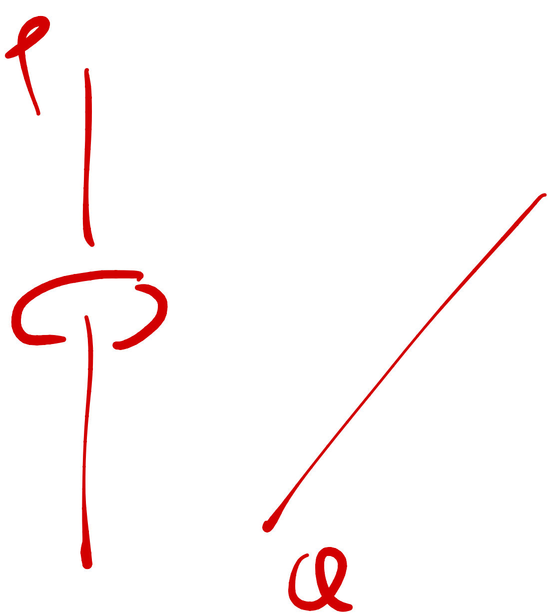
1.6 when $\pi_1(V)$ is nonabelian.

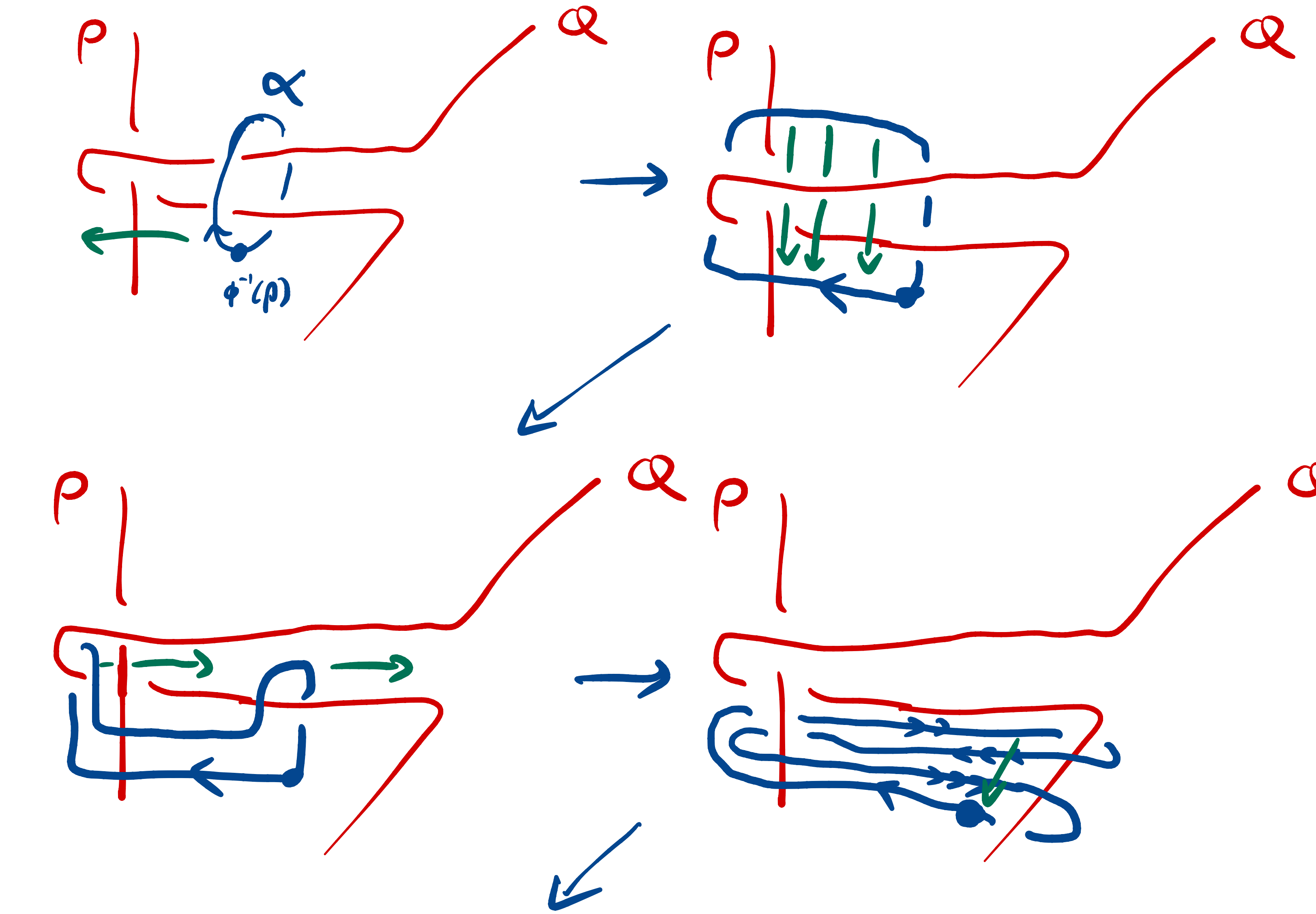
- mobility of line defects in $d=3$ is inhibited.

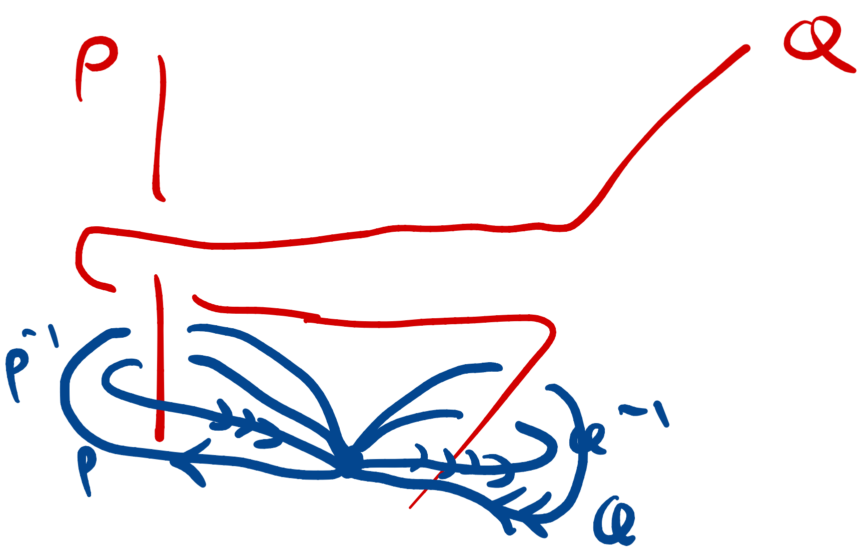


Q:

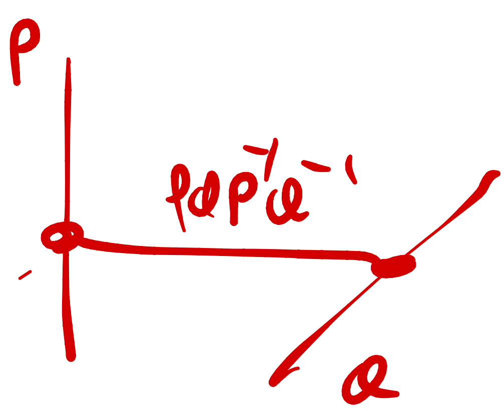






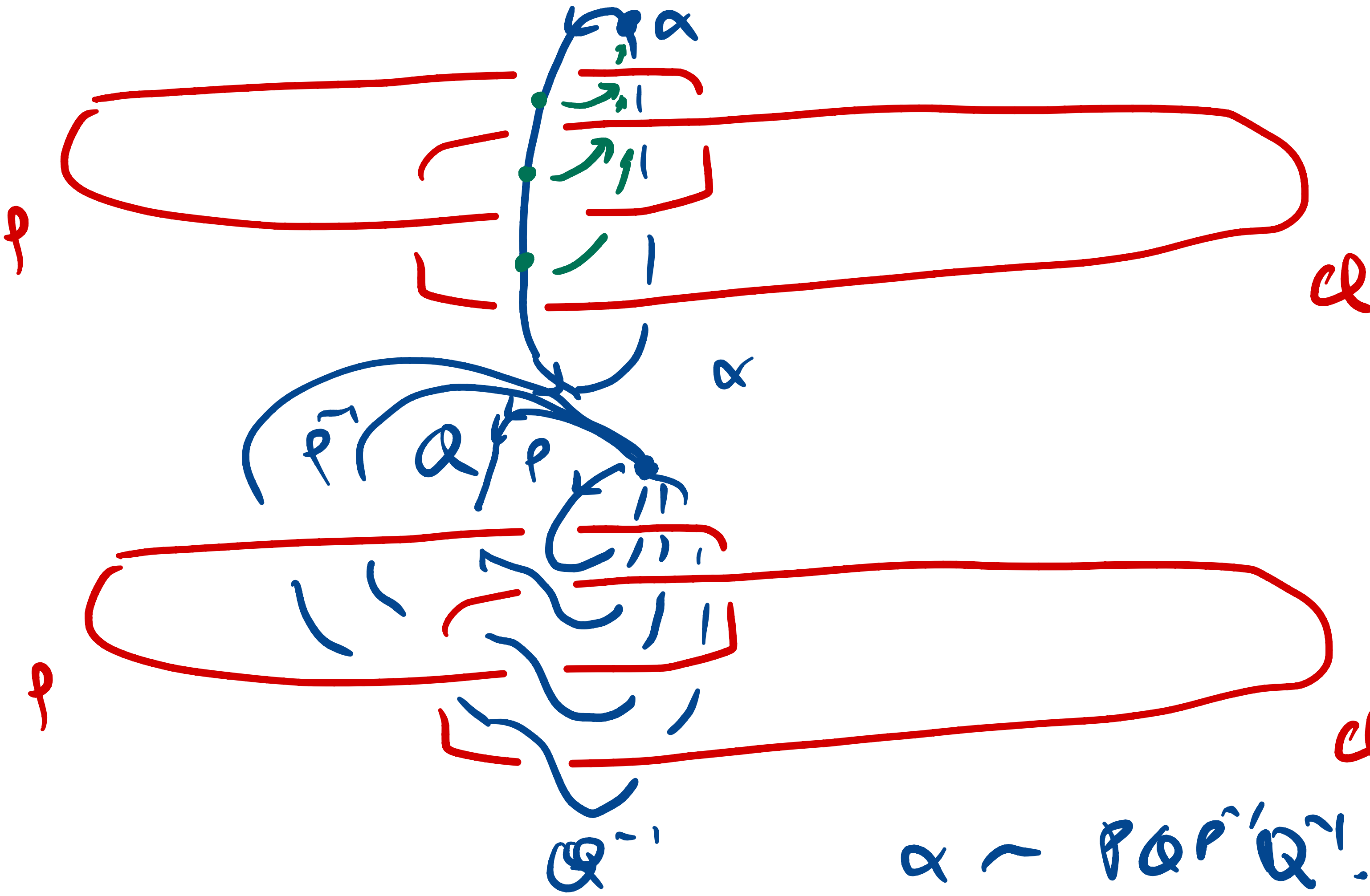


$$\alpha \simeq PQP^{-1}Q^{-1}$$



if $PQP^{-1}Q^{-1} \neq 1 \in \pi_1(V)$ then

(P, Q are confined.)



$\alpha \sim P Q P^{-1} Q^{-1}$.

Network of linked (and/or knotted) defects & smooth on K = space $\setminus$ defects

= "link/knot complement".

$\Leftrightarrow$ homomorphism from

$$\pi_1(K) \longrightarrow \pi_1(V)$$

(a representation of $\pi_1(K)$ in $\pi_1(V)$.)

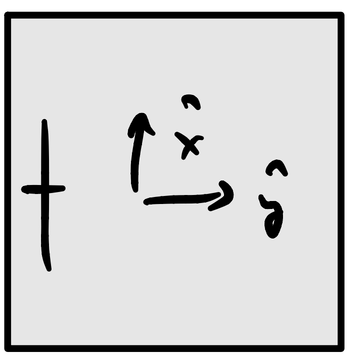
$\longleftrightarrow$ flat vector bundle with structure group $\underline{\pi_1(V)}$.

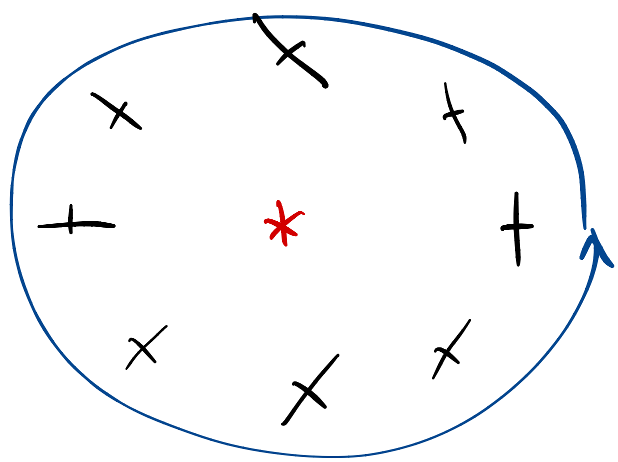
Codim 2 defects of biaxial nematic:

$$V = SU(2)/Q_8 \Longrightarrow \underline{\pi_1(V) = Q_8}.$$

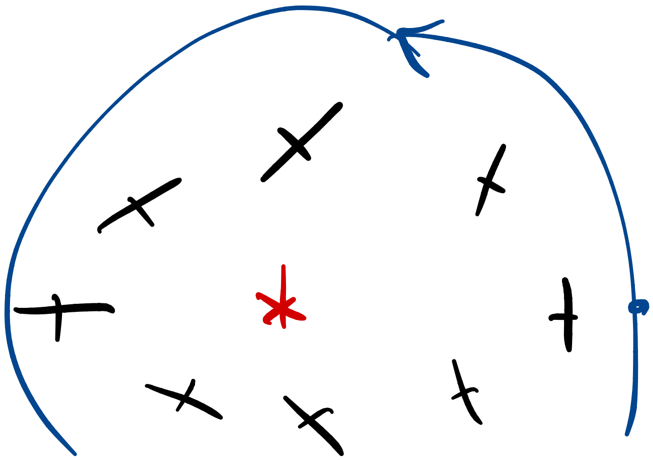
$$Q_8 = \{\{1\}, \{-1\}, \{\pm iX\}, \{\pm iY\}, \underline{\{\pm iZ\}}\}$$

$$= \{C_1, C_{-1}, C_x, C_y, C_z\}.$$



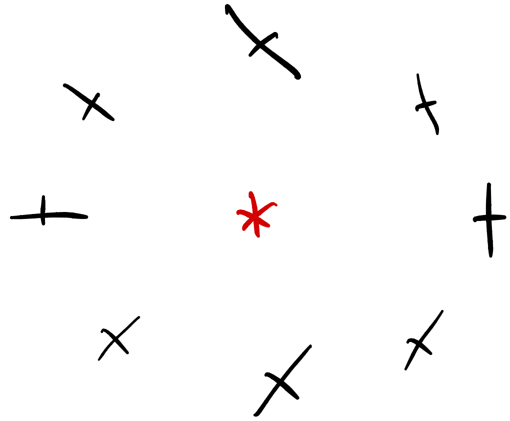


Rot by π about $\hat{z}$

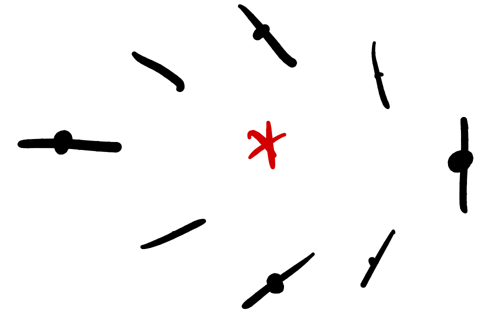


Rot by $-\pi$ about $\hat{z}$.

C_z :

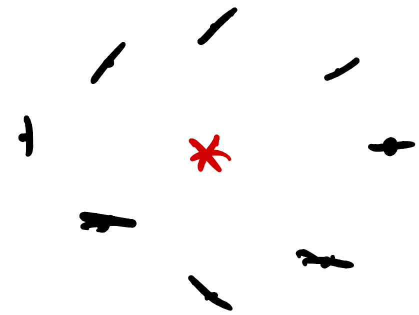


C_y :



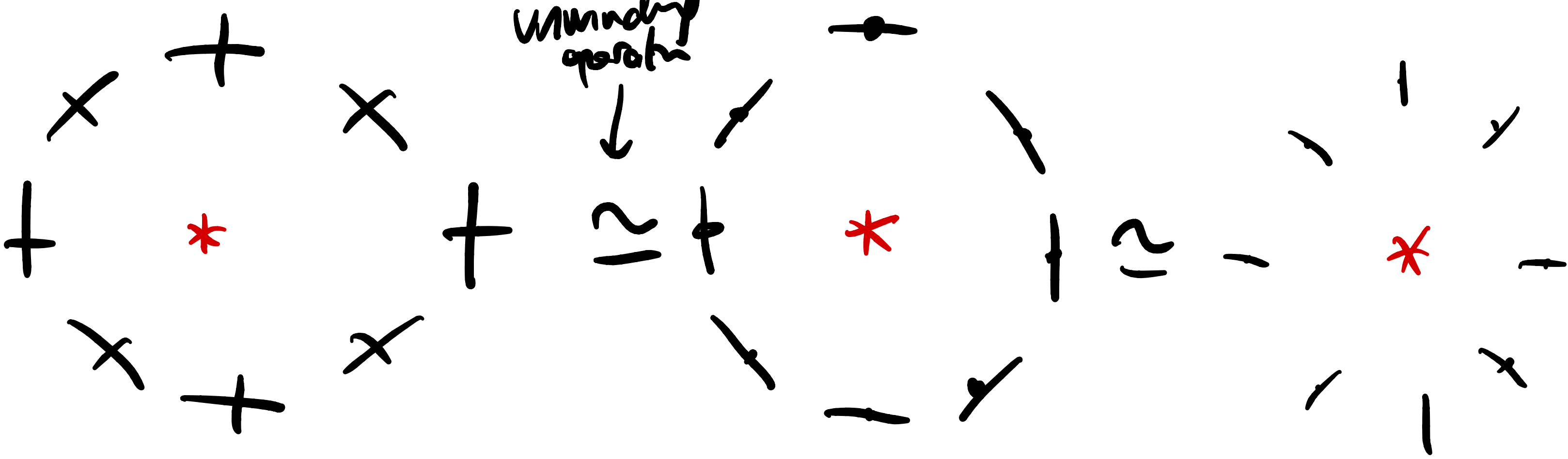
$\underline{iX \cdot iY = -iZ}$,

C_x :



C_{-1} : 2π disclination of either or both axes :

winding operator

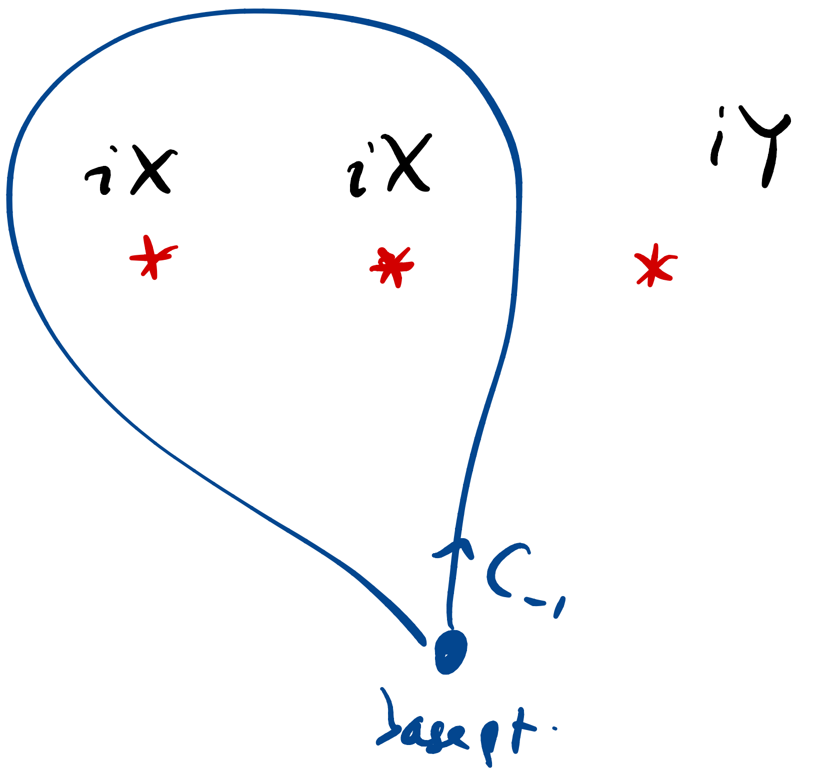


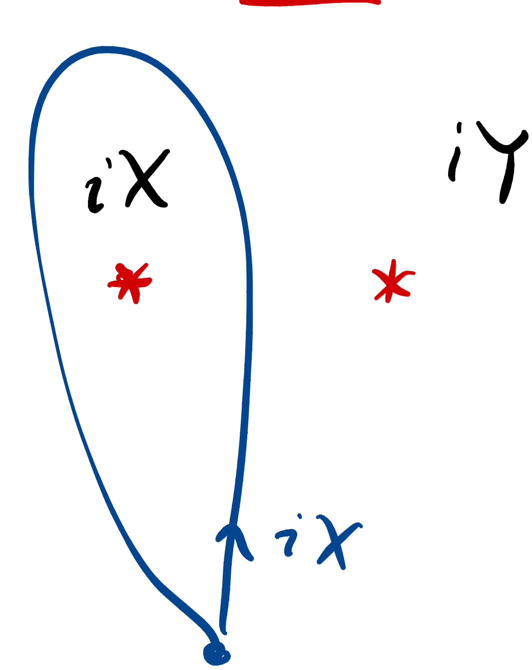
$\underline{\text{group algebra}}$: $\begin{matrix} G \\ g \end{matrix} \longmapsto \begin{matrix} \underline{\underline{\text{group algebra}}} \\ e_g \end{matrix}$ $\quad [e_{g_1} + e_{g_2} \in \text{grp alg.}]$

$e_g e_h = e_{gh}$

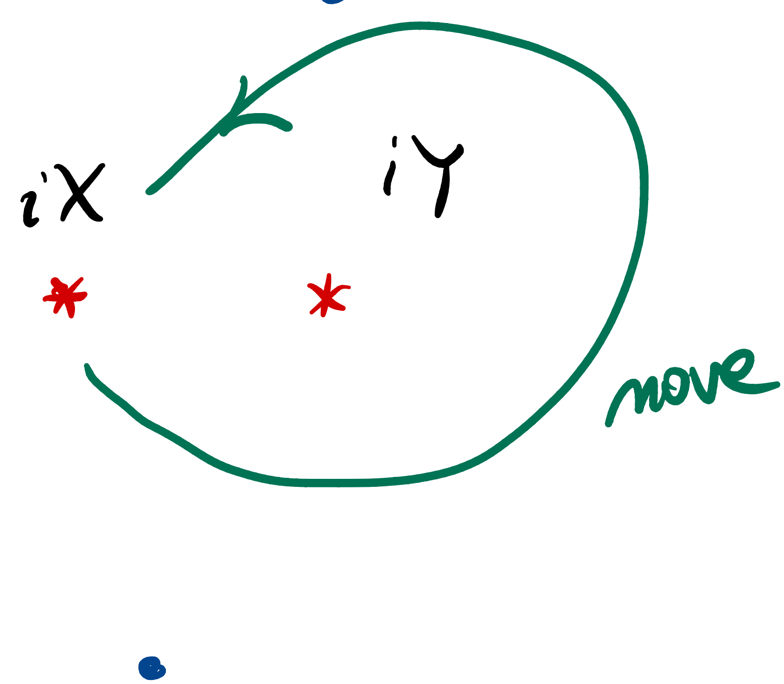
$C = \sum_{g \in C} e_g$.

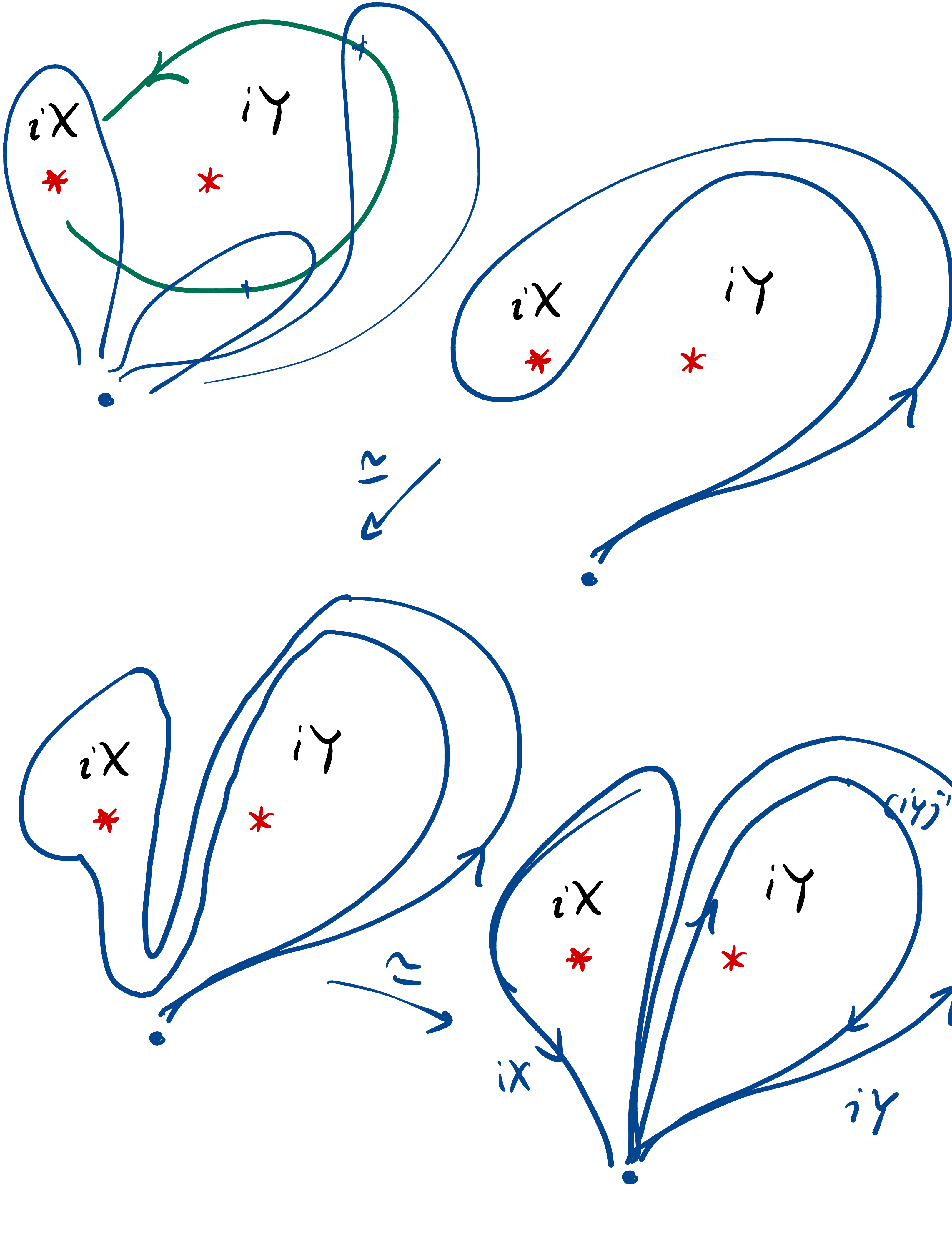
	C_1	C_{-1}	C_x	C_y	C_z
C_1	C_1	C_{-1}	C_x	C_y	C_z
C_{-1}		C_1	$2C_x$	$2C_y$	$2C_z$
C_x			$2C_1+2C_{-1}$	$2C_z$	$2C_y$
C_y				$2C_1+2C_{-1}$	$2C_x$
C_z					$2C_1+2C_{-1}$





$(iX \cdot iX = -1)$





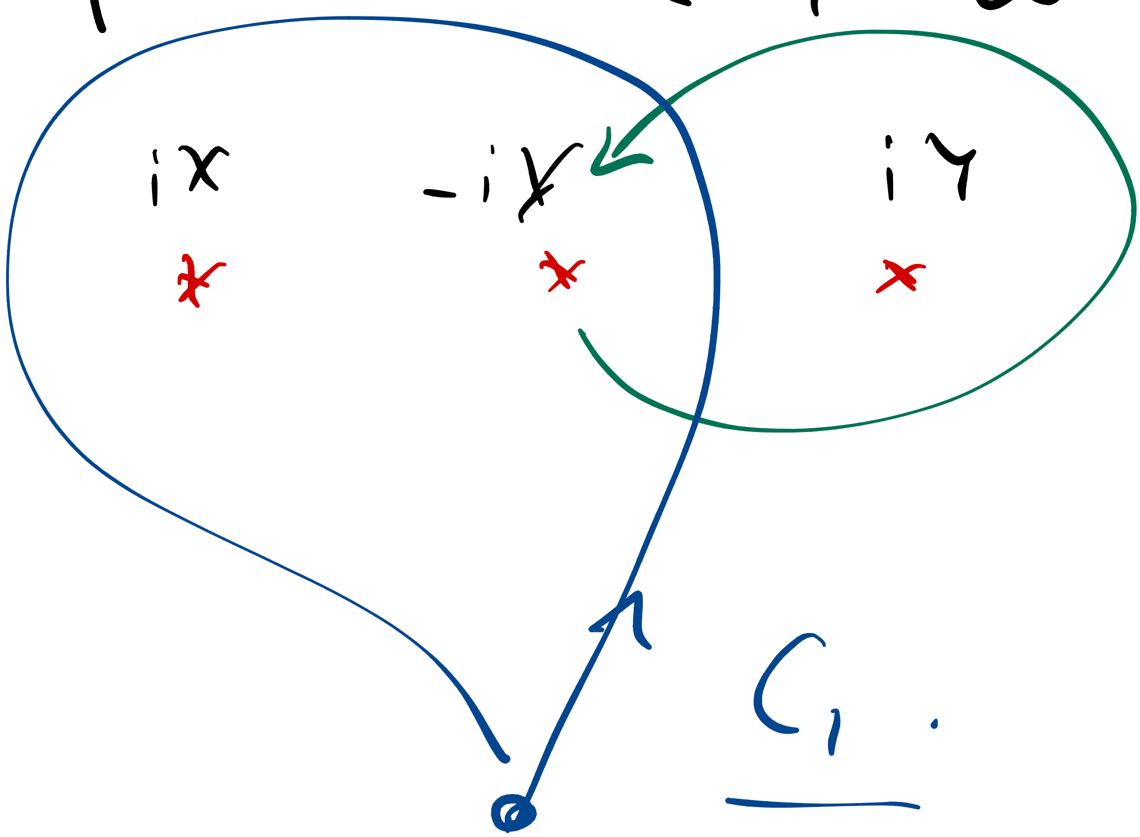
i'X
iY
≃
iX
(iY)⁻¹

transporting iX around iY

$$iX \longrightarrow iY\, iX\, (iY)^{-1} = Y iX Y = -iX.$$

transporting P around Q

$$P \longrightarrow Q\, P\, Q^{-1}.$$



$\underline{C_1}$.

homotopy class of codim 2 defects $\longleftrightarrow$ $\pi_1(V)$ / conjugation = conjugacy classes.

$X \qquad G = \langle g_1 \cdots \rangle$

$$g(\theta) = e^{i\theta A T^A}$$

T^A: generators

$$g(\theta = 0) = \mathbb{1}$$

$$\underline{\text{eg}}: \quad G = U(1)_Y$$

$$U(1)_Y \ni g = e^{i\theta Y}$$

$$H \ni e^{i\theta \underline{\underline{r}} Y}$$

$Y = 1$

$\underline{r \in \mathbb{Z}_{>0}}$

$$Y = \begin{pmatrix} q_1 & & \\ & q_2 & \\ & & \ddots \end{pmatrix} \qquad \begin{cases} \Phi_1 \to e^{iq_1\theta} \Phi_1 \\ \Phi_2 \to e^{iq_2\theta} \Phi_2 \\ \vdots \end{cases}$$

$$g_k = e^{ik2\pi/r} \quad \text{is trivial in } H \subset G$$

$$k = 0 \ldots r-1$$

$$\to \left(e^{ik2\pi/r}\right)^r = \mathbb{1} \quad \text{in } H.$$

$$G/H \overset{?}{\simeq} \langle g_k \rangle \simeq \mathbb{Z}_r.$$

$$\pi_1(G/H) = 0.$$

$$\hat{G} = SU(2) \times SU(2) \times \underline{\mathbb{R}}$$

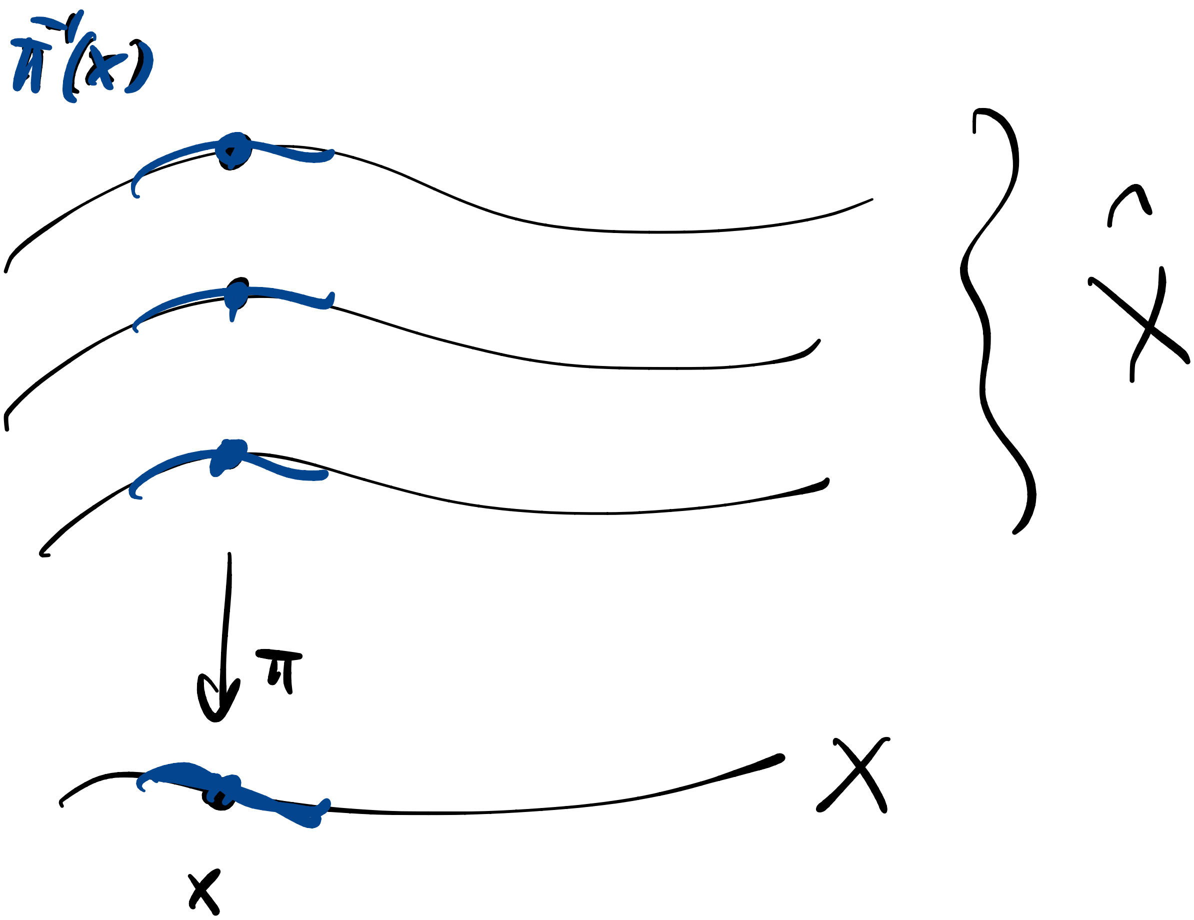
$$\hat{H} = \{ g \in \hat{G} \text{ s.t. } \underline{g\phi = \phi} \}$$

$$U(1)_Y = \mathbb{R}/\underline{\mathbb{Z}}$$

$$\theta \simeq \theta + 2\pi$$

$U(1)_Q$: action on θ is

$$\underline{\theta \to \theta + 2\pi r.}$$



$$G = SU(2) \times U(1)_Y$$

H is gen by $\boxed{pT^3 + rY \equiv Q}$

$$\left(V_\alpha \, , \, \Phi \right)$$

$\in 2$ of $SU(2)$

$$Y : (v, \Phi) \to (v, e^{i\gamma} \Phi)$$

$$Q : (v, \Phi) \to e^{i\theta \hat{Q}} (v, \Phi) = \left(\underline{e^{ip\theta \frac{\sigma^3}{2}} v}, \underline{e^{i\theta r} \Phi} \right).$$

$$\overset{!}{=} (v, \Phi)$$

if $e^{i\theta\hat{Q}} \in H$.

is a condition on v, Φ.

---

suppose $\exists$ order parameter

$\underset{\sim}{A}$ inv't under Q.

$A \in$ (3 of SU(2))

$$A \propto T^3 \longrightarrow e^{i\frac{\theta}{2}T^3} A\, e^{-i\frac{\theta}{2}T^3} = A.$$

Different problem: $G = U(1) \xrightarrow{\Phi} \mathbb{Z}_r = H$

$\langle\Phi\rangle \to e^{ir\theta}\langle\Phi\rangle$.

$G/H \cong U(1)$