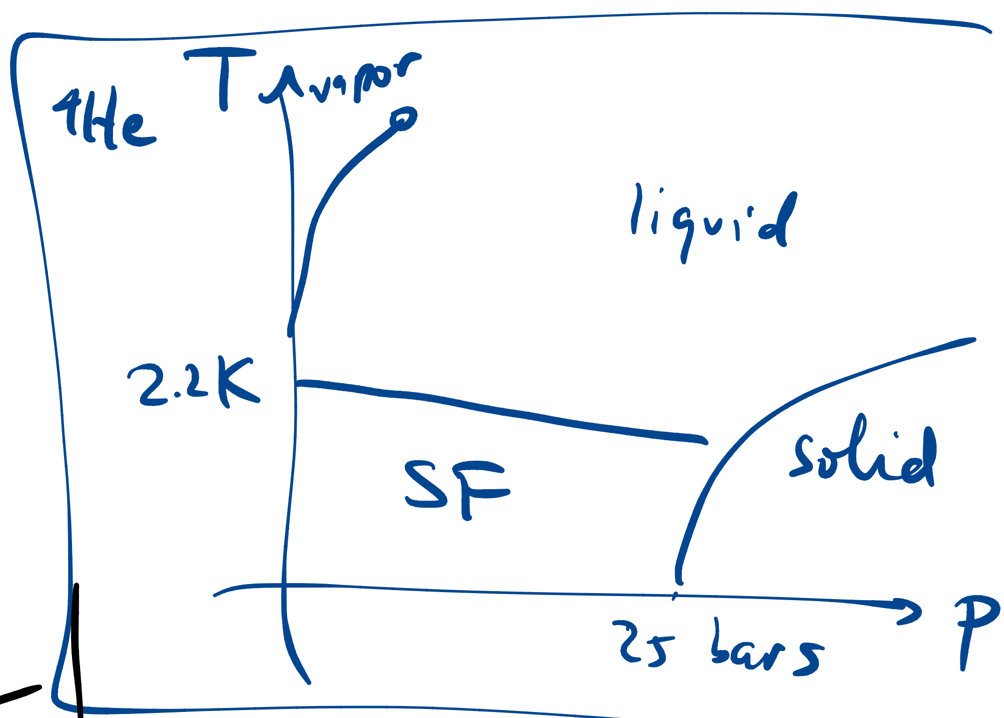
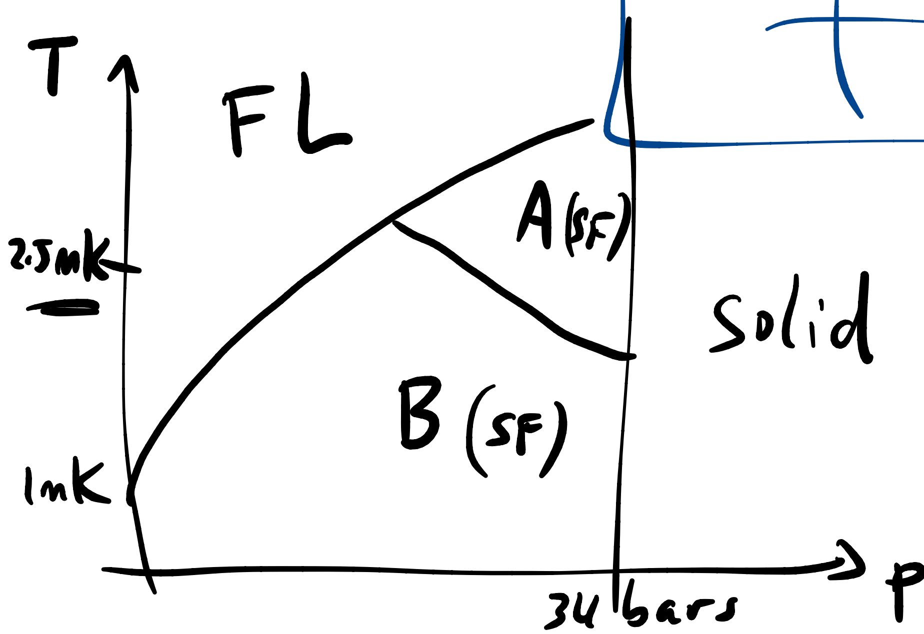
Last time: • Codim-q defects $\simeq \pi_{q-1}(V=G/H)/\pi_1(V)$

• examples
1. planar magnets
2. "
3. nematic $\pi_1(V)=\mathbb{Z}_2$
4. biaxial nematic $\pi_1(V)=Q_8$

non-abelian.

5. ^{3}He.





Sym of disordered phase:

$$G = \underline{\underline{U(1)}}_N \times SO(3)_S \times SO(3)_L$$

particle # — spin rotation — spatial rotations

(^3He has spin 1/2.)

Analog of $\Phi = v e^{i\phi} \to e^{i\alpha} v e^{i\phi}$

$\phi \to \phi + \alpha.$

Φ: transforms linearly under G

ϕ: non-linear

$$A_{\alpha i} \in (3,3)_1$$

$\alpha = 1..3 \in 3$ of $SO(3)_S$

$i = 1..3 \in 3$ of $SO(3)_L$

$$A_{\alpha i} \longmapsto \underbrace{e^{i\gamma}}_{U(1)} \underbrace{R^S_{\alpha\beta}}_{SO(3)_S} \underbrace{R^L_{ij}}_{SO(3)_L} A_{\beta j}.$$

whence $A_{\alpha i}$:

$$\underline{\underline{A_{\alpha i}}} \sim$$

(p-wave SF.)

$\psi_s(x)$ creates a ^3He at x in spin state $s = \pm$

$\psi_s^2 = 0$.

$$\langle \frac{\partial}{\partial x^i} \underbrace{\psi_s(x) \; \psi_{s'}(x)}\rangle \; \underline{\underline{\sigma^{\alpha}_{ss'}}}$$

$2 \otimes 2 = 1 \oplus 3$

$$H_{BdG} = \left(\begin{array}{c|c} \epsilon(k) \mathbb{1} & A_{\alpha i} \underline{\underline{k}}^i \sigma^\alpha \\ \hline A_{\alpha i} \underline{\underline{k}}^i \sigma^\alpha & -\epsilon(k) \mathbb{1} \end{array} \right)$$

$$f_{LG}[A] = r A^*_{\alpha i} A_{\alpha i} + u_1 A^*_{\alpha i} A_{\beta i} A^*_{\alpha j} A_{\beta j}$$

uniform

$+$ 4 other quartic terms.

$r > 0$ disordered liq.

$r < 0$ some ordered phase.

B-phase: $A^0_{\alpha i} = \Delta_B \, \underline{\underline{\delta_{\alpha i}}}$

$$SO(3)_L \times SO(3)_S \longrightarrow SO(3)_{diag} = H_B$$

orbit of A^0 under G

is $\Delta_B \, e^{i\gamma} \underline{\underline{R_{\alpha i}}}$ ← relative rotatn.

not invt under $U(1)_N$. ⟶ SF.

$$V_B = G/H_B = SO(3)_{rel} \times U(1)$$

$$= \mathbb{RP}^3 \times S^1$$

$$\left\{ \begin{array}{l} R = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \in SO(3)_{rel} \\ \times \, e^{i\gamma} = -1 \end{array} \right.$$ is unbroken

A phase: $A^{0}_{i\alpha} = \Delta_A \hat{z}_\alpha (\hat{x}_i + i\hat{y}_i)$

$\xrightarrow{G} \Delta_A \hat{d}_\alpha (\hat{e}^{(1)}_i + i\hat{e}^{(2)}_i)$

spin Rots about $\hat{d}$ are preserved

$SO(2)_S \subset SO(3)_S$

$\hat{e}^{(1)} \perp \hat{e}^{(2)} = 0.$

$\hat{e}^{(1)} \times \hat{e}^{(2)} = \hat{e}^{(3)}$
$= \hat{\ell}$

form a frame.

Completely breaks $SO(3)_L$.

Relative $U(1)_{rel}$ is preserved:

$$\begin{cases} A \to e^{i\gamma} A & U(1)_N \\ e^{(1)} + ie^{(2)} \to e^{-i\gamma}(e^{(1)} + ie^{(2)}) \end{cases}$$

$U(1) \subset SO(3)_L$

$-\gamma$ Rot. about $\hat{\ell}$.

$\mathbb{Z}_2$:
$$\begin{cases} d \xrightarrow{\text{Rotation by } \pi} -d \\ e^{(1,2)} \to -e^{(1,2)} \end{cases}$$

$$\left.\begin{matrix} \mathbb{Z}_2 \subset SO(3)_L \\ \times \mathbb{Z}_2 \subset SO(3)_S \end{matrix}\right\}$$

$\to \mathbb{Z}_2^{\text{diag}}.$

$$H_A = SU(2)_S \times U(1)_{rel} \times \mathbb{Z}_2$$

$$V_A = G/H_A = \frac{SO(3)_L \times SO(3)_S \times \cancel{U(1)_N}}{SO(2)_S \times \cancel{U(1)_{rel}} \times \mathbb{Z}_2}$$

$\hookrightarrow S^2$

$$= \left(S^2 \times SO(3)\right) / \mathbb{Z}_2$$

$$\pi_1(V_A) = \pi_1\left(\left(S^2 \times SU(2)/\mathbb{Z}_2\right)/\mathbb{Z}_2\right)$$

$$= \mathbb{Z}_4 .$$

6. Superconductors & Standard Model & Beyond.

w/ U(1) global sym:

$$S[\Phi] = \int \left(\partial_\mu \Phi^* \partial^\mu \Phi - V(|\Phi|)\right) d^d x\, dt$$

$+$ higher orders.

SF.

A superconductor is a SF w/ the $U(1)$ symmetry gauged $\equiv$ regard as an equivalence relation

$$\begin{cases} \Phi(x) \to e^{i\gamma(x)}\Phi(x) \\ A_\mu \to A_\mu + \partial_\mu \gamma \end{cases}$$

to write a gauge-inv't action
Replace $\partial_\mu \Phi \rightsquigarrow D_\mu \Phi$

$$\equiv (\partial_\mu + A_\mu)\Phi$$

$$\longrightarrow e^{i\gamma(x)} D_\mu \Phi .$$

$S[\Phi, \partial_\mu \Phi]$ (Abelian Higgs model)

$$\longrightarrow S[\Phi, D_\mu \Phi] + S_{kin}[F = dA] .$$

same top. analysis of defects in $\boxed{\text{"ordered phase"}}$

except for energetics.
(vortices have finite energy)

$$S[\Phi = v e^{i\phi}, A]$$

$$= \ldots\ v^2 \underline{(\partial_\mu \phi + A_\mu)}^2$$

choose $\gamma(x)$ s.t. $\phi = 0$.

$$\downarrow$$

$A_\mu A^\mu$ mass for A

$\longrightarrow$ Meissner effect

[Anderson-Higgs mechanism)

$\Phi \equiv$ Higgs field.

<u>SM of particle physics</u>: $G = SU(3) \times SU(2) \times U(1)_Y$

$\Phi \in (1, 2)_{\frac{1}{2}}$

Higgs field

$\langle \Phi \rangle$ preserves

$$G \supset H = SU(3) \times U(1)_Q$$

$$V = G/H = SU(2) \times U(1)_Y / U(1)_Q .$$

1.5 Homotopy groups of coset spaces $V = G/H$.

Relative homotopy groups:

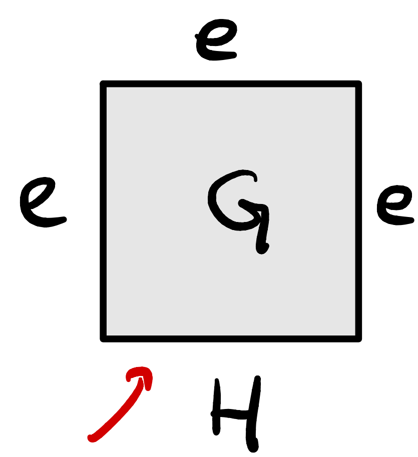
a closed submanifold $e \in H \subset G$

$$\pi_k(G, H, e) \equiv \{\text{maps } \phi : (I^k, \partial I^k) \to (G, H \text{ or } e)\}/\simeq$$

for π_1:

(not a group.)

w/



"bottom face"

Reason 1 to care:

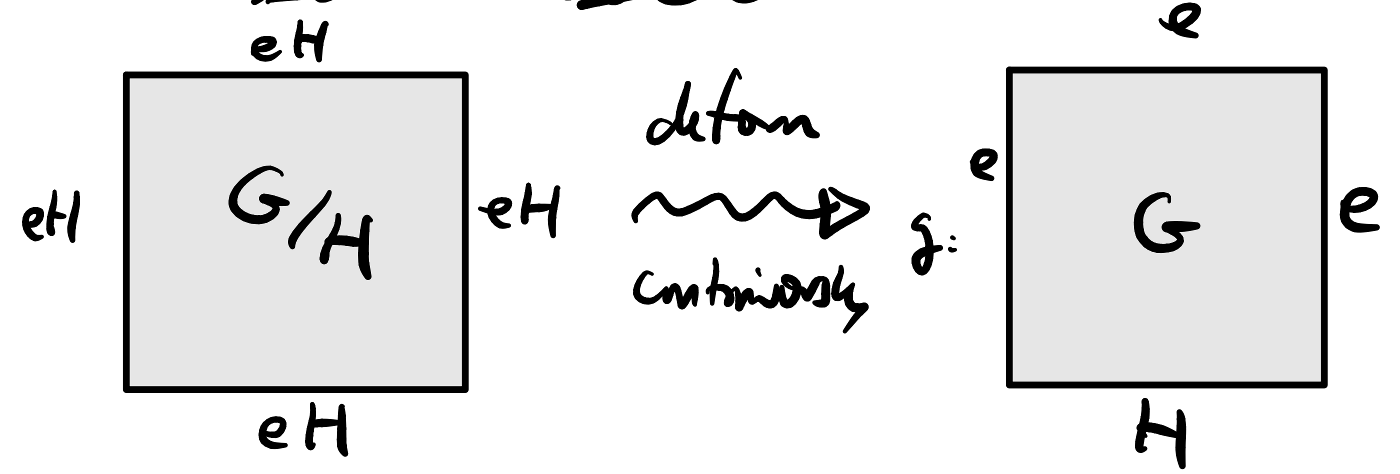
coset w/ identity

$$\pi_k(G, H, e) \cong \pi_k(G/H, eH)$$

given $f : (I^k, \partial I^k) \to (G/H, eH)$

determines g a rep. of $\pi_k(G, H, e)$

by $f(s) = g(s)H$.

$f:$ 

$g:$

$\in \pi_1(G/H, eH)$ $\qquad\qquad\qquad \in \pi_1(G, H, e)$

Reason 2:

$$\xrightarrow{\partial_*} \pi_k(H,e) \xrightarrow{i_*} \underline{\pi_k(G,e)} \xrightarrow{j_*} \pi_k(G,H,e) \xrightarrow{\partial_*} \pi_{k-1}(H,e) \xrightarrow{i_*} \cdots$$

$i : H \hookrightarrow G$. $\qquad e \in H$

$$\partial_* \phi = \phi\big|_{\text{bottom face}} : (I^{k-1}, \partial I^{k-1}) \to (H, e)$$

is an exact sequence of gr. homomorphisms (when groups)

$$\mathrm{Im}\, i_* = \ker j_*$$

$$\mathrm{Im}\, j_* = \ker \partial_*$$

$$\mathrm{Im}\, \partial_\# = \ker i_* \leftarrow$$

$\vdots$

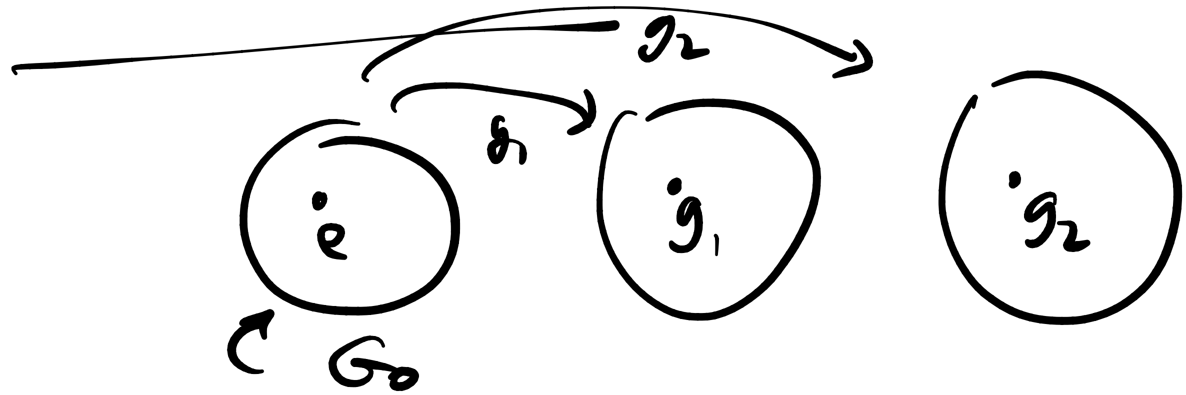
$$\pi_0(G),\ \pi_0(H) \ \overset{\text{exact seq}}{\rightsquigarrow}\ \pi_0(G,H) = \underbrace{\pi_0(G/H)}_{=V}.$$

homotopy groups of Lie groups:

0. let G_0 = component of G w/ e.

is a normal subgroup.

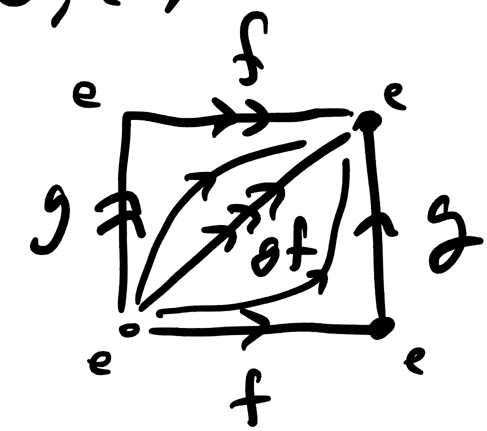
$\pi_0(G) = G/G_0$ (also a group)



1: For any Lie group, $\pi_1(G)$ is abelian.

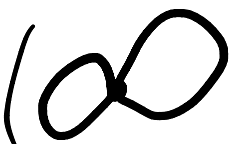
$$f, g : (I, \partial I) \to (G, e)$$

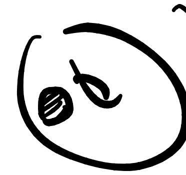
<u>To show:</u> $f * g \simeq g * f$.



Another way to multiply paths:

$$(gf)(s) \equiv g(s) f(s)$$

<u>Conseq</u>: $\pi_1\Big($  $\Big) = \pi_1\Big($  $\Big) = \pi_1(\mathbb{R}^2 \setminus 2\text{pts})$

$= \langle a, b \rangle$ $\qquad = \pi_1\Big($  $\Big)$

$= F_2$

none of these is a Lie group.

$\underline{\underline{2}}.$ $\pi_2(G) = 0$ for any Lie group.

$\underline{3}.$ $\pi_3(G) = \mathbb{Z}$ for any simple Lie group.

$\underline{\underline{4}}.$ $\pi_q(G) = \pi_{q+p}(G)$ — $p=2$ for $G = U(N)$, $p=8$ for $G = SO(N)$

Bott periodicity.

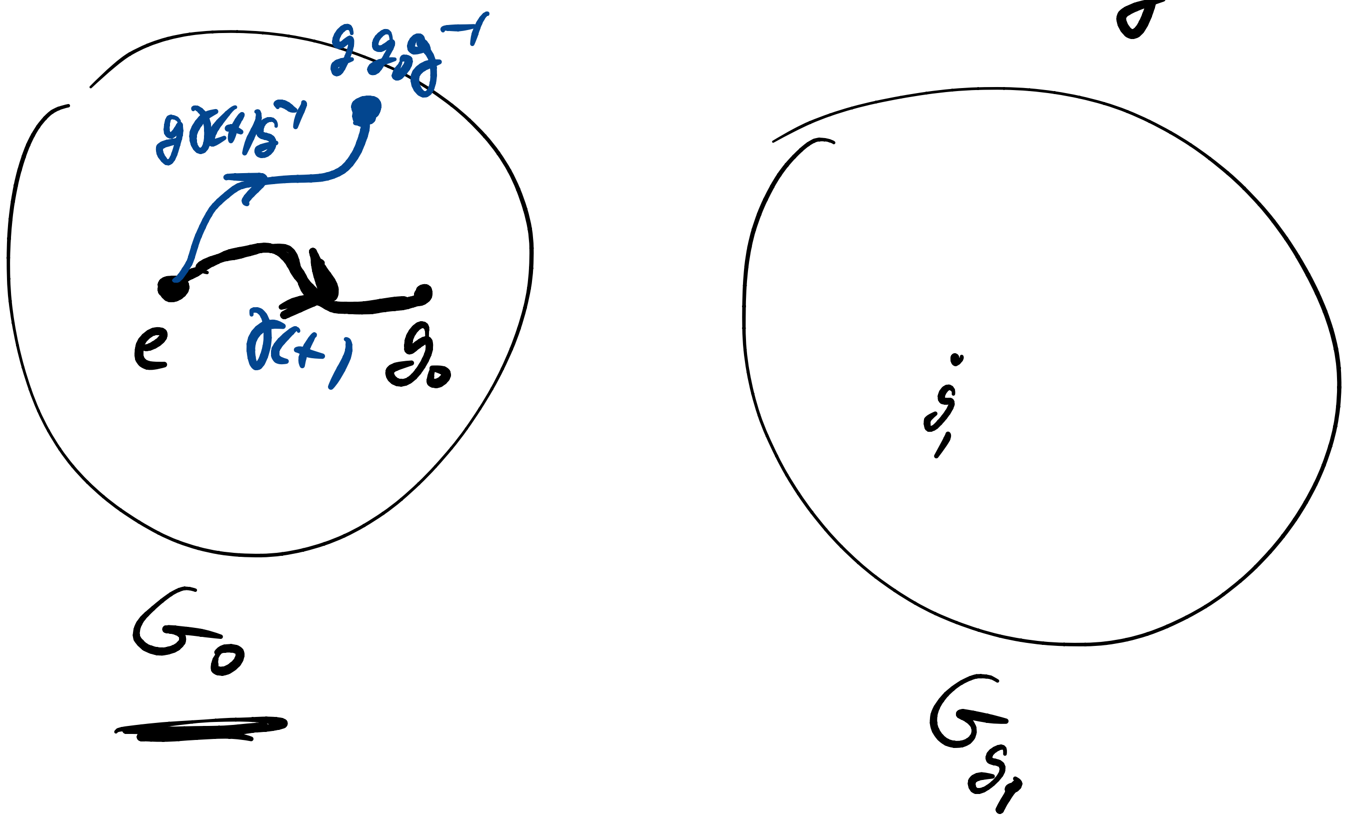
To show:

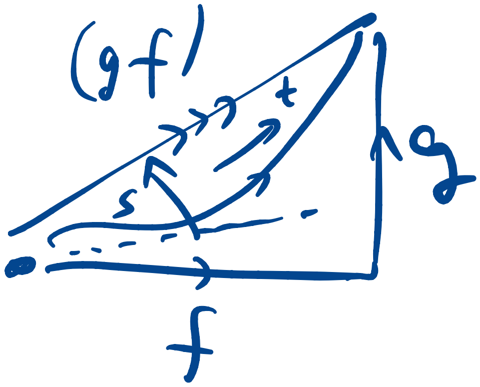
- $\pi_2(G/H) = \pi_1(H_0)$ if $\pi_1(G) = 0$.

- $\pi_1(G/H) = \pi_0(H)$
$= H/H_0$

claim: $G_0 < G$ is normal.

i.e. $g_0 \in G_0$ then $g g_0 g^{-1} \in G_0$ $\forall g \in G$.

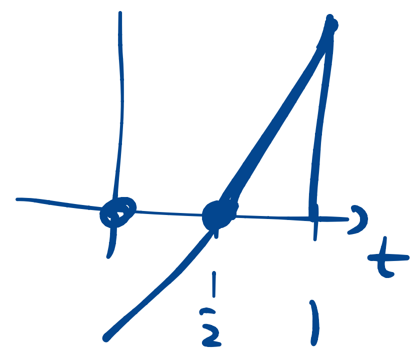




e

$$F(t,s) \overset{?}{=} \begin{cases} \underline{\underline{g(s\cdot 2t)}}\, f(2t) & t < 1/2 \\ g(-1+2t)\, f(\underline{s(-1+2t)}) & t > 1/2 \end{cases}$$

$\underline{s=1:}$ $\qquad \underline{\underline{g(t)\, f(t)}}$



$0 < s < 1$:

$$F\left(\tfrac{1}{2}, s\right) = \begin{cases} g(s)\, f(0) & t \to \frac{1}{2}^{-} \\ g(0)\, f(0) & t \to \frac{1}{2}^{+} \end{cases}$$

$$f * g(t) = \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(-1+2t) & \frac{1}{2} \le t \le 1 \end{cases}$$

$$fg(t) \equiv f(t)\, g(t)$$

<u>claim:</u> $fg(t) \simeq gf(t)$

$$\underbrace{F(t,s) \overset{?}{=} f(t)\, g(ts)}$$

$F(0,s) = e$.

$F(1,s) = g(s) \neq e$.

$$F(t,s) = g^{-1}(ts)\, f(t)\, g(ts)$$

$F(t,0) = f(t)$

$F(t,1) = g^{-1}(t) f(t) g(t)$ $\Rightarrow$ $\boxed{g^{-1} f g \simeq f}$

$$f*g(t) = \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(-1+2t) & \frac{1}{2} \le t \le 1 \end{cases}$$

$$F(t,s) \overset{?}{=} \begin{cases} g^{-1}(st)\, f(2t)\, g(2st) & (0,\tfrac{1}{2}) \\ \underline{\underline{f}}^{-1}((-1+2t)s)\, g(-1+2t)\, \underline{\underline{f}}((-1+2t)s) \end{cases}$$

$\underline{t = \frac{1}{2}}$:

$$g^{-1}(s) f(1) g(s) \qquad \tfrac{1}{2}^-$$

$$= f^{-1}(s) g(1) f(s) \qquad \tfrac{1}{2}^+$$

$$F(t,0) = f*g(t)$$

$$F(t,1) = \begin{cases} g^{-1}(2t) f(2t) g(2t) \\ \underline{\underline{f}}^{-1}(s(-1+2t))\, g(\quad)\, f \end{cases}$$

$$= g^{-1}fg * f^{-1}gf$$

$$f*g \simeq g^{-1}fg * g \simeq f * f^{-1}gf$$

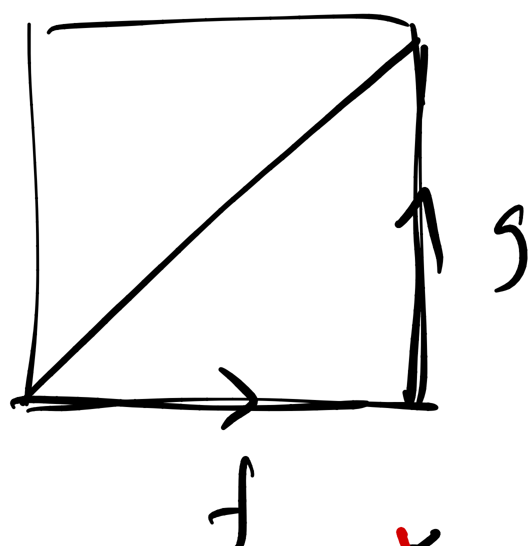
find $h(t)$ s.t. $h(t)f(t) = g(t)$

$h(t) = g(t)f^{-1}(t)$

$$H(t,s) = \begin{cases} e & s=0 \\ h(t) & s=1 \end{cases}$$

$= h(st)$

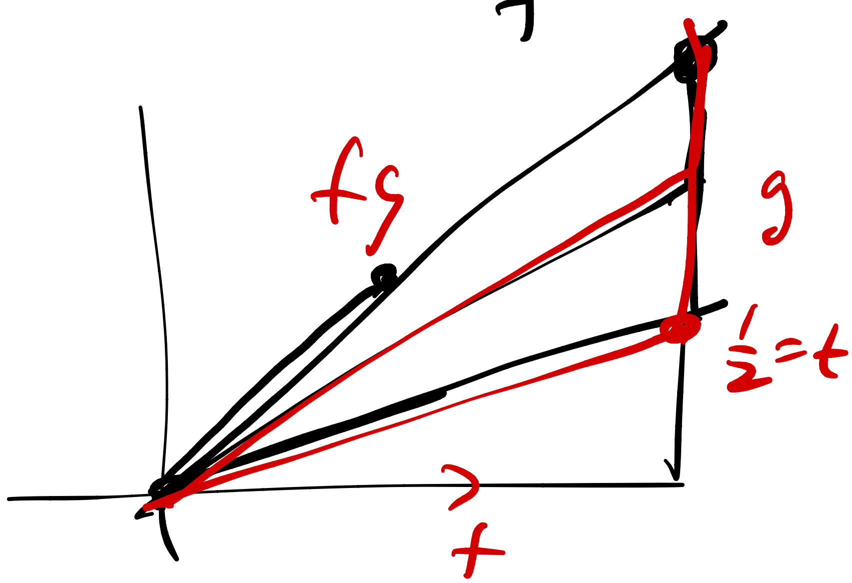
$f * g \overset{?}{\simeq} f * f$



$F(t,s) = f(t)\,g(st)$

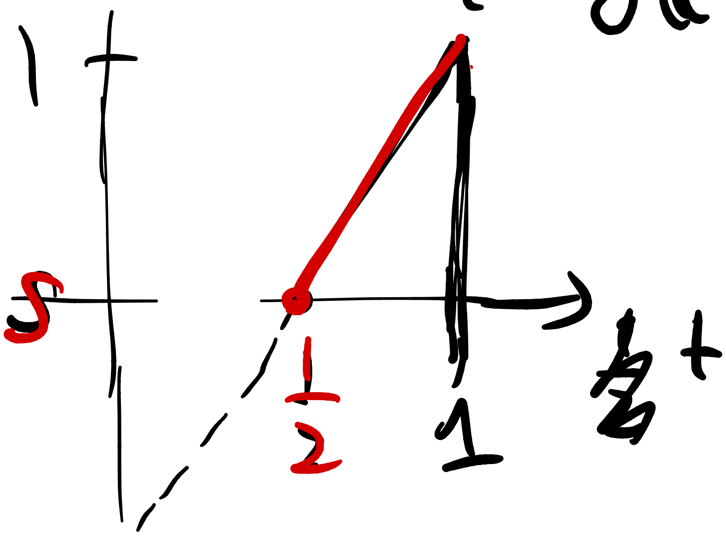
$F(t,0) = f(t)$
$F(t,1) = f(t)g(t)$

<u>But:</u> $F(1,s) = g(s)$



$$F(t,s) = \begin{cases} f(2t)\, g(2ts) & t < \frac{1}{2} \\ g\big((1-s)\,1 + s\big) & t > \frac{1}{2} \end{cases}$$

$(-1+2t)$



$F(t,0) = f * g$ ✓

$F(t,1) = (fg)$ ✓

$F(\frac{1}{2}^-, s) = f(1)\,g(s) = g(s)$

$F(\frac{1}{2}^+, s) = g(s)$ ✓