Recap: continuous G broken $\to$ $\phi(x)$ is the goldstone field
$= U(1)$

Sometimes: $\Phi(x) = v e^{i\phi(x)} \xrightarrow{U(1)} e^{i\alpha}\Phi(x)$

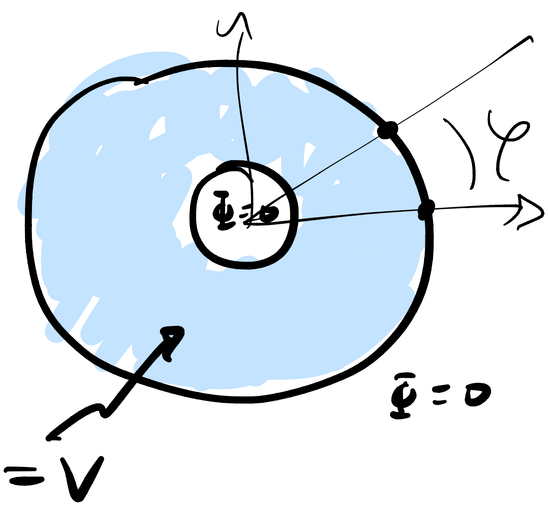
$(\phi \to \phi + \alpha)$

If $\Phi = 0$, ϕ is ill-defined.

ϕ is a phase: $\phi \cong \phi + 2\pi$

$\phi \in S^1$

eg:

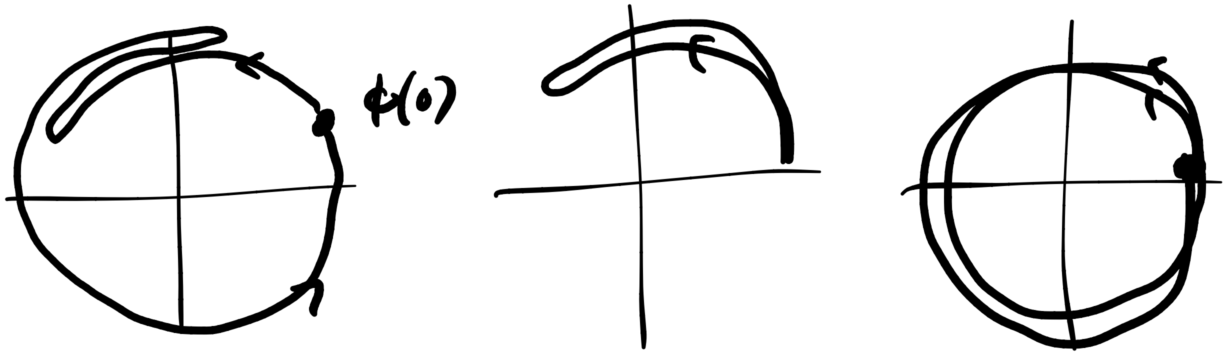


$\varphi \cong \varphi + 2\pi$.

$\phi(\varphi)$ is single-valued

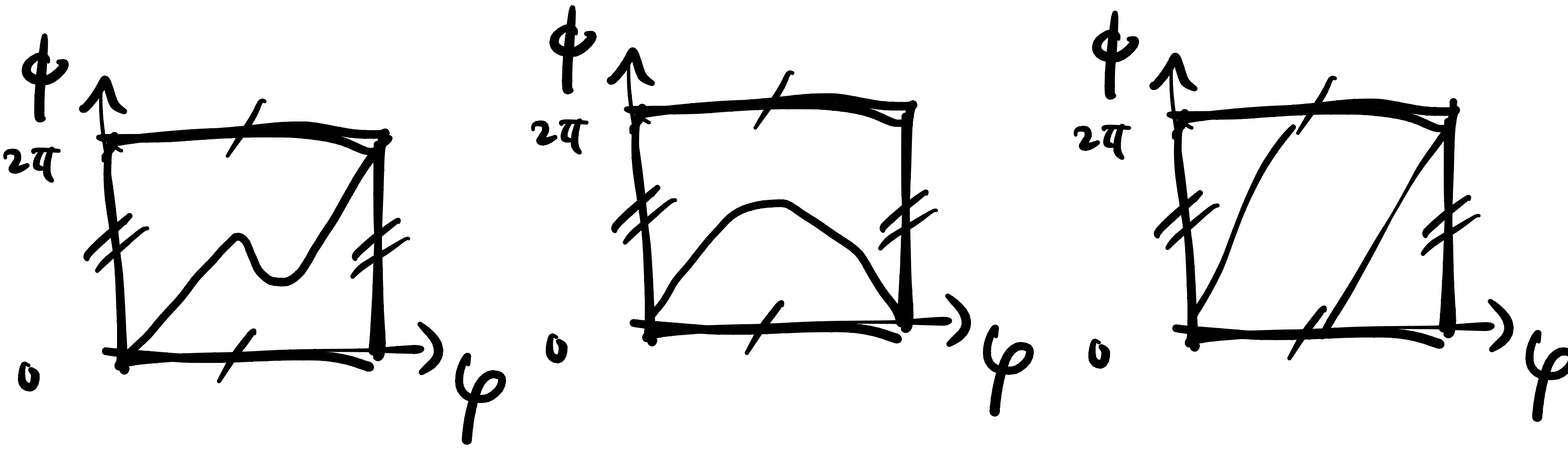
continuous $\phi : S^1 \to S^1$.

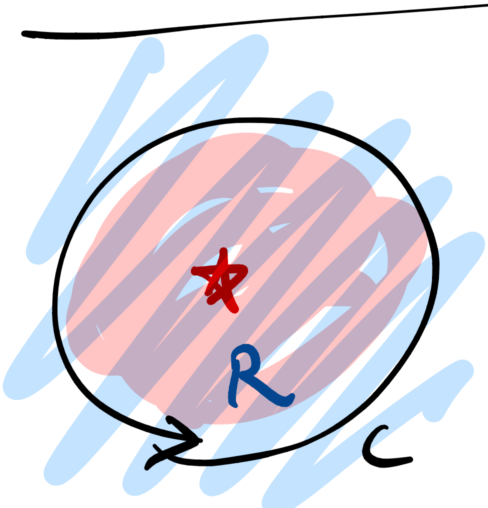
i.e. $\phi(\varphi + 2\pi) \overset{!}{=} \phi(\varphi) + 2\pi w$, $w \in \mathbb{Z}$ $\forall \varphi$.



$w = 1$

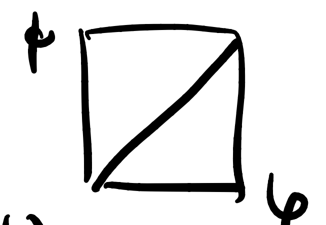
$w = \frac{1}{2\pi}\oint_{\text{unit circle}} d\phi = 1$ $\qquad$ $w = 0$ $\qquad$ $w = 2$.





vortex ≡ codim 2 locus around which $\oint d\phi \neq 0$.

<u>eg:</u> $\phi = \varphi$



$$2\pi W[C] = \oint_C d\phi \overset{\text{stokes}}{=} \int_R d(d\phi) = \int_R d^2\phi$$

$\partial R = C$

$w(C)$ = # of vortices contained in R

$$\frac{1}{2\pi} d^2\phi = \text{density of vortices} \equiv j^0_v$$

$$= \frac{1}{2\pi} \epsilon_{ij} \partial_i \partial_j \phi \quad \neq 0.$$

ϕ is singular in the core of the vortex.

$$j^0_v = \frac{1}{2\pi} \epsilon_{ij} \partial_i \Big(\underbrace{-\frac{1}{2} i \Phi^* \partial_j \Phi + \frac{1}{2} i \partial_j \Phi^* \Phi} \Big)$$

e.g.

$$S[\Phi] = \int dt\, d^dx \left(\underline{\partial_t \Phi^* \partial_t \Phi}^{\,*} - \vec\nabla \Phi^* \cdot \vec\nabla \Phi - V(|\Phi|) \right)$$

$$\left(\text{or } \; i \Phi^* \partial_t \Phi \right)$$

$$\Phi \to e^{i\alpha} \Phi \xRightarrow{\text{Noether}} j_\mu = -\frac{1}{2} i \Phi^* \partial_\mu \Phi + h.c.$$

is conserved $\partial^\mu j_\mu = 0.$

$$\left(\Rightarrow \frac{d}{dt} \underbrace{\left(\int_R j_0 \right)}_{Q \text{ in } R} = \oint_{\partial R} \vec{j} \cdot d\vec{a} \right)$$

similarly: <u>in $D = 2+1$</u>.

$$j^\nu_\rho = \frac{i}{2\pi} \underbrace{\epsilon_{\rho\mu\nu}} \partial_\mu \Phi^* \partial_\nu \Phi$$

(in $D = 3+1$ $\quad j^\nu_{\rho\lambda} = \frac{i}{2\pi} \epsilon_{\rho\lambda\mu\nu} \partial_\mu \Phi^* \partial_\nu \Phi$)

1-form current $\quad \partial^\rho j^\nu_{\rho\lambda} = 0$.

$$\partial^\rho j^\nu_\rho = 0$$

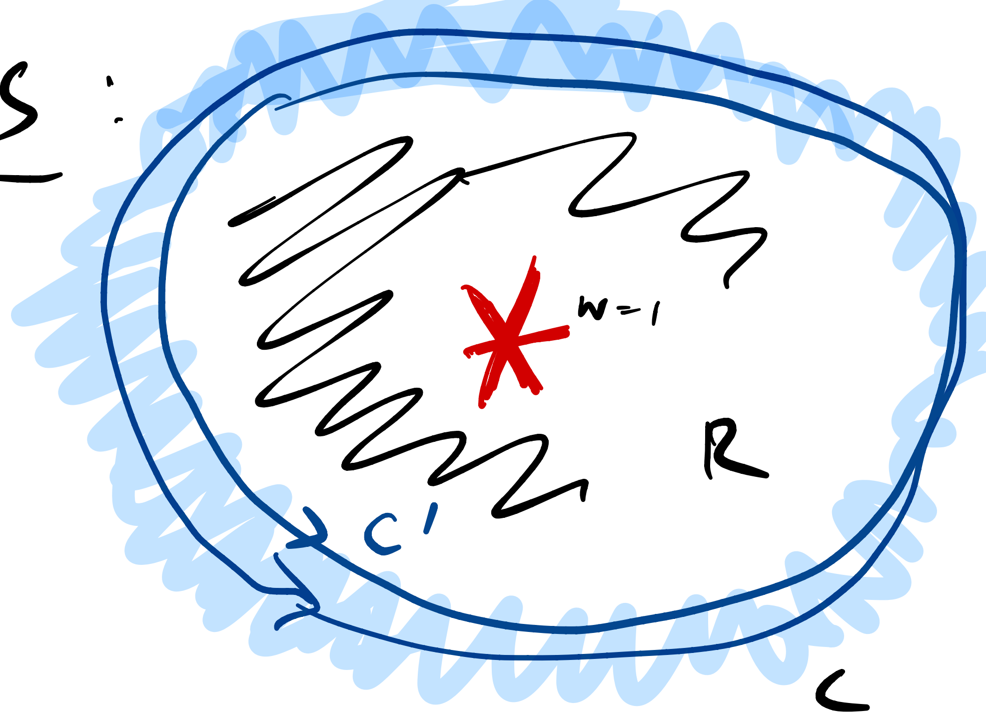
$[\partial_\mu, \partial_\nu] = 0$ on $\underline{\underline{\Phi}}$ ✓

$$\underbrace{\text{topological current density}}$$

vortices don't disappear..

vortex

CRUCIAL OBS :



$w(C) = \frac{1}{2\pi}\oint d\varphi \neq 0$

$= w[C']$

To fix the singularity in R ($\partial R = C$)

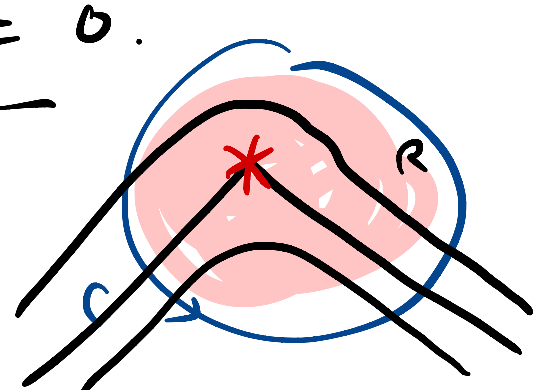
a singularity must pass through every curve C, C'.

$\Longrightarrow$ defect is topologically stable.

(to all operations in R)

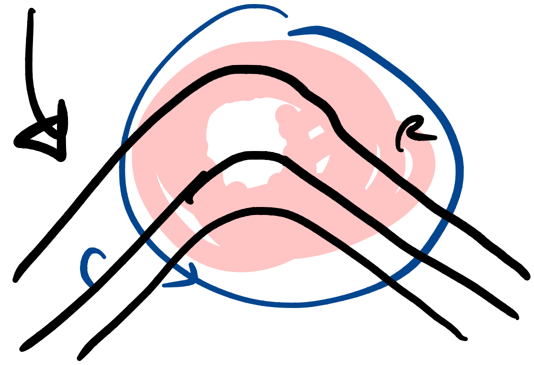
In contrast: if $w[C] = 0$.

can still be singular.



CLAIM: can be repaired locally.

1.2 Q: When are there stable defects?



$$\mathbf{V} \equiv \{ \text{minima of } V(\Phi) \} \equiv \text{vacuum manifold.}$$

$$\equiv F_{LG}\left[\begin{matrix}\text{uniform} \\ \text{configs}\end{matrix} \right]$$

(previously $V = S^1$)

(focus on internal syms for a bit)

acts on each point independently

Fact: There are no stable topological defects of codim q if $\pi_{q-1}(\mathbf{V}) = 0$.

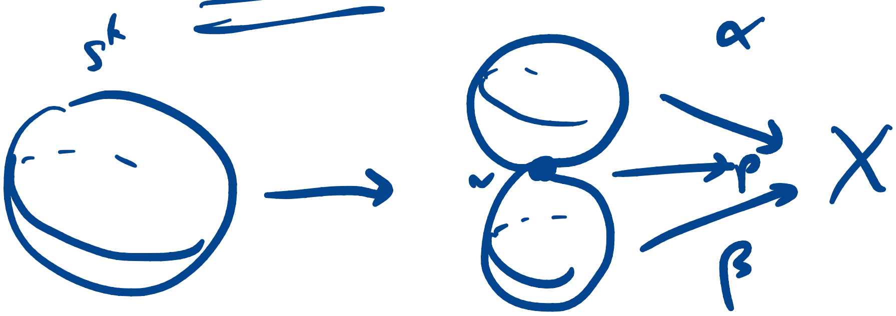
Reminder:

basepoint

$$\pi_k(X,p) \equiv \left\{ \text{maps } \alpha : \begin{array}{c} S^k \to X \\ N \mapsto p \end{array} \right\} \Big/ \simeq$$

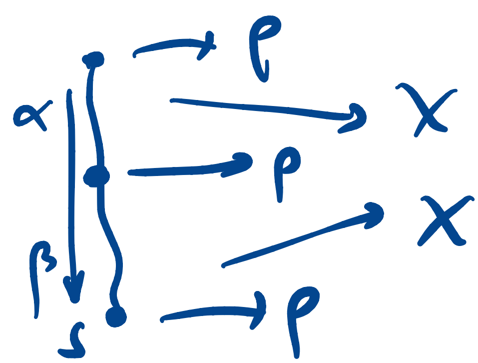
homotopy equivalence

is a group under



$\alpha * \beta$.

for $k=1$:



order matters.

$\alpha * \beta \neq \beta * \alpha$.

identity $= p$, constant map to p

π_1 can be nonabelian.

If $\pi_k(X) = 0$ then any map $\alpha : S^k \to X$

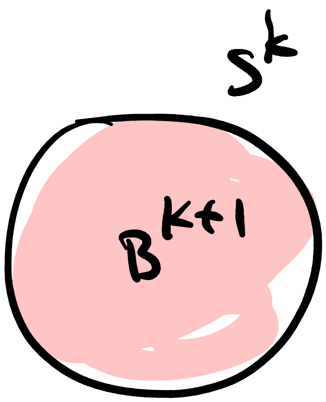
$\simeq p$.

Lemma: A continuous map $\phi : S^k \to V$ is homotopic to a constant map

$\iff$ ϕ can be extended to

$$\hat{\phi} : B^{k+1} \longrightarrow V$$

w/ $\hat{\phi}\big|_{\partial B^{k+1} = S^k} = \phi$.



$\pi_k(V)$ measures obstruction to lifting from S^k to its interior.

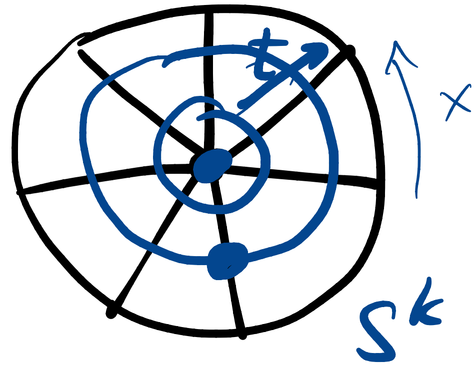
Pf: $\boxed{\Rightarrow}$ given a homotopy

$\phi_t : S^k \to V \quad \forall t \in I = [0,1]$

w/ $\phi_0 = \phi$ and $\phi_1 = p$, constant map.

crucial obs:

A point in B^{k+1} is uniquely labeled as $(x, t) \in S^k \times I$



$\underline{y = tx} \in \mathbb{R}^{k+1}$.

$$\hat{\phi} : B^{k+1} \to V$$

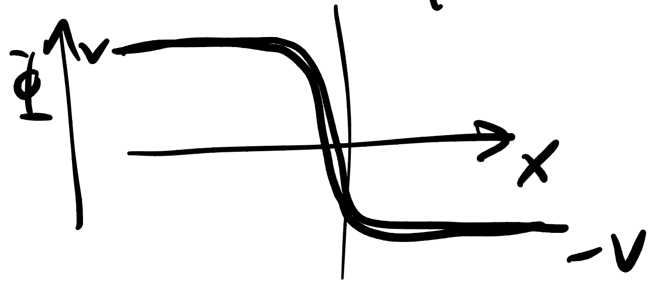
$$(x, t) \mapsto \phi_{1-t}(x) \equiv \hat{\phi}(tx)$$

continuous b/c: $\phi_0(x) = p$ $\forall x$. ✓

∎

$\pi_0(V) = \{\text{connected components}\}$

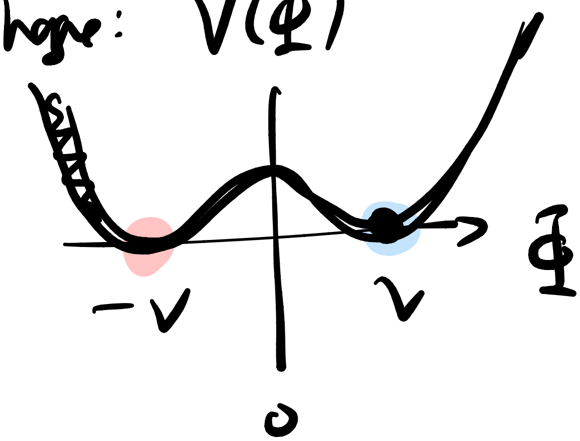
defect of codim 1 = domain wall

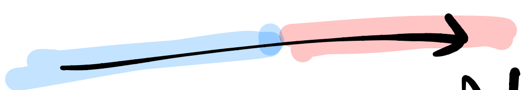


eg: $G = \mathbb{Z}_2$

$\Phi \to -\Phi$.

Broken phase: $V(\Phi)$





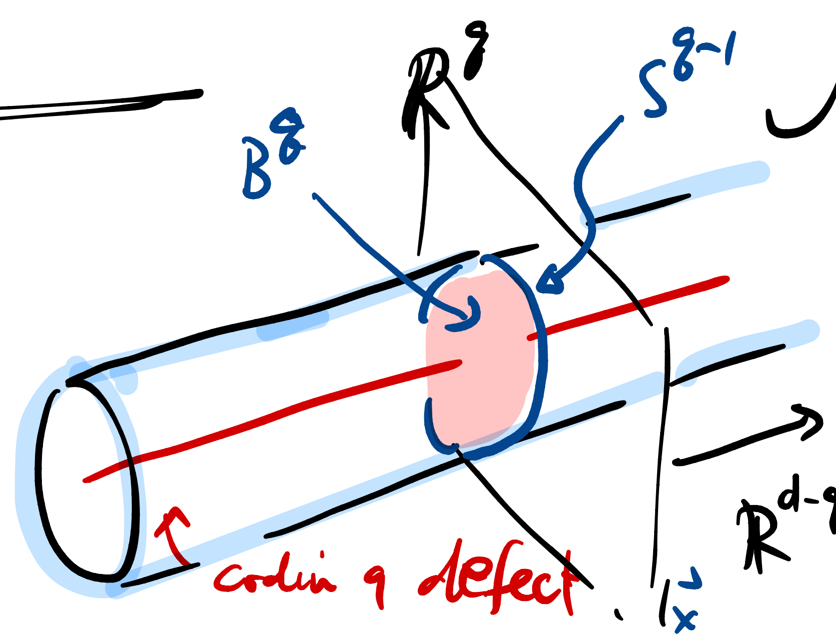
$\underline{V = \mathbb{Z}_2}$.

$\pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$.

$$\boxed{\pi_{q-1}(V) = 0 \equiv \pi_{q-1}(V) \text{ has } \underline{\text{one element}}.}$$

<u>Pf of fact</u>.

$S^{q-1} \times \mathbb{R}^{d-q} \rightsquigarrow$



ϕ is continuous on $S^{q-1} \times \mathbb{R}^{d-q}$.

$\ni (S^{q-1}, x)$

$$\phi\Big|_{(S^{q-1}, x)} : S^{q-1} \longrightarrow V$$

represents an element of $\pi_{q-1}(V)$

If $\pi_{q-1}(V) = 0 \implies \phi\Big|_{(S^{q-1}, x)}$

is homotopic to p. the constant map.

$\overset{\text{Lemma}}{\implies} \exists\ \hat{\phi}\Big|_{(B^q, x)}$ <u>continuous</u>.

$\implies$ <u>no singularity</u> of $\hat{\phi}$

$\hat{\phi}$ differs from ϕ by local surgery. $\boxtimes$

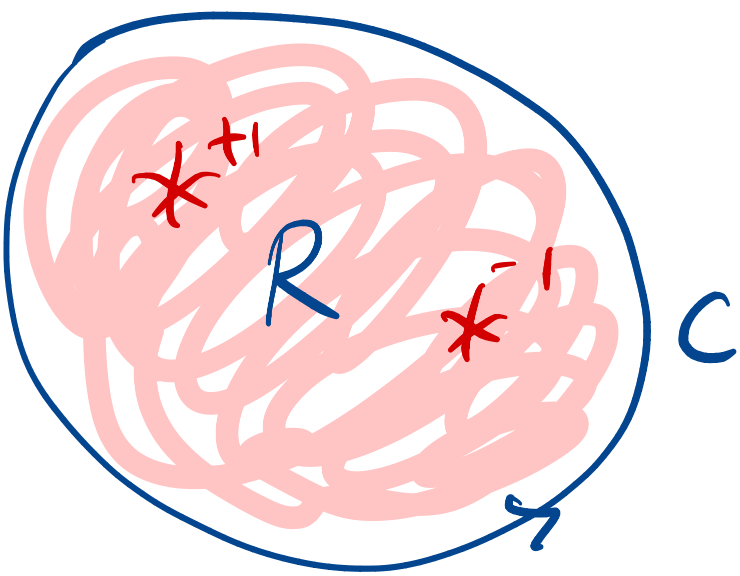
Pf of if $w[C]=0$ then repair by local surgery:

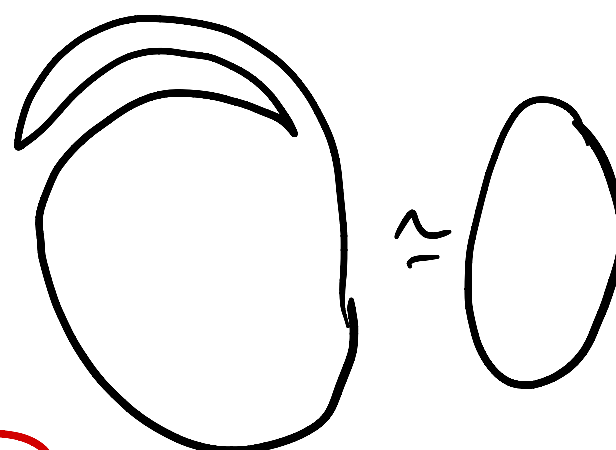
$w[C]=0 \Rightarrow \phi|_C : S^1 \to S^1$

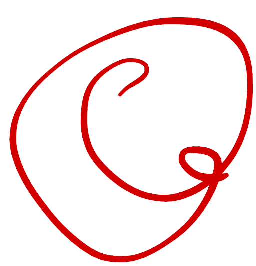
$\phi|_C \simeq$ constant map.

i.e. by modifying ϕ only in R, $\partial R = C$. $\Rightarrow \exists\ \hat{\phi}|_R$ continuous.

$w\ \partial R = 0$ ∎







homotopy equiv.

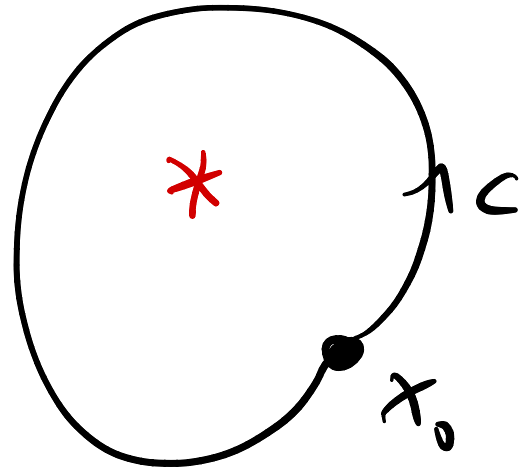
e.g.

Q: Does this mean top. defects of codim q

$\Longleftrightarrow \pi_{q-1}(V)$?

A: almost.

eg: codim 2. $(q=2)$



given ϕ

$\phi(x_0) = p$.

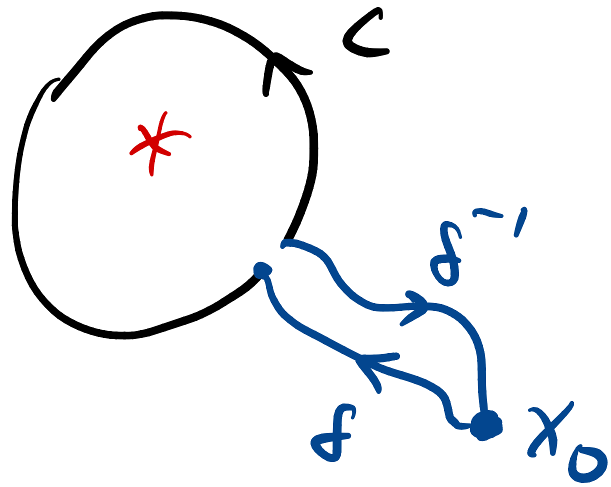
$\underbrace{[\phi|_C]}_{\equiv\alpha} \in \pi_1(V,p)$.

problem: some other $\phi|_C$

or $\phi|_{C'}$ will not hit p.

fix #1: suppose V is path connected.

demand that $\phi(x_0) = p$.



$\phi(\delta^{-1} \circ C \circ \delta)$

$= \beta^{-1} * \alpha * \beta$:

$(I, \partial I) \to (V, p)$

$\alpha \longrightarrow \beta^{-1} \alpha \beta$ conjugation.

stable defects of codim 2 $\Longleftrightarrow$ conjugacy classes of $\pi_1(V)$.

$= \{ g \in \pi_1(V) \} / g \sim hgh^{-1}$

$h \in \pi_1(V)$.

$$G = \bigsqcup_i C_i \qquad C_g = \{g' = hgh^{-1}\}$$

$$h \in G.$$

$$= C_e \sqcup \cdots$$

$g, g' \in C_g$

if $g = hgh^{-1}$.

<u>eg</u>: $Q_8 = \{\mathbb{1}, -\mathbb{1}, \pm i\sigma_x, \pm i\sigma_y, \pm i\sigma_z\}$

$$= C_{\mathbb{1}} \sqcup C_{-\mathbb{1}} \sqcup C_x \sqcup C_y \sqcup C_z$$

$$C_j = \{i\sigma^j, -i\sigma^j\}$$

$$\epsilon_{ij} \partial_i \partial_j \phi$$

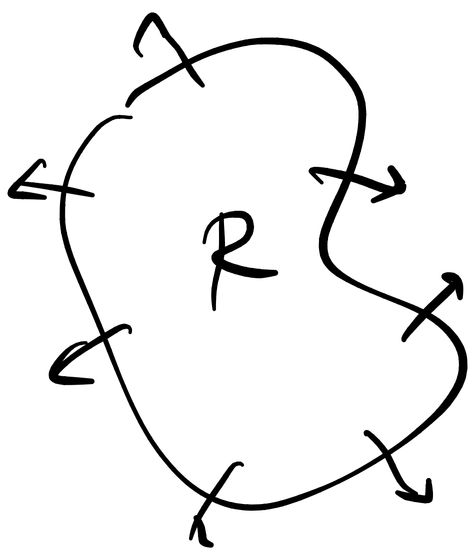
$$= \partial_x \partial_y \phi - \partial_y \partial_x \phi$$

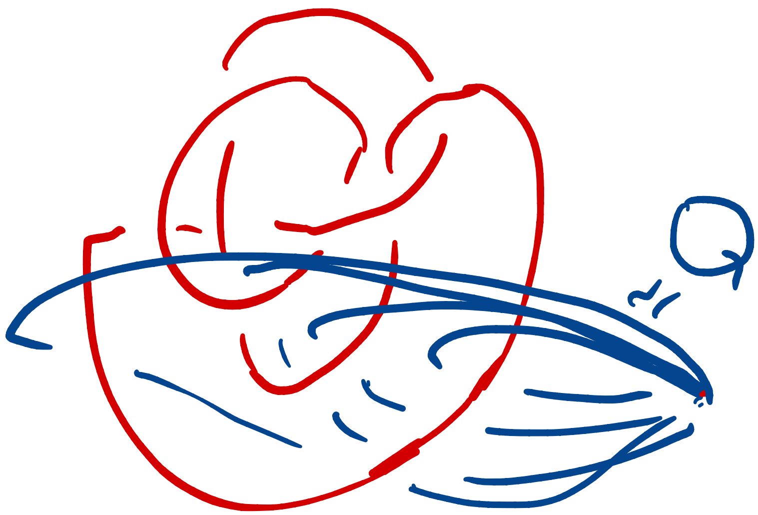
$$\partial^\mu j_\mu = 0$$

$$= \partial_t j_0 + \vec{\nabla} \cdot \vec{j}$$

$$\frac{d}{dt}\Big(\underbrace{\int_R j_0 \, d^d x}_{Q} \Big) = \int_R \vec{\nabla} \cdot \vec{j} \, d^d x$$

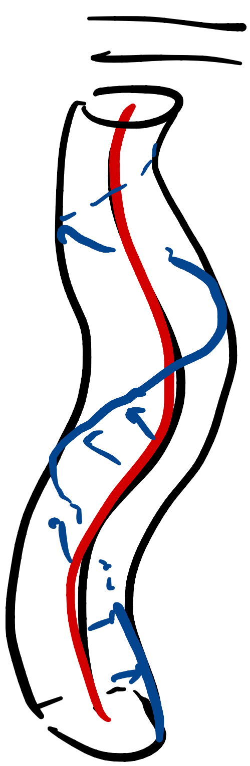
$$\underset{\text{Stokes}}{=} \int_{\partial R} \vec{j} \cdot d\vec{S}$$





$\pi_1(V)$-bundle on $\mathbb{R}^3 \setminus K$

Rep $\rho : \underbrace{\pi_1(\mathbb{R}^3 \setminus K) \to \pi_1(V)}$



$\phi^{-1}(0)\Big|_{\partial B}$