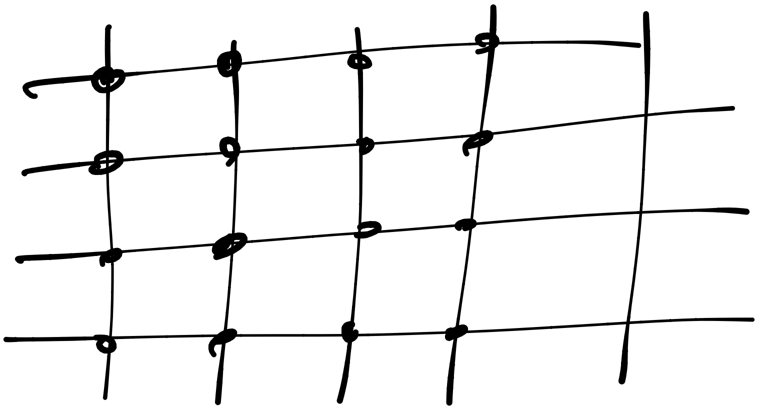
Non-BCS superconductivity



$$H = H_t + H_U$$

$$U > t.$$

Half-filling $U \gg t$:

$$H_{eff} = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

$$J = 4t^2/U$$

"Dope with holes"

filling $< \frac{1}{2}$.

$$H_{tJ} = \sum t\, c^{\dagger}_{i\sigma} c_{j\sigma} + h.c. + J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j.$$

$$\vec{S}_1 \cdot \vec{S}_2 = -\frac{1}{2} \underbrace{\overbrace{\left(\epsilon_{\alpha\beta}\, c^{\dagger}_{1\alpha} c^{\dagger}_{2\beta}\right)}^{\Delta_2^* \equiv} \left(\epsilon_{\gamma\delta}\, c_{1\gamma} c_{2\delta}\right)} + \frac{1}{4} \left(c^{\dagger}_{1\alpha} c_{1\alpha}\right)\left(c^{\dagger}_{2\beta} c_{2\beta}\right)$$

$$\sum_a \sigma^a_{\alpha\beta}\, \sigma^a_{\gamma\delta} = -2\, \epsilon_{\alpha\gamma} \epsilon_{\beta\delta} + \delta_{\alpha\beta} \delta_{\gamma\delta}.$$

$$\boxed{\vec{S}_1 = \frac{1}{2} c^{\dagger}_{1\alpha} \vec{\sigma}_{\alpha\beta} c_{1\beta}}$$

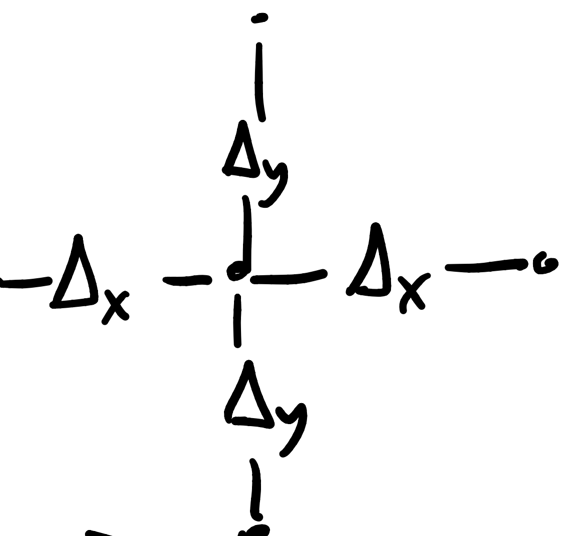
$$\Delta_{ij} \equiv -\langle \epsilon_{\alpha\beta}\, c_{i\alpha}\, c_{j\beta} \rangle$$

$$H_{MF} = \sum t_{ij}\, c_i^\dagger c_j + h.c. - \mu \sum c_i^\dagger c_i$$

$$- \frac{J}{2} \sum_{\langle ij \rangle} \Big(\Delta_{ij}\, \epsilon_{\alpha\beta}\, c_{i\alpha}^\dagger c_{j\beta}^\dagger + h.c - |\Delta_{ij}|^2 \Big)$$

$$\begin{cases} \Delta_{ij} \to e^{i\phi_i} \Delta_{ij} e^{i\phi_j} \\ c_i \to e^{i\phi_i} \end{cases}$$

Assume transl inv: $\Delta_{ij} = \Delta_{i-j} = \Delta_{j-i}$



$$\Delta_k = \Delta_x \cos k_x + \Delta_y \cos k_y$$

$$H = \sum_k \Big[\epsilon_k\, c_{k\sigma}^\dagger c_{k\sigma} - J\, \Delta_k\, c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + h.c. \Big] - |\Delta_x|^2 - |\Delta_y|^2$$

$$\begin{cases} d_{k\uparrow} = c_{k\uparrow} \\ d_{k\downarrow} = c_{-k\downarrow}^\dagger \end{cases}$$

Sol'n: $\boxed{\text{"d-wave SC"}}$

$\Delta_x = -\Delta_y$.

$$= \sum \begin{pmatrix} d_{k\uparrow}^\dagger & d_{k\downarrow}^\dagger \end{pmatrix} \begin{pmatrix} \epsilon & -J\Delta \\ -J\Delta^* & -\epsilon \end{pmatrix} \begin{pmatrix} d_{k\uparrow} \\ d_{k\downarrow} \end{pmatrix}.$$

5 Linear Response

step 0: $\rho_0 = |\Phi_0\rangle\langle\Phi_0|$ or $e^{-\beta H_0}$

step 1: $H = H_0 + V(t)$

$$V(t) = \int d^d x \; \phi_B(t,x)\, \mathcal{O}_B(x)$$

step 2: measure response:

$$\langle \mathcal{O}_A(t,x)\rangle = \mathrm{tr}\, \rho(t)\, \mathcal{O}_A(x)$$

$$\rho(t) = e^{-iHt} \rho_0 e^{+iHt} \equiv U_H(t)\rho_0 U_H^{-1}(t).$$

$$\overset{*}{=} \mathrm{tr}\left[\rho_0 U^{-1}(t)\, \mathcal{O}_A(t,x)\, U(t)\right]$$

$$\mathcal{O}_A(t,x) \equiv \underbrace{e^{-iH_0 t}}_{U_0(t)} \mathcal{O}_A(x) e^{iH_0 t}$$

$$\mathcal{U}(t) \equiv \mathcal{U}_0^{-1}(t)\, \mathcal{U}_H(t).$$

claim: $= T e^{-i\int^t V(t')dt'}$.

"interaction picture"

$$\begin{cases} i\partial_t U_H = U_H H = H U_H \\ i\partial_t U_0 = H_0 U_0 \end{cases}$$

$$\Rightarrow i\partial_t \mathcal{U}(t) = \mathcal{U}_0^{-1}(-H_0+H)\mathcal{U}$$

$$= \mathcal{U}_0^{-1} V \underset{U_0 U_0^{-1}}{\wedge} \mathcal{U} = V(t)\mathcal{U}$$

$$\underline{V(t) \equiv U_0^{-1} V U_0}.$$

$$\boxed{i\partial_t \mathcal{U}(t) = V(t)\mathcal{U}(t)}$$

sol'n:

$$U(t) = U(0) - i\int_0^t dt_1 V(t_1) \underbrace{U(t_1)}$$

$$= U(0) - i\int_0^t dt_1 V(t_1)\left(\mathcal{U}(0) - i\int^{t} dt_2 V(t_2)\mathcal{U}(t_2)\right)$$

$$\vdots$$

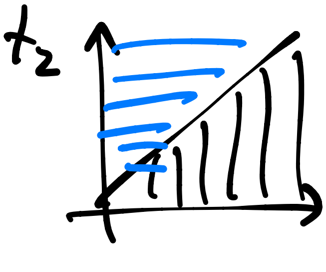
$$= \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \cdots \int_0^{t_{n-1}} V(t_1)V(t_2)\cdots V(t_n)U(0)$$

$$\mathcal{U}(t) = U(t,0)U(0)$$

$$U(t,t_i) = \sum_{n=0}^{\infty} (-i)^n \int_{t_i}^t dt_1 \int_{t_i}^{t_1} \cdots \int_{t_i}^{t_{n-1}} \underbrace{V(t_1)\cdots V(t_n)}_{= T(V(t_1)\cdots V(t_n))}$$

$$t_n \geq t_{n-1} \geq t_{n-2} \cdots \geq t_1$$

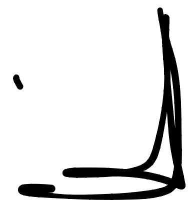
$$T(V(t)V(t')) \equiv \begin{cases} V(t)V(t') & \text{if } t \leq t' \\ V(t')V(t) & \text{else} \end{cases}.$$



$\times \frac{1}{2!}$

$$U(t,t_i) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_i}^{t} dt_1 \cdots \int_{t_i}^{t} dt_n \, T\big(V(t_1)\cdots V(t_n)\big)$$

$$\equiv T e^{-i\int_{t_i}^{t} dt' \, V(t')}$$



$$\langle \mathcal{O}_A(t,x)\rangle = \mathrm{tr}\, \rho_0 \, U^{-1}(t)\, \mathcal{O}_A(t,x)\, U(t)$$

V is small $= \mathrm{tr}\, \rho_0 \left(1 + i\int^t V + \ldots\right) \mathcal{O}_A(t,x) \left(1 - i\int^t V + \ldots\right)$

$$= \underbrace{\mathrm{tr}\, \rho_0 \, \mathcal{O}_A(t,x)}_{\text{answer if } V=0} - i \, \mathrm{tr}\, \rho_0 \int^t [\mathcal{O}_A(t,x), V(t')]\, dt'$$

$$\delta\langle \mathcal{O}_A(t,x)\rangle = -i\, \mathrm{tr}\, \rho_0 \int^t dt' \, [\mathcal{O}_A(t,x), V(t')]$$

$$= \langle \mathcal{O}_A\rangle - \langle \mathcal{O}_A\rangle\big|_{V=0} = -i\, \mathrm{tr}\, \rho_0 \int d^d x' \int^t dt' \, \phi_B(x',t') \times \langle [\mathcal{O}_A(t,x), \mathcal{O}_B(t',x')]\rangle$$

$$\equiv \int d^d x' dt \; G^R_{\mathcal{O}_A \mathcal{O}_B}(xt, x't')\, \phi_B(x't').$$

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(tx) = -i\theta(t)\langle[\mathcal{O}_A(t,x), \mathcal{O}_B(00)]\rangle$$

transl. inv.

$\theta(t) = \begin{cases} 1 & t>0 \\ 0 & t\le 0. \end{cases}$

(Response f'n
Retarded Green's f'n)

eqbm property.

$\langle\ldots\rangle = \mathrm{tr}(\rho_0 \ldots)$

$$\delta\langle\mathcal{O}_A(k\omega)\rangle = G^R_{\mathcal{O}_A\mathcal{O}_B}(k,\omega)\,\phi_B(k,\omega)$$

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(k,\omega) = -i\int d^dx\,dt\, e^{i\omega t - ikx}\,\theta(t)\langle[\mathcal{O}_A(t,x), \mathcal{O}_B(00)]\rangle$$

$\mathbb{1} = \sum_n |n\rangle\langle n|$

eg: $V = \int d^dx\; e^{iq\cdot x - i\omega t}\,\phi(q,\omega)\,\mathcal{O}_B(q,\omega)$
(one mode)

START IN EQBM: $-i\omega t \rightsquigarrow (-i\omega+\eta)t$

$\omega \rightsquigarrow \omega + i\eta.$ $\quad \eta = 0^+.$

eg 1 : perturbation : $E_x = i\omega A_x$

$$V = \int \underline{A_x} J^x$$

ie $\mathcal{O}_B \equiv J_x$

response : current $\mathcal{O}_A \equiv J_x$.

$$\delta\langle \mathcal{O}_A \rangle (k,\omega) = G^R_{\mathcal{O}_A \mathcal{O}_B}(k\,\omega)\ \phi_B(k,\omega)$$

$\underline{\langle J^x \rangle_{E=0} = 0}$. (Bloch's theorem)

$$\langle J^x(k,\omega) \rangle = G^R_{J^x J^x} A_x(k\,\omega)$$

$$= G^R_{JJ} \frac{E_x}{i\omega} . \quad \star$$

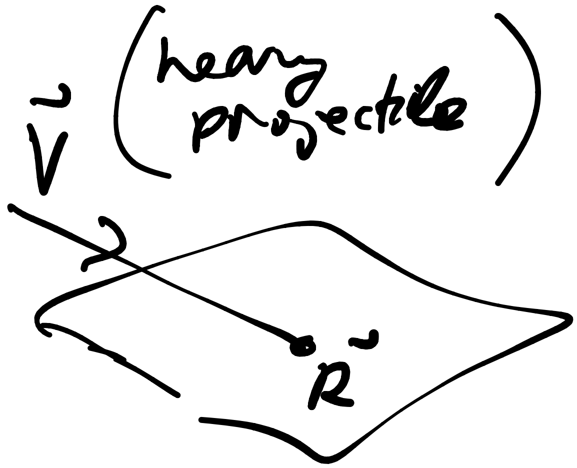
Ohm's Law: $\vec{J} = \sigma \vec{E}$ $\star$

$$\sigma^{xx}(k,\omega) = \frac{G^R_{J^x J^x}(k,\omega)}{i\omega}$$ Kubo formula.

<u>eg 2</u>: $U_A = U_B = \rho$ density.

$$V = \sum_q \int d\omega \; \rho_q^\dagger \, \underbrace{\varphi(q,\omega)}_{\text{scalar potential.}} e^{-i\omega t} + h.c.$$

<u>eg</u>:



$$\varphi(q,\omega) = 2\pi V_q \, e^{-i\vec{q}\cdot\vec{R}} \, \delta(\omega - q\cdot V)$$

<u>small V_q</u>:

$$\delta\langle \rho(q,\omega)\rangle = G^R_{\rho\rho}(q\omega)\,\varphi(q,\omega) \qquad (\mathrm{Im}\,\omega > 0)$$

- $\mathbb{1} = \sum_n |n\rangle\langle n|$ $\qquad H_0|n\rangle = (E_0 + \omega_n)|n\rangle$.

- $\theta(t) = -i \int d\epsilon \, \dfrac{e^{+i\epsilon t}}{\epsilon - i\eta}$ $\qquad \eta = 0^+$

<u>spectral rep of G^R</u>:

$$G^R_{\rho\rho}(q,\omega) = \sum_n |\langle n|\rho_q^\dagger|0\rangle|^2 \left(\frac{1}{\omega - \omega_n + i\eta} - \frac{1}{\omega + \omega_n + i\eta} \right)$$

defines G^R for $\omega \in LHP$

$$= \underline{\underline{\int_0^\infty d\omega' \, S_{\rho\rho}(q,\omega')}} \left(\frac{1}{\omega - \omega' + i\eta} - \frac{1}{\omega + \omega' + i\eta} \right)$$

$$S_{\rho\rho}(q,\omega) = \sum_n |\langle n| \mathcal{O}_q |0\rangle|^2 \, 2\pi\,\delta(\omega-\omega_n)$$

$$G^R_{\rho\rho}(q,\omega) \xrightarrow{\omega\to\infty} \frac{2}{\omega^2}\int_0^\infty \omega' d\omega' \, S(q,\omega')$$

$$\underset{=}{\text{f-sumrule}} \quad \frac{Nq^2}{m\omega^2} \qquad \text{ind of } H_0.$$

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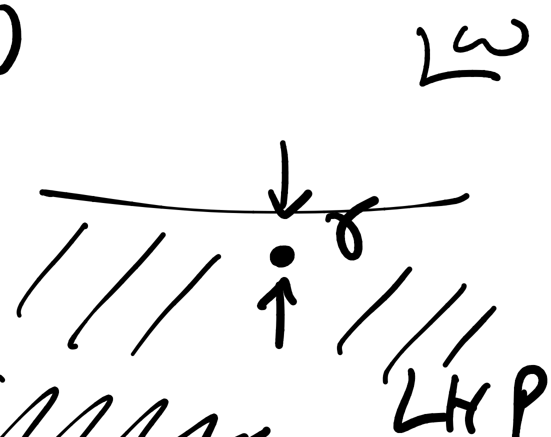
stable eqbm : $\delta\langle \mathcal{O}\rangle(t) \sim e^{-\gamma t}$ $\quad \underline{\gamma > 0.}$

$\wedge$ $e^{i\omega_R t}$

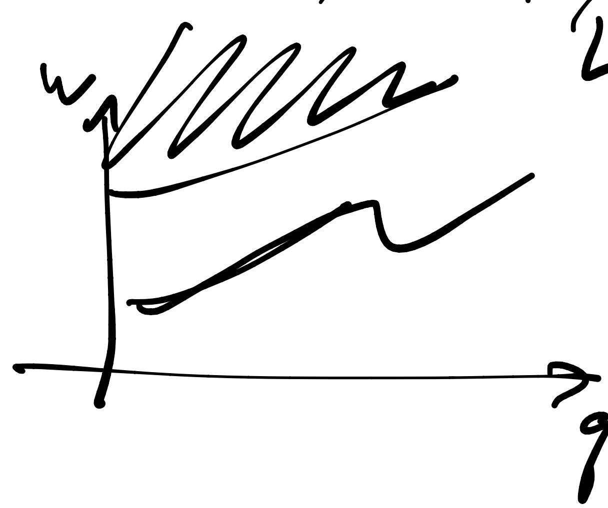
$$G^R(\omega) \sim \frac{1}{\omega - (\omega_R - i\gamma)}$$

$\underline{\text{pole}}$ at $\underline{\omega = \omega_R - i\gamma.}$



G^R has pole at

$\omega = \pm\omega_n - i\eta$



$G^R(\omega)$ is analytic in $\underline{\text{UHP.}}$

$\Longleftrightarrow \quad G^R(q,t) = 0 \quad t < 0.$ $\quad \underline{\text{Causality.}}$

$$\lim_{\eta\to 0} \frac{1}{x-a+i\eta} = P\frac{1}{x-a} - i\pi\delta(x-a)$$

$$\mathrm{Re}\, G^R_{\rho\rho}(q\omega) = \int_0^\infty d\omega' \, S(q,\omega')\, P\left(\frac{2\omega'}{\omega^2-(\omega')^2}\right)$$

$$\mathrm{Im}\, G^R_{\rho\rho}(q,\omega) = -\pi\left(S(q,\omega) - S(q,-\omega)\right)$$

$\Rightarrow \left\{ \begin{array}{l} \mathrm{Re}\, G^R \text{ is even under } \omega \to -\omega \\ \mathrm{Im}\, G^R \text{ is odd} \end{array} \right.$ $\quad \omega \to -\omega \Rightarrow \mathrm{Im}\, G^R(0) = 0$

$\forall q,\omega$

$\Rightarrow \quad -\omega\, \mathrm{Im}\, G^R(q,\omega) > 0.$

$|\langle n|\rho_q^\dagger|0\rangle|^2 \geq 0$

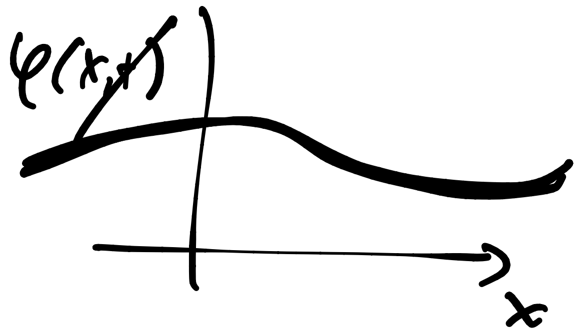
$\Rightarrow$ no anti-damping in eqbm.

skip: Kramers Kronig

" $\mathrm{Re}\, G^R(\omega) = \int d\omega' \dfrac{\mathrm{Im}\, G^R(\omega')}{\omega-\omega'}$ "

Compressibility sum rule ☆

$\omega = 0$ static $\quad$ q small.



$\vec{F} = -\vec{\nabla}\varphi$. (felt by each particle)

$$\mathcal{F} = -iq\,\varphi(q,0)\,e^{iq\cdot r} + h.c.$$

Macroscopic (hydro) picture $\longrightarrow \delta\rho$

$$\rightarrow \delta P(r) = \frac{\delta\rho}{\kappa N}$$

def of κ compressibility

in eqbm: (no net acceleration)

$$0 = -\vec{\nabla}\delta P + N\vec{F}$$

$$\Rightarrow \langle \delta\rho(r)\rangle = -N^2\kappa\,\varphi(q,0)\,e^{iqr} + h.c.$$

$$= G^R_{\rho\rho}\,\varphi(r)$$

$$\boxed{G^R_{\rho\rho}(q,0) \xrightarrow{q\to 0} -N^2\kappa = -\frac{N}{m v_s^2}.}$$ ☆

why : $v_s^2 = m\kappa N$?

① $0 = \dot{\rho} + \vec{\nabla}\cdot\vec{J} = \dot{\rho} + \vec{\nabla}\cdot(\rho\vec{u})$

u small

② Newton's Law: $-\vec{\nabla}p = m\rho\left(\partial_t\vec{u} + \cancel{(\vec{u}\cdot\vec{\nabla})\vec{u}}\right)$

$\vec{u}$ small, $\rho = \rho_0(1+s(t,x))$ ← small.

1d

① $\Rightarrow$ $\underline{\partial_x u = -\partial_t s}$.

② $\Rightarrow$ $\partial_t u = -\dfrac{\partial_x p}{m\rho} = -\dfrac{1}{m\kappa N}\partial_x s$

$$\boxed{\partial_x p = \frac{1}{\kappa N}\partial_x \rho.}$$ → def

$\partial_x(\text{BHS}) \Rightarrow \ddot{s} = \underbrace{\dfrac{1}{m\kappa N}}_{\equiv v_s^2} s''$.

$$\frac{N}{2mv_s^2} \overset{*}{=} -\frac{1}{2} \lim_{q\to 0} G^R_{\rho\rho}(q,0)$$

$$= -\frac{1}{2} \lim_{q\to 0} \left(\mathrm{Re}\, G^R_{\rho\rho}(q,0) + \underbrace{i\, \mathrm{Im}\, G^R_{\rho\rho}(q,0)}_{0} \right)$$

$$= -\frac{1}{2} \lim_{q\to 0} \int_0^\infty d\omega' S(q,\omega')\, P\Big(\underbrace{\frac{2\omega'}{0-\omega'^2}}_{-\frac{2}{\omega'}} \Big)$$

$$= \lim_{q\to 0} \int_0^\infty d\omega \frac{S(q,\omega)}{\omega} .$$

<u>κ sum rule</u>.

Classification of Atoms

- Hydrogen
- Helium
- Everything else.