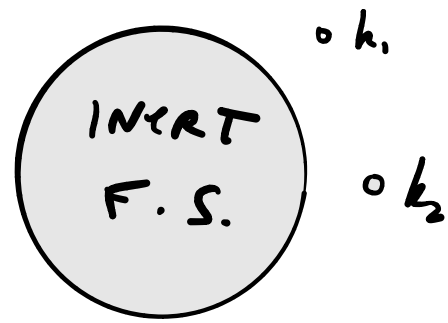
Cooper Problem:



$$|\Psi\rangle = \sum_{k_1 k_2} a_{k_1,k_2}\, \psi^\dagger_{k_1}\psi^\dagger_{k_2}|FS\rangle$$

$$H_c|\Psi\rangle = E|\Psi\rangle \Longrightarrow$$

$$E\, a_{k_1k_2} = (\epsilon_{k_1} + \epsilon_{k_2})\, a_{k_1,k_2} + \sum_{k_1'k_2'} \langle k_1 k_2|\hat{V}|k_1' k_2'\rangle\, a_{k_1'k_2'}$$

$V_0 > 0$ (attractive)

$$\langle k_1 k_2|\hat{V}|k_1' k_2'\rangle = \begin{cases} -V_0/V & \text{if } k_F < k_1, k_2, k_1', k_2' < k_a \\ 0 & \text{else} \end{cases}$$

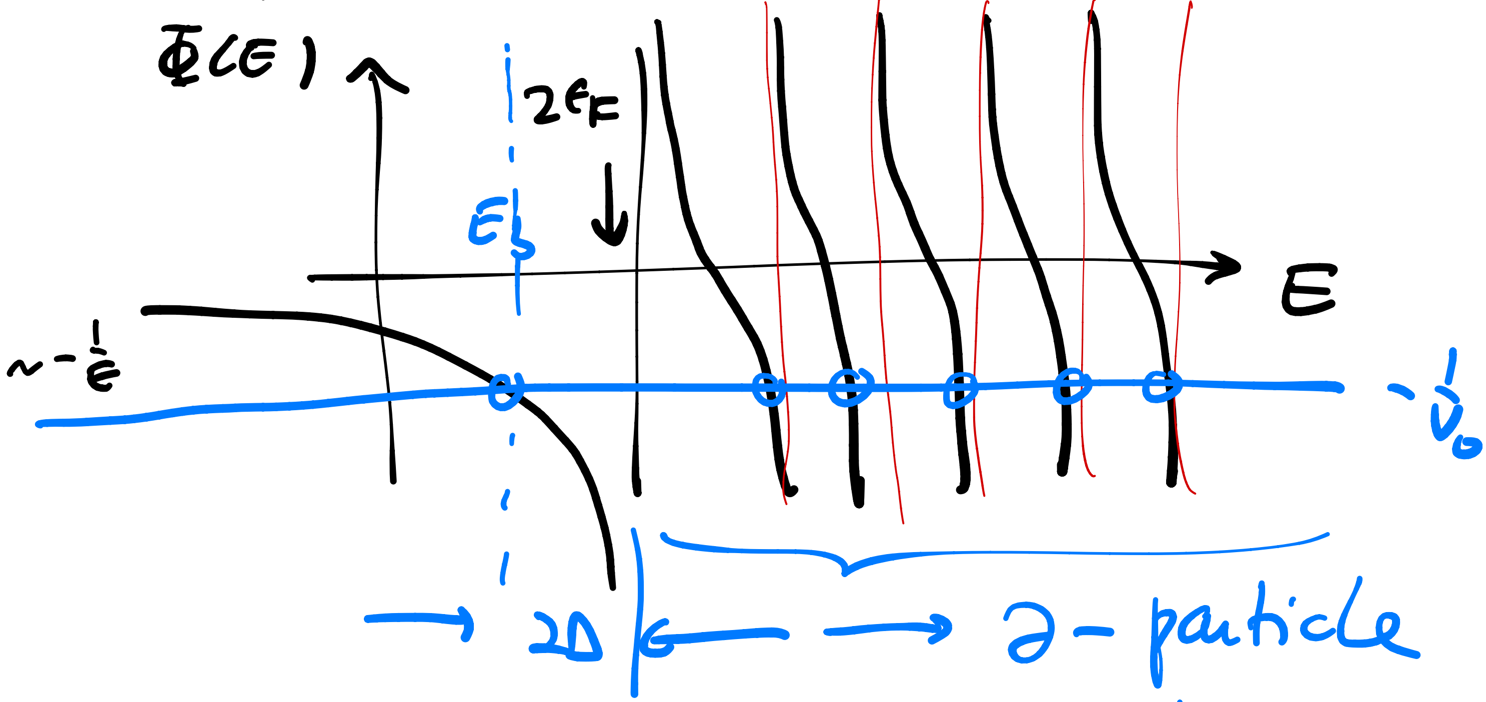
$$\Longleftrightarrow \quad -\frac{1}{V_0} = \frac{1}{V}{\sum_k}' \frac{1}{E - \epsilon_{k_1} - \epsilon_{k_2}} \equiv \Phi(E)$$

$${\sum_k}' \equiv \sum_{k_F < |\frac{K}{2} \pm k| < k_a}$$

$K = k_1 + k_2$ CoM

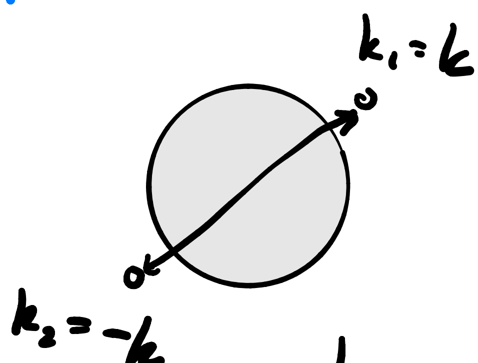
$k = \dfrac{k_1 - k_2}{2}$ relative

$$\vec{k}_{1,2} \in \left\{ \frac{2\pi}{L}\vec{n} \right\} \cap \left\{ k_F < \left| \frac{K}{2} \pm k \right| < k_a \right\}$$



$$E_b = 2\epsilon_F - 2\Delta$$

$\underline{K=0}$.



$$\rightarrow \sum_k{}' = \sum_{k_F < |k| < k_a}$$

$$\Phi_{K=0}(E) \underset{V\to\infty}{=} \int_{k_F}^{k_a} \frac{d^d k}{E - 2\epsilon_k} = \int_{\epsilon_F}^{\epsilon_a} \frac{d\epsilon\, g(\epsilon)}{E - 2\epsilon}$$

$$\boxed{g(\epsilon) \equiv \int d^d k\, \delta(\epsilon_k - \epsilon)}$$

$$\Phi_{K=0}(E) \simeq g(\epsilon_F)\int_{\epsilon_F}^{\epsilon_a} \frac{d\epsilon}{E-2\epsilon}$$

$$\overset{E<2\epsilon_F}{=} -\frac{g(\epsilon_F)}{2}\log\left|\frac{2\epsilon_a - E}{2\epsilon_F - E}\right| \overset{!}{=} -\frac{1}{V_0}$$

$$E_b \equiv 2\epsilon_F - 2\Delta$$

$$\Delta = \frac{\frac{\epsilon_a - \epsilon_F}{2}}{e^{\frac{2}{V_0 g(\epsilon_F)}} - 1} \underset{V_0 g < 1}{\simeq} \epsilon_D e^{-2/V_0 g(\epsilon_F)}$$

① Non-perturbative in V_0.

② $\Delta > 0$ for any V_0.

$$a_k(K) \propto \frac{1}{E - \epsilon_{k_1} - \epsilon_{k_2}}$$

$$\psi(r_1, r_2) \equiv \sum_{k_1 k_2} e^{ik_1 r_1 + k_2 r_2} a_{k_1 k_2}$$

$$\sim \underbrace{e^{iK\cdot\frac{r_1+r_2}{2}}} \frac{1}{V}\sum_k{}' \frac{e^{ik\cdot(r_1 - r_2)}}{E - \epsilon_{k_1} - \epsilon_{k_2}}$$

$$\xrightarrow{V\to\infty} \int' d^d k \frac{e^{ik\cdot r_{12}}}{E - \epsilon_{k_1} - \epsilon_{k_2}} \overset{K\to 0}{\simeq} \frac{\sin k_F |r_{12}|}{|r_{12}|} \sin\frac{|r_{12}|}{\xi}$$

<u>s-wave</u>.

$$= \psi(|r_{12}|)$$

$\xi = \dfrac{2k_F}{m\Delta}$. 'size' of the Cooper pair.

Warning: FS is not really inert.

→ not a bound state ($E < 2\epsilon_F$)

but an instability $E = 2\epsilon_F + i\gamma$

$\gamma > 0$.

Poetry: every electron experiences this.

Q: what is the role of FS?

Redo w/ $k_F = 0$.

requires a threshold attraction $V_0 > V_0^c$.

to have $\Delta > 0$.

4.5 Instabilities of a FS to attractive interactions

$$H = -t\sum_{\langle xy\rangle} c^\dagger_{x\sigma}c_{y\sigma} + h.c. + U\sum_x (n_x - 1)^2$$

$$\equiv H_t + H_U$$

$$n_x = \sum_\sigma c^\dagger_{x\sigma}c_{x\sigma}$$

$$H_t = \sum_k \epsilon_k c^\dagger_{k\sigma}c_{k\sigma}$$

$$\epsilon_k = -2t(\cos k_x a + \cos k_y a) - \mu$$

($\mu = 0$ is half-filling)

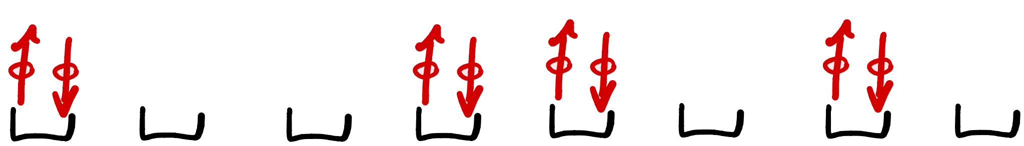
Recall $U > 0$: AFM.

$\boxed{U<0}$ First, $U \to -\infty$.

$$U(n_x - 1)^2 = -|U|(n_x - 1)^2$$

minimized when $n_x = 0$ or 2 $\forall x$.

2^V states



BEC of $b_x^\dagger = c_{\uparrow x}^\dagger c_{\downarrow x}^\dagger$. $(\forall \mu)$.

$\underline{|U| < \infty}$. $\boxed{\text{MFT}}$

$$U(n_x - 1)^2 = \underbrace{U n_x^2} - \underbrace{2U n_x}_{-\delta\mu\, n_x} + \underbrace{U}_{\text{cst}}$$

$$U n_x^2 = U (c_{x\uparrow}^\dagger c_{x\uparrow} + c_{x\downarrow}^\dagger c_{x\downarrow})^2$$

$c^2 = 0$

$$= 2U\, c_{x\uparrow}^\dagger c_{x\uparrow} c_{x\downarrow}^\dagger c_{x\downarrow}$$

$$= -2U\, c_{x\uparrow}^\dagger c_{x\downarrow}^\dagger c_{x\uparrow} c_{\downarrow}$$

$$H_{MF} = \sum_k (\epsilon_k - \mu) c_{k\sigma}^\dagger c_{k\sigma}$$

$$- 2U \sum_x \Big(\underbrace{\langle c_{x\uparrow}^\dagger c_{x\downarrow}^\dagger \rangle}_{\equiv \Delta} c_{x\uparrow} c_{x\downarrow} + c_{x\uparrow}^\dagger c_{x\downarrow}^\dagger \overbrace{\langle c_{x\uparrow} c_{\downarrow} \rangle}^{-\Delta^*}$$

$$- \underbrace{\langle c_{\uparrow}^\dagger c_{\downarrow}^\dagger \rangle}_{\Delta} \underbrace{\langle c_{\uparrow} c_{\downarrow} \rangle}_{-\Delta^*} \Big)$$

$$H_{MF} = \sum_k c^\dagger_{k\sigma} c_{k\sigma}(\epsilon_k - \mu) - 2U \sum_x \left(\Delta c_{x\uparrow} c_{x\downarrow} + \Delta^* c^\dagger_{x\downarrow} c^\dagger_{x\uparrow} \right) - 2UV|\Delta|^2$$

$$c_{x\sigma} \equiv \frac{1}{\sqrt{V}} \sum_k e^{ikx} c_{k\sigma}$$

$$= \sum_k \Big[c^\dagger_{k\sigma} c_{k\sigma} (\epsilon_k - \mu) + 2U \left(\Delta^* c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + \Delta c_{k\downarrow} c_{-k\uparrow} \right) \Big] - \frac{2U|\Delta|^2}{V}$$

$U(1)$: $c \to e^{i\theta} c$ symmetry of H. Not a symmetry of H_{MF}.

only $c \to -c$, $\mathbb{Z}_2 \subset U(1)$.

<u>Solve H_{MF}.</u> <u>Bogoliubov.</u>

$$\begin{cases} d_{k\downarrow} \equiv c^\dagger_{-k\downarrow} \\ d_{k\uparrow} \equiv c_{+k\uparrow} \end{cases} \Rightarrow \begin{cases} d^\dagger_{k\downarrow} = c_{-k\downarrow} \\ d^\dagger_{k\uparrow} = c^\dagger_{k\uparrow} \end{cases}$$

<u>CANONICAL</u>.

$\{d_k, d^\dagger_{k'}\} = \delta^d(k-k')$, $\{d_k, d_{k'}\} = 0$.

$$H_{MF} = \sum_k (\epsilon_k - \mu)\, d^\dagger_{k\uparrow} d_{k\uparrow} - \sum_k (\epsilon_k - \mu)\, d^\dagger_{k\downarrow} d_{k\downarrow} + 2U \sum_k \left(d^\dagger_{k\uparrow} d_{k\downarrow} \Delta^* + h.c. \right) - 2UV|\Delta|^2$$

$$c^\dagger_{k\downarrow} c_{k\downarrow} = d_{-k\downarrow} d^\dagger_{-k\downarrow} = -d^\dagger_{-k\downarrow} d_{-k\downarrow}.$$

assume $\boxed{\epsilon(k) = \epsilon(-k)}$

$$= \begin{pmatrix} d^\dagger_{k\uparrow} & d_{k\downarrow} \end{pmatrix} \underbrace{\begin{pmatrix} \epsilon_k - \mu & 2\Delta^* U \\ 2\Delta U & -(\epsilon_k - \mu) \end{pmatrix}}_{= Z(\epsilon_k - \mu) + X(2U\,\mathrm{Re}\Delta) + Y(2U\,\mathrm{Im}\Delta) = \vec{h}\cdot\vec{\sigma} + \cancel{h_0 \mathbb{1}}} \begin{pmatrix} d_{k\uparrow} \\ d_{k\downarrow} \end{pmatrix} - 2UV|\Delta|^2$$

evals are $\pm\sqrt{\vec{h}^2} = \pm\sqrt{(\epsilon_k - \mu)^2 + 4U^2|\Delta|^2}$

$$\underbrace{a^\dagger_\varphi} = \sum_k \varphi(k)\, \underbrace{a^\dagger_k}$$

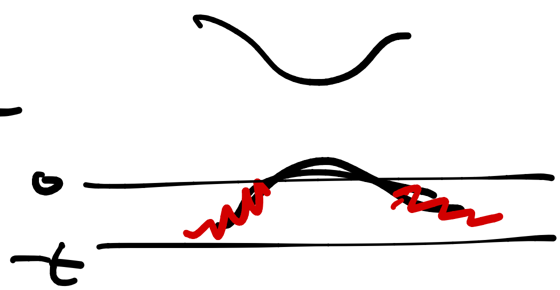
$$|\text{gs of } H_{MF}\rangle = \prod_k \left(u d^+_{\uparrow k} + v d^+_{\downarrow k}\right) |\tilde{0}\rangle \qquad d_k |\tilde{0}\rangle = 0$$

$$E_0^{MF} = +|U||\Delta|^2 V - \sum_k^{k_F} \sqrt{(\epsilon_k - \mu)^2 + 4U^2|\Delta|^2}$$

Minimize over Δ.

$$= \langle MF | \hat{H} | MF \rangle.$$

$$1 = |U| \int_?^0 \frac{d\epsilon\, g(\mu+\epsilon)}{\sqrt{\epsilon^2 + 4U^2|\Delta|^2}}$$



$$\simeq |U| g(\epsilon_F) \int_{-t}^0 \frac{d\epsilon}{\sqrt{\epsilon^2 + 4U^2|\Delta|^2}}$$

$$\cong |U| g(\epsilon_F) \log \frac{t}{2|\Delta| U} \qquad (\leftarrow t \gg 2U\Delta)$$

$$\rightarrow |\Delta| \cong \frac{t}{2|U|} e^{-\frac{1}{4 g(\epsilon_F)|U|}} \qquad (U<0)$$

$$\Delta = \langle c^+_{x\uparrow} c^+_{x\downarrow} \rangle$$

condensation of fermion pairs.

(superconductivity.)

$U \rightsquigarrow \lambda$ Coulomb interactions + phonons
screened, repulsive

~~(-1) shoulders of giants~~

5 Linear Response (0) ...

what's an experiment?

$\boxed{\text{(1) poke} \quad \text{(2) see what happens.}}$

(1): $H = H_0 + V(t)$ $\quad V(t) = \begin{cases} 0 & \text{before expt} \\ \ldots & \text{during \& after.} \end{cases}$

(2) measure $\langle \mathcal{O} \rangle$.

specialize: A) system starts in eqbm.

$|\Phi_0\rangle\langle\Phi_0|$ or $\rho_0 = \frac{e^{-H_0/T}}{Z}$.

gs of H_0.

B) small V $\longrightarrow$ $\langle \mathcal{O} \rangle$ is linear in ϵ.

$V \propto \epsilon$ $\Longrightarrow$ determined by eqbm correlation f'ns.