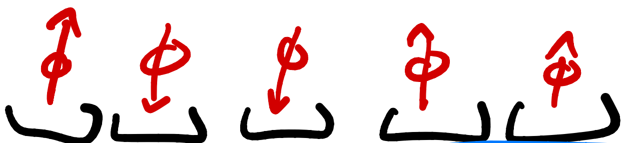
Hubbard Model:

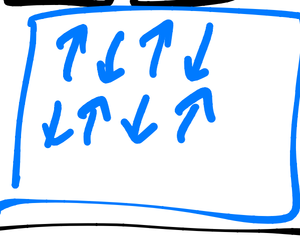
$$H = H_t + H_U$$

$$= -t \sum_{\langle xy\rangle\sigma} c^\dagger_{x\sigma} c_{y\sigma} + h.c. + U \sum_x \underbrace{(n_x - 1)^2}$$

$\boxed{U/t \gg 1}$ Mott insulator

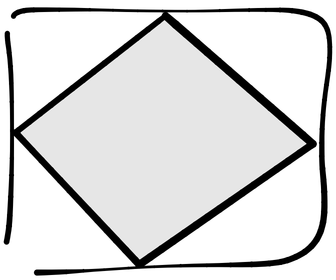


$$H_{eff} = \omega t + \frac{4t^2}{U} \sum_{\langle xy\rangle} \vec{S}_x \cdot \vec{S}_y \rightarrow$$



$\boxed{\text{At half-filling}}$

$\boxed{U/t \ll 1}$



At $t=0$ Metal.

$$(n-1)^2 = 1 - (c^\dagger_\sigma \sigma^z_{\sigma\sigma'} c_{\sigma'})^2 = 1 - (n_\uparrow - n_\downarrow)^2$$

$n_\uparrow, n_\downarrow$	0 0	0 1	1 0	1 1
LHS	1	0	0	1
RHS	1	0	0	1

large U favors $1 = (c^\dagger \sigma^z c)^2 = (S^z)^2$

$$S^z = \pm 1.$$

Mean field theory :

$$(c^\dagger\sigma^z c)^2 \rightsquigarrow 2\langle c^\dagger\sigma^z c\rangle c^\dagger\sigma^z c - \langle c^\dagger\sigma^z c\rangle$$

$$= 2\langle S^z\rangle c^\dagger\sigma^z c - \langle S^z\rangle^2$$

Recall: for TFIM

m.F. ansatz : $\bigotimes_x |\phi_x\rangle$

analog for fermions : $\prod_\alpha a^\dagger_\alpha |0\rangle$

Alternative PoV [Hubbard-Stratonovich]

$$U(S^z)^2 = -S^z\sigma + \frac{\sigma^2}{2U}$$

$$0 \overset{!}{=} \frac{\delta S}{\delta\sigma} = \frac{\sigma}{U} - S^z$$

Guess : $\langle S^z_{(x)}\rangle = \langle c^\dagger\sigma^z c\rangle = M(-1)^{x+y}$

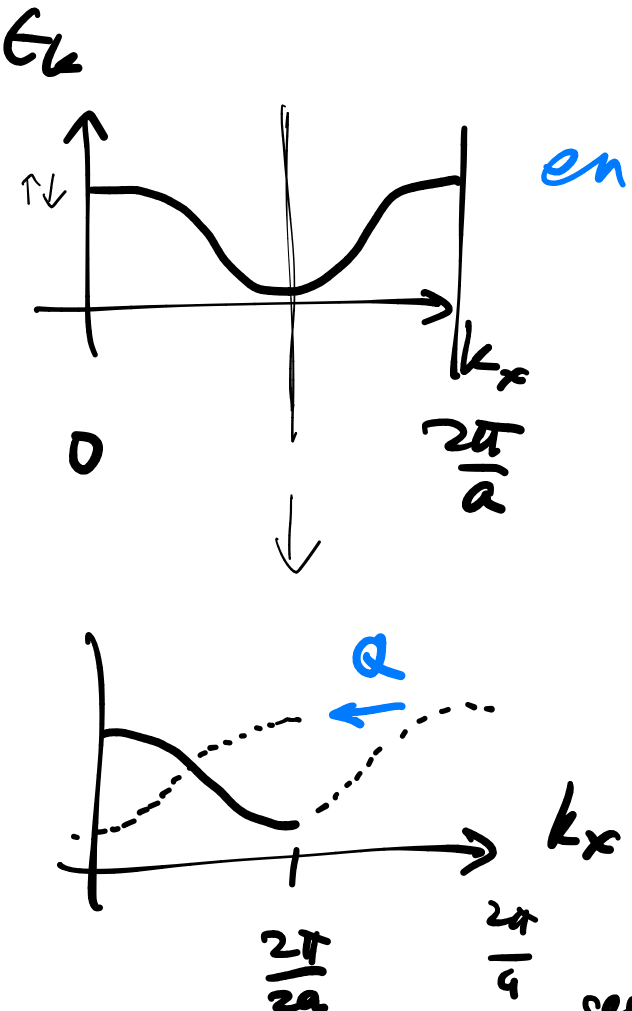
↑ ↓ ↑ ↓
↓ ↑ ↓ ↑

Find E_g for $\qquad \epsilon_k = -2t(\cos k_x + \cos k_y)$

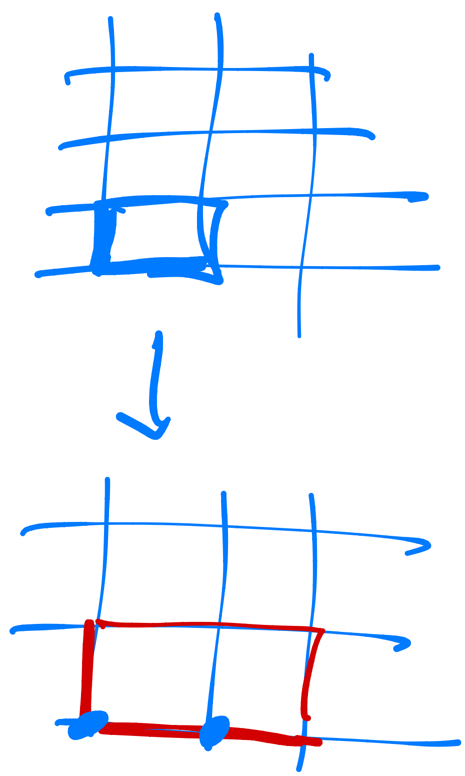
$$H_{MF} = \sum_{k\sigma} c^\dagger_{k\sigma}\, \epsilon_k\, c_{k\sigma} - 2UM\sum_x (-1)^{x+y} c^\dagger_x \sigma^z c_x + UM^2 V.$$

$$(-1)^{x+y} = e^{i\pi(x+y)}$$

Breaks transl. inv.

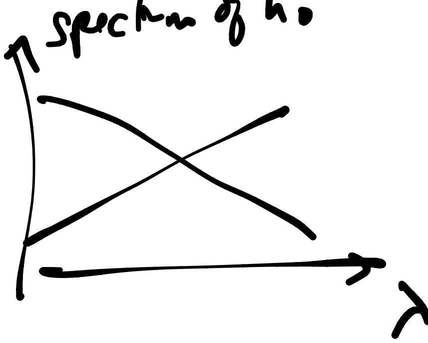


enlarges unit cell

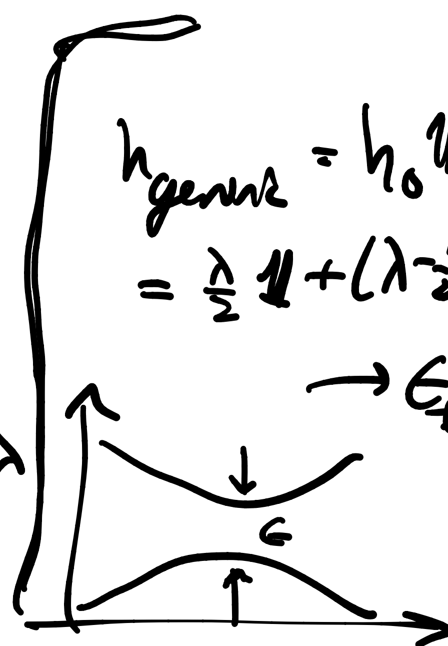


Level crossings:

$$h_0 = \begin{pmatrix} 1-\lambda & \\ & \lambda \end{pmatrix} = \frac{1}{2}\mathbb{1} + (\lambda - \tfrac{1}{2})Z$$



$$h_{generic} = h_0 \mathbb{1} + \vec{h}\cdot\vec{\sigma} = \frac{\lambda}{2}\mathbb{1} + (\lambda - \tfrac{1}{2})Z + \epsilon X$$

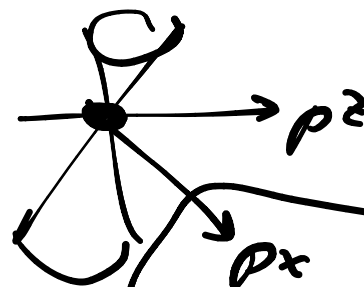


$\rightarrow E_\pm = h_0 \pm \sqrt{|\vec{h}|^2}$

level crossings don't happen at codimension one.

Dirac point in 3d: $h(\vec{p}) = p^x\sigma^x + p^y\sigma^y + p^z\sigma^z = \vec{p}\cdot\vec{\sigma}$

$\epsilon_\pm = \pm\sqrt{\vec{p}^2}$ at $p^y=0$:



$$\sum_x (-1)^{x+y} c_x^\dagger \sigma^z c_x$$

$$= \sum_k{}' c_k^\dagger \sigma^z c_{k+Q} + h.c. \qquad \vec{Q} = (\pi,\pi)$$

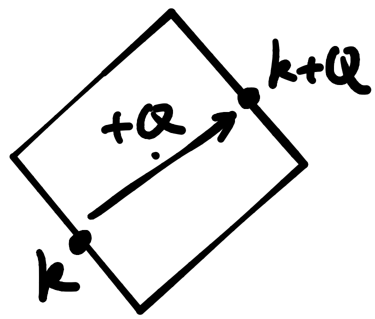
(half of original BZ)

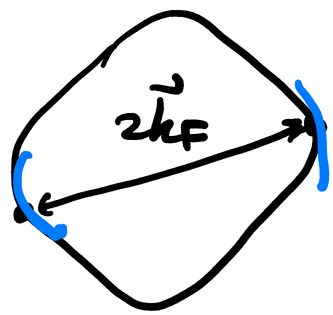
$$e^{i(x+y)\pi} \equiv e^{i\vec{Q}\cdot\vec{x}}.$$

$$H_{MF} = \sum_k{}' \left(c_k^\dagger, c_{k+Q}^\dagger\right) \underbrace{\begin{pmatrix} \epsilon_k & -2UM\sigma^z \\ -2UM\sigma^z & \epsilon_{k+Q} \end{pmatrix}}_{h(k)} \begin{pmatrix} c_k \\ c_{k+Q} \end{pmatrix} + h.c.$$

$$h(k) = \begin{pmatrix} \epsilon_k & -2UM\sigma^z \\ -2UM\sigma^z & \underbrace{\epsilon_{k+Q}} \end{pmatrix} \equiv h_0 \mathbb{1} + \vec{h}\cdot\vec{\sigma}$$

At half filling: $\boxed{\epsilon_{k+Q} = -\epsilon_k}$ "NESTING".





Breaks away from $\frac{1}{2}$ - filling in $d>1$.

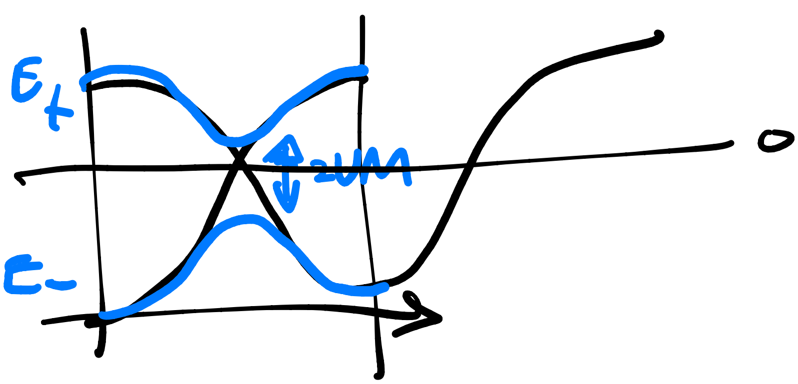
(In $d=1$ this happens at any μ.)

$$\rightarrow h(k) = \begin{pmatrix} \epsilon_k & -2UM\sigma^z \\ -2UM\sigma^z & -\epsilon_k \end{pmatrix} = \epsilon_k Z + (2UM\sigma^z) X$$

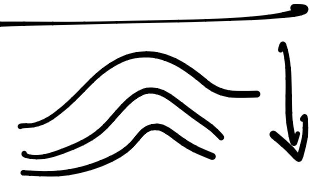
i.e. $h_0 = 0$.

eval:

$$\Rightarrow \pm E(k) = \pm\sqrt{\epsilon_k^2 + \underline{4UM^2}}$$



larger M lowers E.



$$H_{MF} = \sum_k{}' \left(E_k\, d^\dagger_{+\sigma}(k)\, d_{+\sigma}(k) - E_k\, d^\dagger_{-\sigma}(k)\, d_{-\sigma}(k) \right) + VUM^2$$

$$\begin{cases} d_{+\sigma}(k) = -v\sigma^z c(k) + u\, c(k+Q) \\ d_{-\sigma}(k) = u\, c(k) \\ \quad v\sigma^z c(k+Q) \end{cases}$$

|gs of H_{MF} ⟩

$$= \prod_k d^\dagger_{-\sigma}(k)\,|\tilde{0}\rangle$$

whole lower band

like a band insulator!

$d_{+\sigma}(k)|\tilde{0}\rangle = 0$.

not $c|0\rangle = 0$

$c|\tilde{0}\rangle \neq 0$.

$$E_0(M) = -2\sum_k{}' E(k) + M^2 UV$$

$$0 = \partial_M E_0 \Rightarrow 0 = UV - \sum_k{}' \frac{4U^2}{E_k}$$

$$\boxed{\frac{4U}{V} \sum_k{}' \frac{1}{E_k} = 1}$$

requires

$U > 0$

(repulsive)

$\underline{V\to\infty}$: $$1 = 4U\int_{BZ'} \frac{d^dk}{\sqrt{\epsilon_k^2+4U^2M^2}}$$

$$= 4U\int \frac{d\epsilon\, g(\epsilon)}{\sqrt{4U^2M^2+\epsilon^2}}$$

$$\boxed{g(\epsilon)\equiv\int_{BZ'} d^dk\,\delta(\epsilon-\epsilon_k)}$$

Near $\epsilon=\epsilon_F$
$g(\epsilon)\simeq g(\epsilon_F)$

$$\simeq 4U\, g(\epsilon_F)\int_{-t}^{t}\frac{d\epsilon}{\sqrt{\epsilon^2+4U^2M^2}}$$

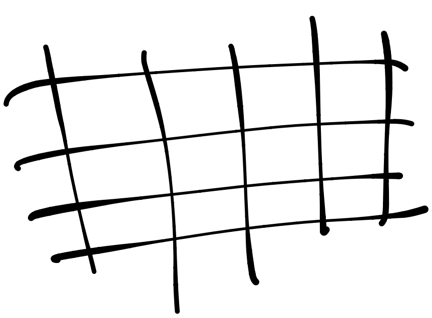
$$\cong 8U g(\epsilon_F)\log\frac{t}{2|UM|}$$

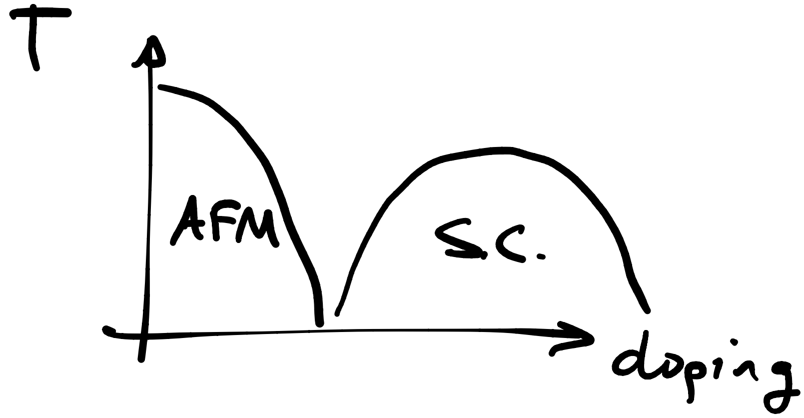
$$\Rightarrow |M| \simeq \frac{t}{U}\, e^{-1/8g(\epsilon_F)U}$$

U is small $\Rightarrow$ $e^{-1/U}$ is small

But <u>NON-perturbative</u>.

<u>Cuprates</u>:





Attractive interactions ($U<0$)?

claim BEC of $b^\dagger(x) \sim c^\dagger_\uparrow(x)\, c^\dagger_\downarrow(x)$.

4.2 Why attractive interactions between electrons

① Coulomb interactions are screened.

② phonons mediate attraction.

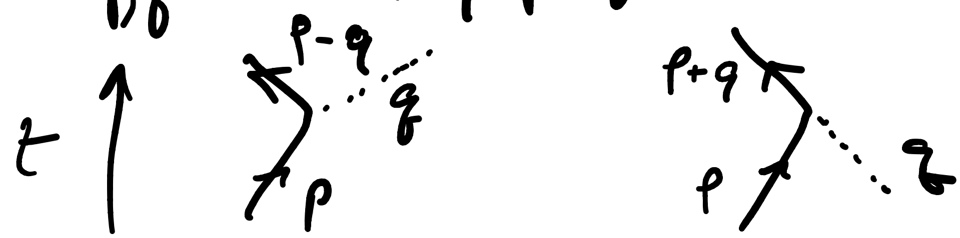
$$H_{e\text{-}ph} = -t\sum_{\langle ij\rangle} c^\dagger_i c_j + h.c. + H(u) \qquad u_{ij} \equiv q_i - q_j$$

$$+\, g \sum_{\langle ij\rangle} c^\dagger_i c_j u_{ij}$$

small.

$$|\text{gs at } g=0\rangle = |\text{gs of FS}\rangle \otimes |\overbrace{\text{gs of lattice}}^{a|0\rangle = 0.}\rangle$$

$$\Delta H = g\sum_{\langle ij\rangle} c_i^\dagger c_j u_{ij} = \sum_{p,q} g(q)\, c^\dagger_{p-q} c_p a^\dagger_q + h.c.$$

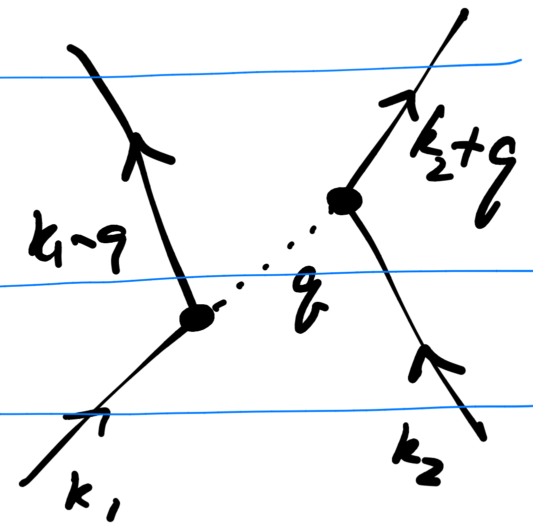


$$\mathcal{M}_{fi}\cdot(2\pi)^d\delta^d(p_f-p_i) = \cancel{\langle f|\Delta H|i\rangle} - \sum_n \frac{\langle f|\Delta H|n\rangle\langle n|\Delta H|i\rangle}{E_n - E_i - i\epsilon} + \mathcal{O}(\Delta H^3)$$

$$|f\rangle = c^\dagger_{k_1-q}\, c^\dagger_{k_2+q}\,|FS\rangle\otimes|0\rangle$$

$$|n\rangle = c^\dagger_{k_1-q}\, c^\dagger_{k_2}\, a^\dagger_q\,|FS\rangle\otimes|0\rangle$$

$$|i\rangle = c^\dagger_{k_1}\, c^\dagger_{k_2}\,|FS\rangle\otimes|0\rangle$$



$$E_n - E_i = \epsilon_{k_1-q} + \epsilon_{k_2} + \omega_q - (\epsilon_{k_1} + \epsilon_{k_2})$$

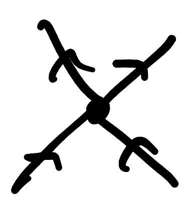
end result:

$$|k_1 k_2\rangle \longrightarrow |k_1-q,\, k_2+q\rangle$$

w/ amplitude $$V_{q k_1 k_2} = -\frac{g^2(q)}{\epsilon_{k_1-q} + \omega_q - \epsilon_{k_1}} < 0.$$

same result:

$$\Delta H_{\text{eff}} = \sum_{k_1 k_2 q} V_{q k_1 k_2}\, c^\dagger_{k_1-q}\, c^\dagger_{k_2+q}\, c_{k_1}\, c_{k_2}$$



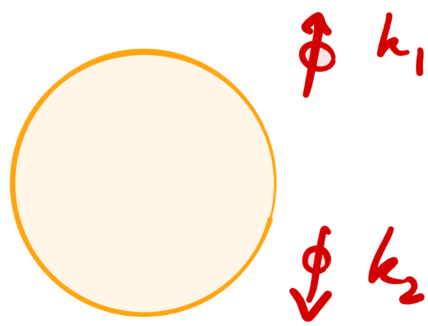
OR: $$\int [Dq]\, e^{iS[c] + i\int (\dot{q}^2 - (\vec{\nabla} q)^2 - g q c^\dagger c)}$$

$$= e^{iS[c] + \int\int c^\dagger c\, D\, c^\dagger c}$$

gaussian!

w/ $D = \langle q q \rangle$

Cooper problem



$$|\psi\rangle = \sum_{k_1 k_2} a_{k_1,k_2}\, \psi^\dagger_{k_1\uparrow} \psi^\dagger_{k_2\downarrow} |FS\rangle \equiv \sum_{k_1 k_2} a_{k_1,k_2} |k_1 k_2\rangle$$

$$H_c = \sum \epsilon_k \psi^\dagger_{k\sigma} \psi_{k\sigma} + \underline{\underline{V_c}}$$

$H_c|\psi\rangle = E|\psi\rangle$

$$E\, a_{k_1 k_2} = (\epsilon_{k_1} + \epsilon_{k_2})\, a_{k_1 k_2} + \sum_{k_1' k_2'} \langle k_1 k_2 | V | k_1' k_2' \rangle\, a_{k_1' k_2'}$$

assumptions: transl. sym. $\quad K \equiv k_1 + k_2,\ K' = k_1' + k_2'$

$$\langle k_1 k_2 | V | k_1' k_2' \rangle = \delta_{K,K'} V_{kk'}(K) \qquad k = k_1 - k_2/2 \quad k' = k_1' - k_2'/2.$$

$$V_{kk'}(K) = \begin{cases} -V_0/V & k_F < k_1, k_2, k_1', k_2' < k_a \\ 0 & \text{else.} \end{cases}$$

$\underline{V_0 > 0}$.

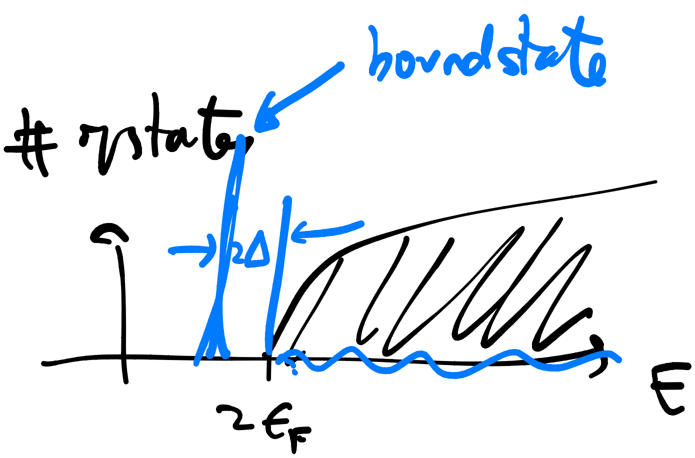
(attractive)

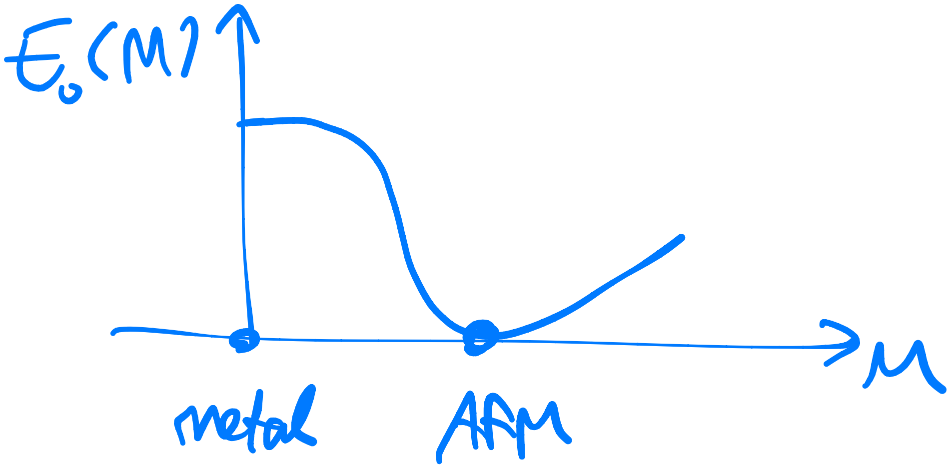
$$\rightarrow \quad (E - \epsilon_{k_1} - \epsilon_{k_2})\, a_k(K) = \sum_{k'}{}' \, a_{k'}(K)$$

$$k_F < \left| \frac{K}{2} \pm k' \right| < k_a .$$

$$\rightarrow \quad \sum_k{}' \, a_k(K) = -\frac{V_0}{V} \left(\sum_k{}' \frac{1}{E - \epsilon_{k_1} - \epsilon_{k_2}} \right) \sum_{k'} a_{k'}(K)$$

$$\Longleftrightarrow \quad \boxed{1 = -\frac{V_0}{V} \sum_k{}' \underbrace{\frac{1}{E - (\epsilon_{k_1} + \epsilon_{k_2})}}_{\text{has poles at free 2-particle energies.}}}$$





Could be: