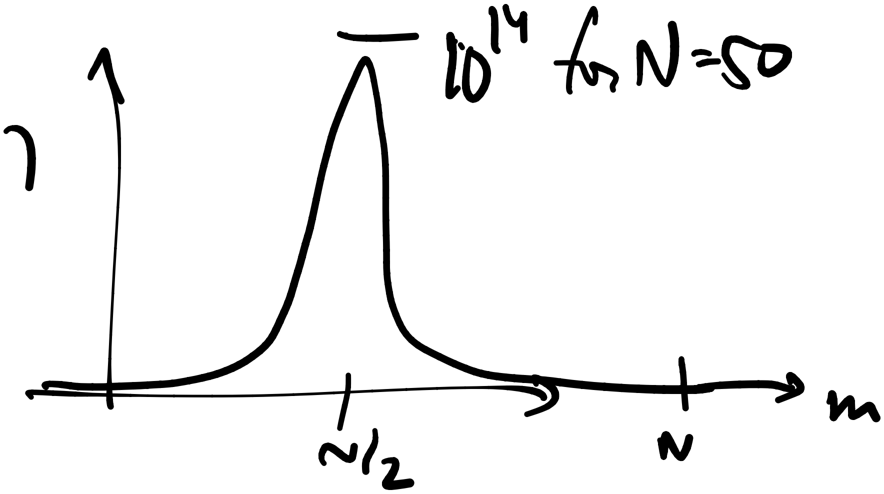
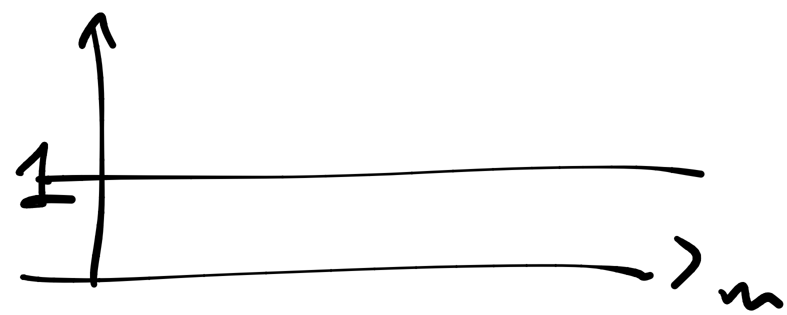
Why are phonons the only low-E excitations of the Bose liquid?

① $W_{\text{distinguishable}}(m)$

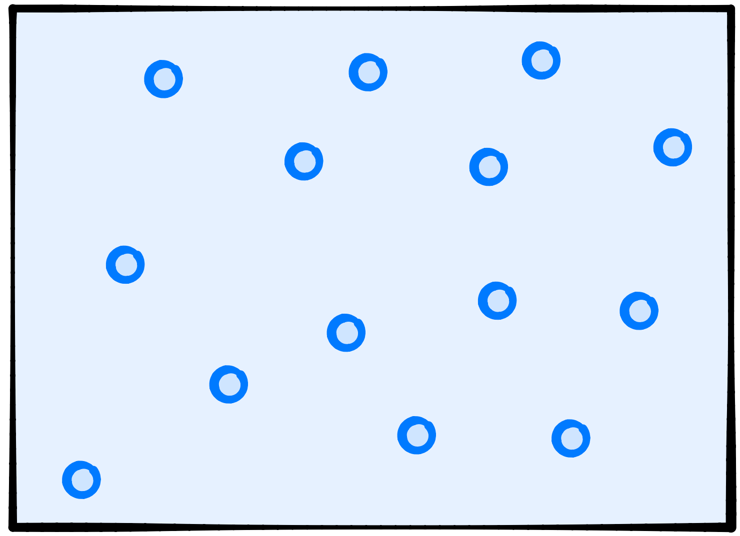


vs $W_{\text{bosons}}(m)$



② [Feynman]

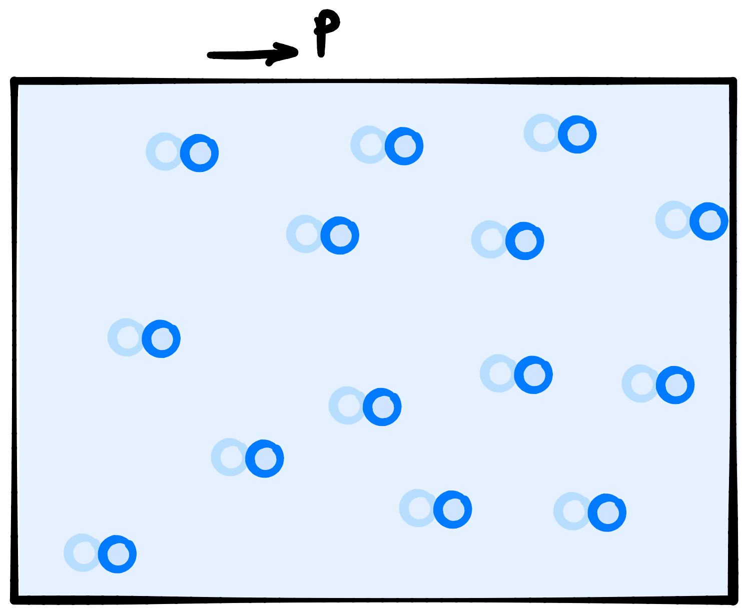
$\Phi_0(r)$, E_0



phonon w/ wavenumber 0 :

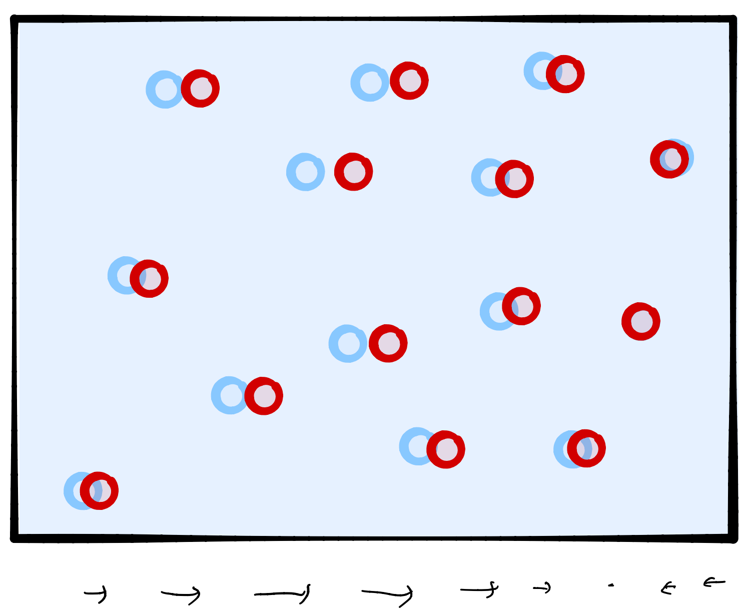
$E_0 + \frac{p^2}{2M}$

$M = Nm$

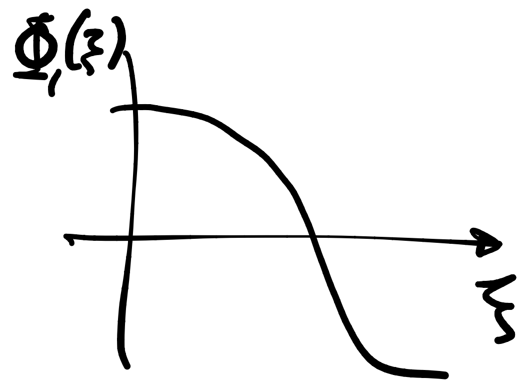


Compression wave.

phonon w/ wavenumber $k\check{x}$.



what else?



Consider $\Phi_1 \in \mathbb{R}$.

$\Phi_1(\xi) = \Phi_1(\xi, r_2 \dots r_N)$

$\Longleftarrow \int \Phi_1 \Phi_0 = 0.$

Suppose Φ_1 is NOT a phonon. $\Longrightarrow \int \Phi_1 \Phi(\text{phonon}) = 0$.

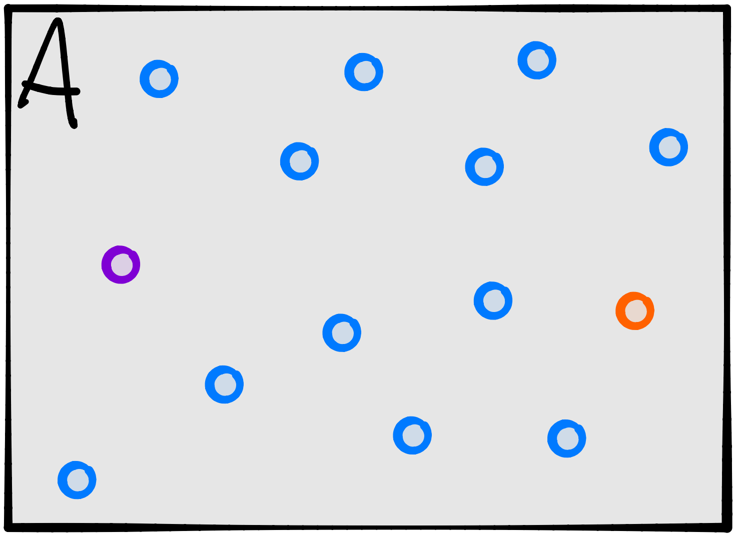
want Φ_1 to vary slowly

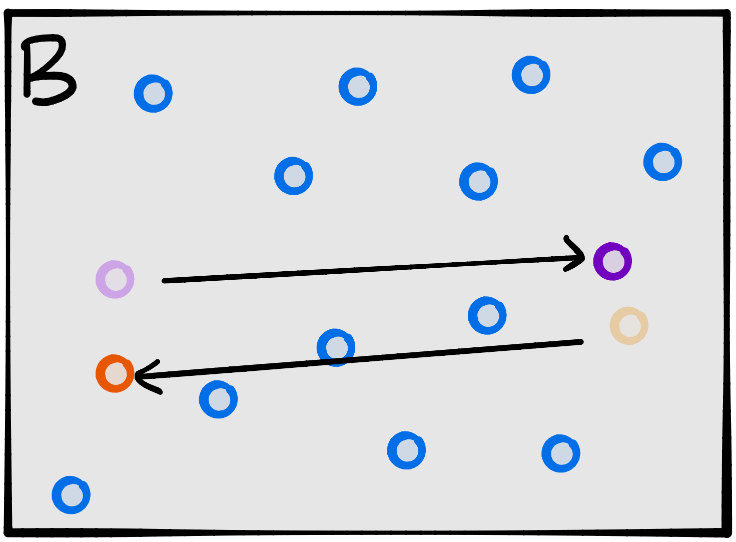
A $\equiv$ config for which Φ_1 is max

B $\equiv$ " " " Φ_1 " min.

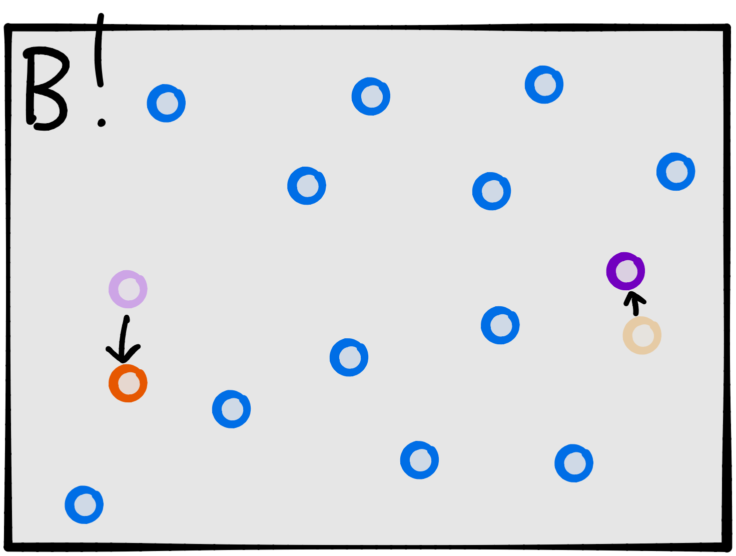
want A & B as different as possible.

(while still avoiding collisions & near collisions)





for bosons:



If sign changes,
$\int (\nabla \Phi_1)^2$ big

If not, it's
a phonon.

Excitation Spectrum (single-mode approximation)

$$\psi(r_1 \ldots r_N) = \underbrace{\sum_{i=1}^{N} f(r_i)}_{F} \Phi_0(r_1 \ldots r_N) = F\,\Phi_0$$

variational principle in each k sector:

$$\langle \psi_k | H | \psi_k \rangle \geqslant \langle gs_k | H | gs_k \rangle$$

$$\langle \psi | H | \psi \rangle = \frac{\int d^{3N}r \left(\sum_i \frac{\hbar^2}{2m} |\vec{\nabla}_i \psi|^2 + V|\psi|^2 \right)}{\int d^{3N}r \, |\psi|^2}$$

$$\boxed{\begin{array}{c} H\Phi_0 = E_0 \Phi_0 \\ \text{IBP} \end{array}}$$

$$= E_0 + \epsilon$$

$$\epsilon = \frac{\sum_i \frac{\hbar^2}{2m} \int d^{3N}r \, |\vec{\nabla}_i F|^2 \Phi_0^2}{\int d^{3N}r \, |F|^2 \Phi_0^2} \equiv \frac{N}{D}$$

$$\boxed{F = \sum_j f(r_j)}$$

$$N = \sum_j \frac{\hbar^2}{2m} \int |\nabla_j f(r_j)|^2 \Phi_0^2(r_1 \ldots r_N)\, d^{3N}r$$

$$N = \underbrace{\sum_{j=1}^{N}}_{=N} \frac{\hbar^2}{2m} \int d^d r_1 \, |\nabla_1 f(r_1)|^2 \underbrace{\int \Phi_0^2(r_1 \ldots r_N) \, d^d r_2 \ldots d^d r_N}$$

$$\int d^d r_2 \ldots d^d r_N \, \Phi_0^2(r_1 r_2 \ldots r_N)$$

$$= \rho_1(r_1, r_1) = \frac{1}{N} \langle \psi^\dagger(r_1) \psi(r_1) \rangle$$

$$\left(\rho_0 = \frac{N}{V}\right) \qquad = \frac{1}{N} \langle \rho(r_1) \rangle_0 = \frac{\rho_0}{N} = \frac{1}{V}.$$

$$\Rightarrow N = \frac{\hbar^2}{2m} \rho_0 \int |\vec{\nabla} f(r)|^2 d^d r.$$

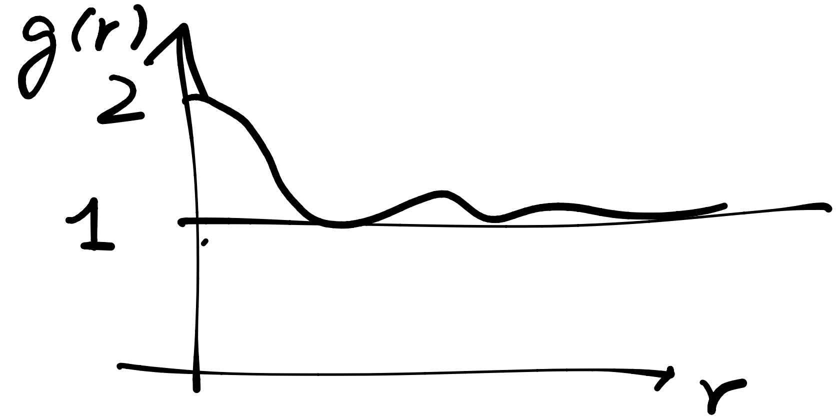
$$D = \sum_{ij} \int d^{dN} r \; f^*(r_i) f(r_j) \Phi_0^2(r_1 \ldots r_N)$$

$$= \rho_0 \int d^d r_1 \, d^d r_2 \, f^*(r_1) f(r_2) g(r_1 - r_2)$$

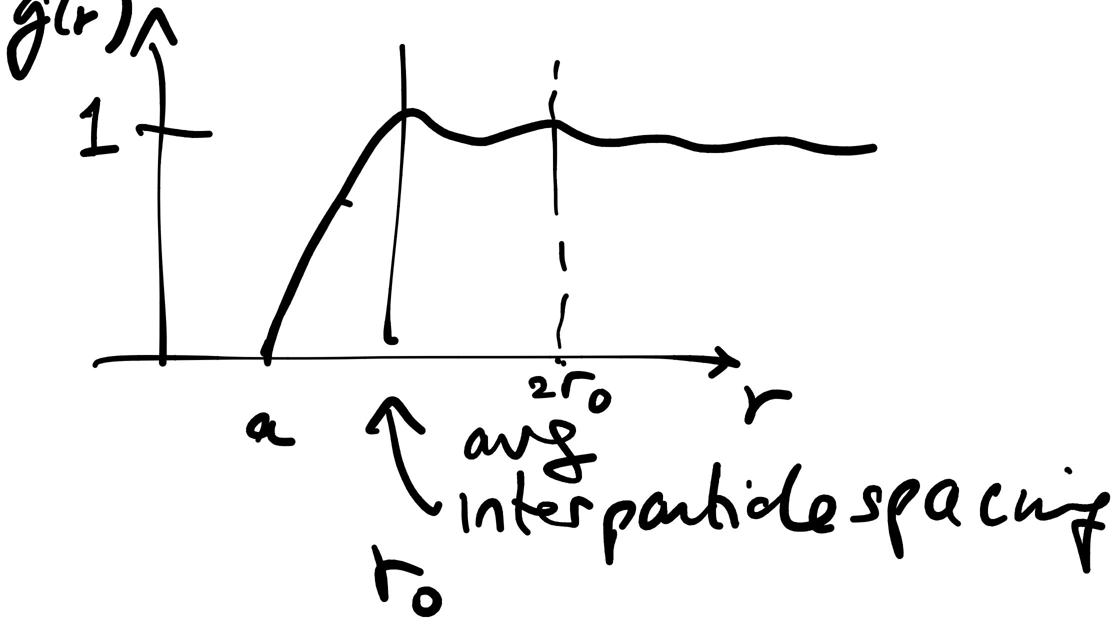
$$\rho_0 g(r_1 - r_2) = \sum_{ij} \int d^{dN} r' \, \Phi_0^2 \, \delta^d(r_i' - r_1) \delta^d(r_j' - r_2)$$

pair correlation = prob of finding a particle at r_2 given one is at r_1.

free bosons:



interacting bosons:



$$0 = \frac{\delta E[f]}{\delta f^*(r)} = \frac{-\frac{\hbar^2}{2m}\nabla^2 f}{D} - \frac{N}{D^2}\int d^d r_2\, f(r_2)\, g(r_{12})$$

$$= \frac{1}{D}\left(-\frac{\hbar^2}{2m}\nabla^2 f - \epsilon \int d^d r'\, g(r-r')\, f(r')\right)$$

- linear in f (at fixed ϵ)
- transl. inv't.

$\Big\}\Longrightarrow$ solve by Fourier

$$\boxed{f(r) = e^{ik\cdot r}}$$

$$\Rightarrow \boxed{\epsilon(k) = \frac{\hbar^2 k^2}{2m S(k)}}$$

$$S(k) = \int d^d r \, g(r) e^{-ik\cdot r} = \frac{\langle \Phi_0 | \rho_k^\dagger \rho_k | \Phi_0 \rangle}{N}$$

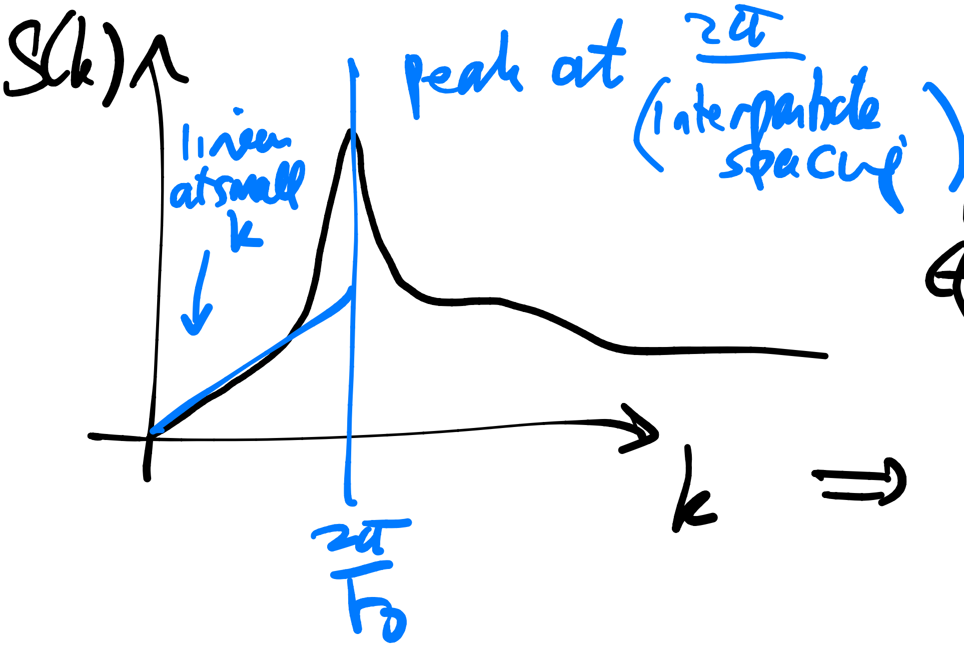
static structure factor

measure by elastic neutron scattering.

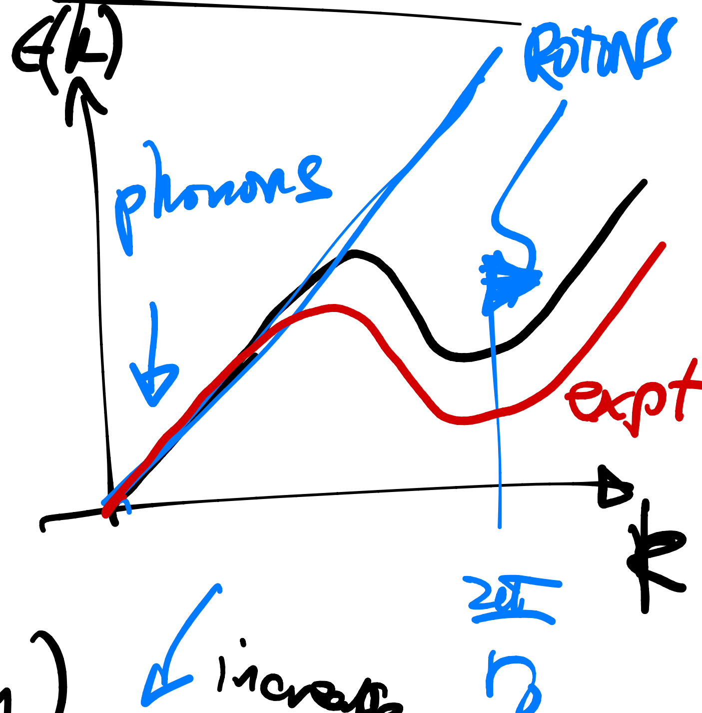
$$\rho_k = \sum_j e^{-ikr_j}$$

$$= \int d^d r \, e^{ikr} \rho(r)$$

$$\rho(r) = \sum_{j=1}^{N} \delta^d (r - r_j)$$

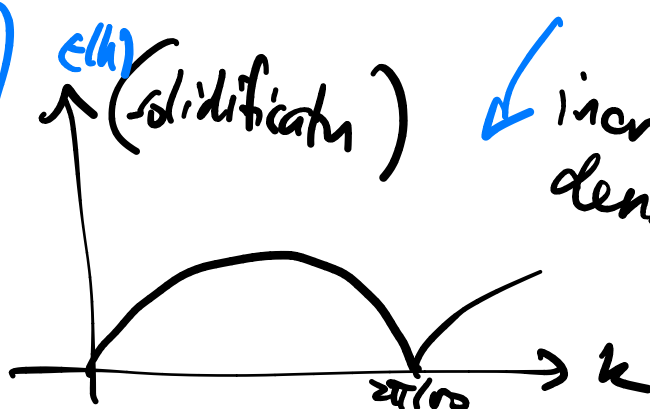


$\Rightarrow$



increase density

In a solid
$S(k) \sim \delta(k - \frac{2\pi}{r_0})$



$$\psi_k = \left(\sum_{i=1}^{N} e^{i k \cdot r_i} \right) \Phi_0$$

$$= \rho_k \, \Phi_0$$

$$\epsilon(k) = \frac{\langle \psi_k | (H - E_0) | \psi_k \rangle}{\langle \psi_k | \psi_k \rangle} = \frac{\langle \Phi_0 | \rho_k^+ [H, \rho_k] | \Phi_0 \rangle}{\langle \Phi_0 | \rho_k^+ \rho_k | \Phi_0 \rangle}$$

$= N S(k)$

Why "Single-mode approx" (SMA)?

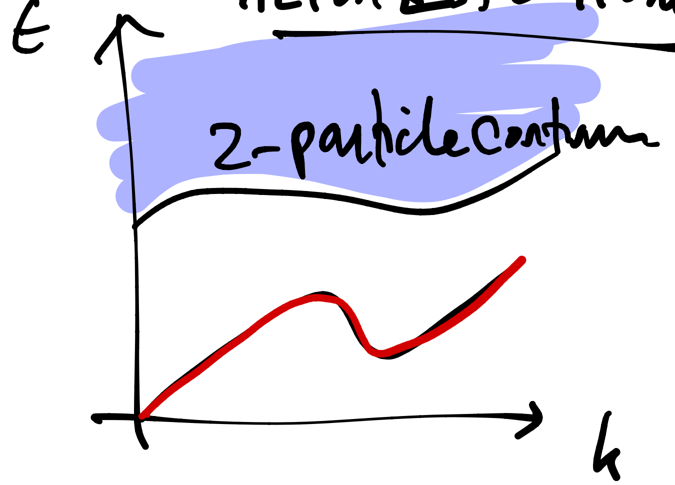
Dynamical structure factor:

$$S(k,\omega) = \sum_n \delta(\omega - \omega_n) \left| \langle \Phi_n | \rho_k | \Phi_0 \rangle \right|^2$$

$$\left(\omega_n \equiv E_n - E_0 \,, \quad (H - E_0) | \Phi_n \rangle = \omega_n | \Phi_n \rangle \right)$$

$$\propto \int d\omega \, e^{-i\omega t} \langle \Phi_0 | \rho_k(t) \rho_k(0) | \Phi_0 \rangle$$

ACTUAL SPECTRUM:



$$S(k,\omega) = S_{SMA} + S_{continuum}$$

$$S(k,\omega) \cong S_{SMA}(k,\omega) = Z_k\, \delta(\omega - \epsilon_k)$$

$$S(k) = \frac{1}{N}\int d\omega\, S(k,\omega) = Z_k$$

True facts (particle # conservation ⟹)

$$\int d\omega\, \omega\, S(k,\omega) = \frac{N\hbar^2 k^2}{2m} \qquad (f \text{ sum rule})$$

$$\lim_{k\to 0} \int d\omega\, \frac{S(k,\omega)}{\omega} = \frac{N}{2mV_s^2} \qquad (\text{compressibility sum rule})$$

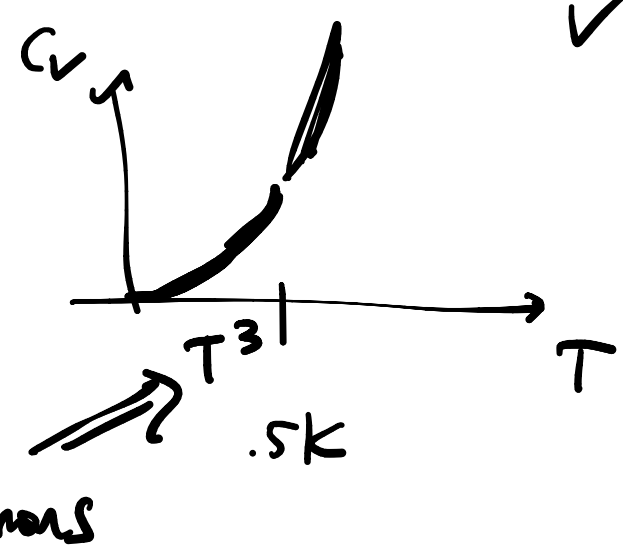
$$\epsilon(k) = \frac{\int d\omega\, \omega\, S(k,\omega)}{\int d\omega\, S(k,\omega)} = \frac{\cancel{N}\, \frac{\hbar^2 k^2}{2m}}{\cancel{N}\, S(k)}$$

$\underline{f}$: $\frac{k^2}{2m} = S(k)\epsilon_k \Rightarrow \epsilon_k = \frac{k^2}{2mS(k)}$

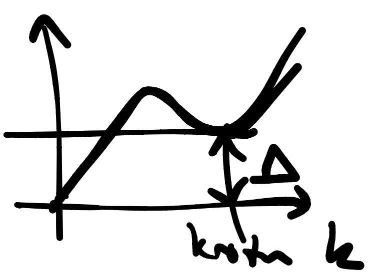
$\underline{\text{compr}}$: $\frac{1}{2mv_s^2} = \lim_{k\to 0} \frac{S(k)}{\epsilon(k)}$

$\Rightarrow S(k) \overset{k\to 0}{\simeq} \frac{|k|}{2mv_s}$.

$\Rightarrow \epsilon_k \overset{k\to 0}{\simeq} v_s|k|$. (phonons move at speed of sound.)

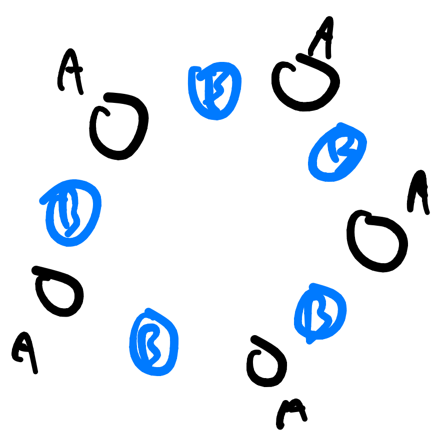


$\underline{\text{"Roton"}}$: [Landau].

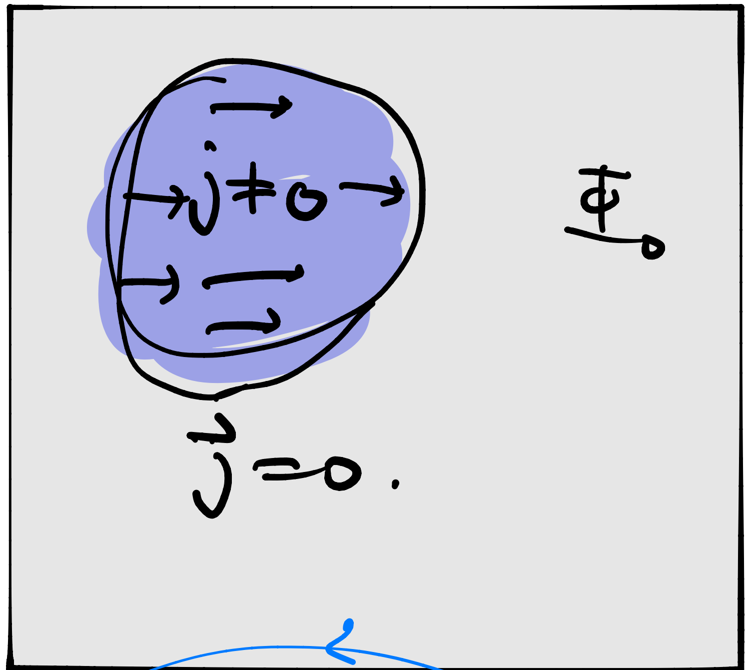


$\Delta = \epsilon(k_{roton})$

$$\boxed{\begin{aligned} U &= \int d^3k \frac{\omega_k}{e^{\omega_k/kT}-1} \\ &\approx \int \frac{d\omega\, \omega^3}{e^{\omega/kT}-1} \sim T^4. \\ c_v &= \frac{\partial U}{\partial T} \sim T^3. \end{aligned}}$$



$$\left.\frac{\partial \epsilon}{\partial k}\right|_{k_{\text{roton}}} = 0 .$$



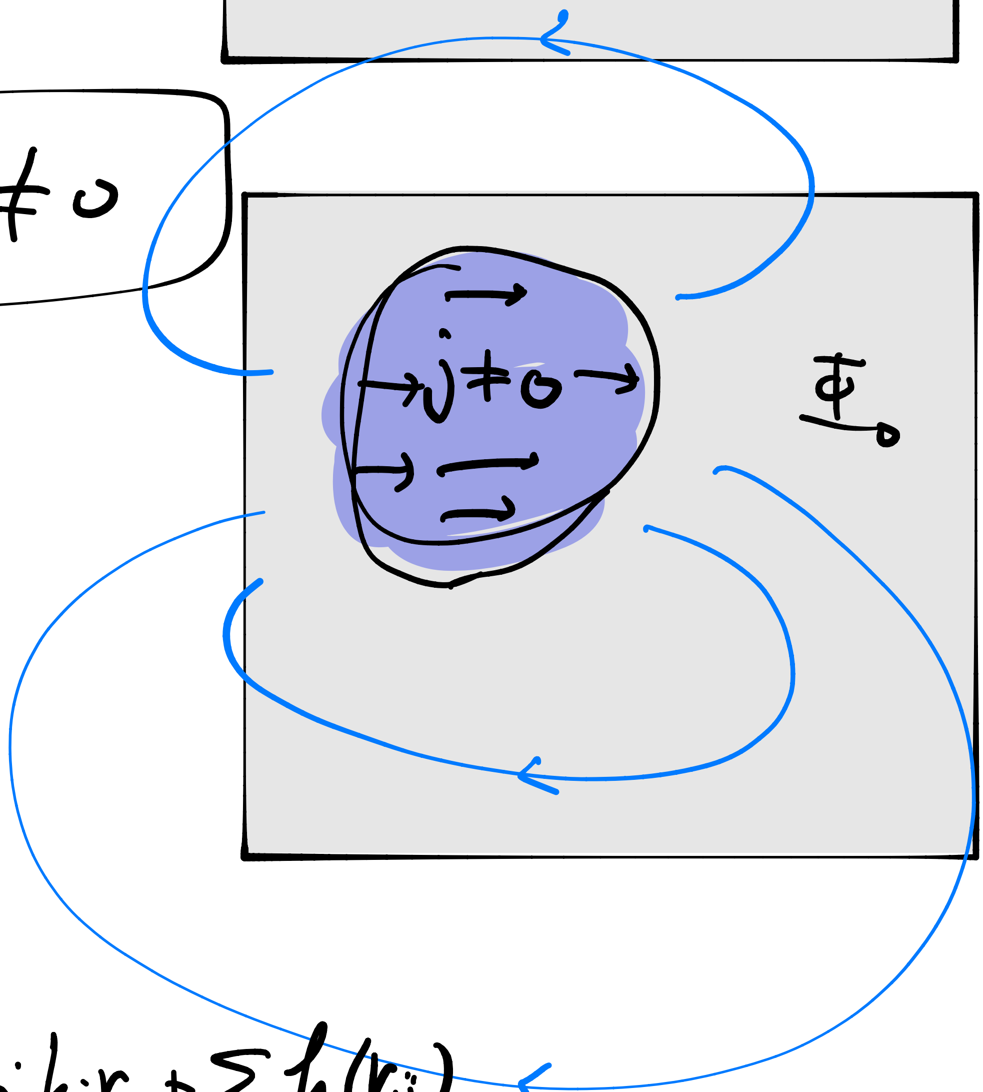
$$\rho \sim \text{const}$$

$$\vec{j} = i\,\psi^* \nabla \psi + h.c.$$

$$\propto \vec{k}$$

$$\psi \sim e^{ikx}$$

$$\Rightarrow \boxed{\partial_t \rho + \vec{\nabla}\cdot\vec{j} \neq 0}$$



$$\psi_{\text{backflow}} = \sum_i e^{\,i k\cdot r_i + \sum_{ij} h(r_{ij})}\, \Phi_0$$

4 Interacting fermions

4.1 Q: whence spin-spin interactions?

2 Distinguishable spinless particles.

$$H = h_0(1) + h_0(2) + \underbrace{V(r_1, r_2)}_{= V(r_2, r_1)}$$

$$\begin{cases} h_0 \psi_\alpha = \epsilon_\alpha \psi_\alpha \\ h_0 \psi_\beta = \epsilon_\beta \psi_\beta \end{cases}$$

TRIAL WAVE f'ns:

$$\Psi_{S/A} \equiv \frac{(\Psi_{\alpha\beta} \pm \Psi_{\beta\alpha})}{\sqrt{2}}$$

$$\Psi_{\alpha\beta}(r_1, r_2) \equiv \psi_\alpha(r_1)\psi_\beta(r_2)$$

$$E_{S/A} = \langle \Psi_{S/A} | \hat{H} | \Psi_{S/A} \rangle = \epsilon_\alpha + \epsilon_\beta + \int d^d r_1 \int d^d r_2 \, \Psi^*_{S/A}(r_1, r_2) V(r_1, r_2) \Psi_{S/A}(r_1, r_2)$$

$$= \epsilon_\alpha + \epsilon_\beta + I \mp J \, .$$