Last time: if $V(r_{ij}) = U_0 f(r_{ij})$

compare $E_{int} \equiv \langle \hat{V} \rangle$ in $\underbrace{\frac{(b_\alpha^+)^{N_\alpha}}{\sqrt{N_\alpha!}} |0\rangle}$

$\boxed{\text{"Hartree-Fock"}}$

$$E_{int} = \frac{1}{2} \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta,\gamma\delta} \underbrace{\langle b_\alpha^+ b_\beta^+ b_\gamma b_\delta \rangle}$$

$\left(\begin{array}{l} N \gg 1 \\ \frac{N}{V} \text{ fixed} \end{array} \right)$ if $\alpha \neq \beta$: $\cong \langle b_\alpha^+ b_\delta \rangle \langle b_\beta^+ b_\gamma \rangle + \gamma \langle b^+_\alpha b_\gamma \rangle \langle b_\beta^+ b_\delta \rangle$

$$= (\delta_{\alpha\delta} \delta_{\beta\gamma} + \gamma \delta_{\alpha\gamma} \delta_{\beta\delta}) \, n_\alpha n_\beta$$

if $\alpha = \beta$:

$$\langle (b_\alpha^+)^2 b_\gamma b_\delta \rangle = \delta_{\alpha\gamma} \delta_{\alpha\delta} \, n_\alpha (n_\alpha - 1)$$

$$\hookrightarrow E_{int} = \frac{1}{2} \sum_{\alpha\beta} (V_{\alpha\beta,\alpha\beta} + \gamma V_{\alpha\beta,\beta\alpha}) n_\alpha n_\beta$$

$$+ \sum_\alpha V_{\alpha\alpha,\alpha\alpha} n_\alpha (n_\alpha - 1). \longrightarrow \text{prev. formula.}$$

Bose-Hubbard model:

$$H_{BH} = -t \sum_{\langle ij \rangle} (b_i^\dagger b_j + h.c.) + U \sum_i (\hat{n}_i - \bar{n})^2$$

$$= H_t + H_U \qquad \underline{U > 0}$$

$$\hat{n}_i = b_i^\dagger b_i$$

Two dimless params: t/U, $\bar{n}$ — like a chemical potential.

H_t wants bosons to delocalize

H_U wants n_i = integer ~~closest~~ to $\bar{n}$.

$$\boxed{t/U \gg 1, \ \hat{n} \gg 1} : \quad \underbrace{\hat{b}_i^\dagger = \sqrt{\hat{n}_i}\, e^{-i\hat{\varphi}_i}} \ : \hat{\varphi} \cong \hat{\varphi} + 2\pi$$

$$\hbar = [b, b^\dagger] \iff [\varphi_i, n_j] = -i \delta_{ij} \hbar$$

$$[\varphi, f(n)] = -i f'(n)$$

$$[\varphi, \sqrt{n}] = -i \frac{1}{2\sqrt{n}}.$$

$$H_{BH} = -t \sum_{\langle ij \rangle} \left(\sqrt{n_i}\, e^{i(\varphi_i - \varphi_j)} \sqrt{n_j} + h.c. \right) + \sum_i U(\hat{n}_i - \bar{n})^2$$

$$\boxed{\begin{aligned} b^\dagger &= \sqrt{n}\, e^{i\varphi} \\ b &= e^{-i\varphi} \sqrt{n} \end{aligned}}$$

If $\bar{n} \gg 1 \Rightarrow \langle \hat{n} \rangle \simeq \bar{n} \gg 1$

$$b_i^\dagger = \sqrt{n_i}\, e^{i\varphi_i} = \sqrt{\bar{n}}\, e^{i\varphi_1} + \text{small}$$

$$n_i \equiv \bar{n} + \Delta n_i \quad \langle \Delta n_i \rangle \ll 1$$

$$\rightarrow H_{BH} \simeq -2t\bar{n} \sum_{\langle ij \rangle} \cos(\varphi_i - \varphi_j) + U \sum_i (\Delta \hat{n}_i)^2$$

(quantum rotors)

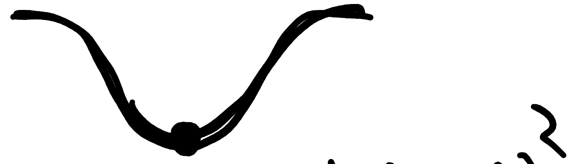
$\boxed{t \gg U}$

$-\cos(\varphi_i - \varphi_j)$ wants $\varphi_i \sim \varphi_j \ \forall ij$.

$$\langle b_i^\dagger \rangle \simeq \sqrt{\bar{n}} \langle e^{i\varphi_i} \rangle \neq 0.$$

BEC

elementary excitations:



<u>if $\Delta\varphi_i$ is small</u> $\quad \cos(\varphi_i-\varphi_j) \cong 1 - \frac{1}{2}(\varphi_i-\varphi_j)^2 + \dots$

$$\boxed{\begin{aligned}\Delta H_{BH} &\simeq t\bar{n}\sum_{\langle ij\rangle}(\varphi_i-\varphi_j)^2 + U\sum_i(\Delta n_i)^2 \\ [\Delta n_i, \varphi_j] &= -i\delta_{ij}\hbar .\end{aligned}}$$

$p \to \Delta n \qquad \frac{1}{2m} \to U$

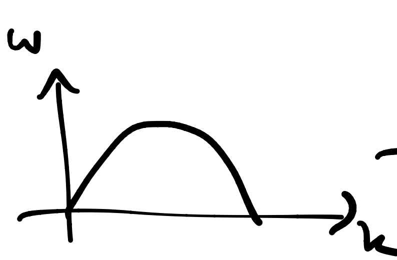
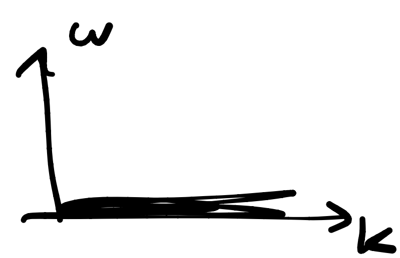
$q \to \varphi \qquad \frac{1}{2}m\omega^2 \to t\bar{n}$

$$\omega_k = \omega_0 \left|\sin \frac{ka}{2}\right|$$

$$\overset{ka\ll 1}{\simeq} \frac{\omega_0 a k}{2} \Rightarrow \boxed{v_s = \sqrt{t\bar{n}U}\,a}$$

$$U(1):\ b_i^\dagger \to e^{i\alpha} b_i^\dagger \Rightarrow \underline{\varphi_i \to \varphi_i + \alpha}.$$

phonon, Goldstone boson.

free limit  $\xrightarrow{U\to 0}$ 

claim: the sound mode at $ka \ll 1$ transcends the lattice.

$\boxed{t/U \ll 1}$ (any $\bar{n}$)

<u>t = 0</u> $\quad H_{BH} = U \sum_i (n_i - \bar{n})^2$ wants $n_i = N$

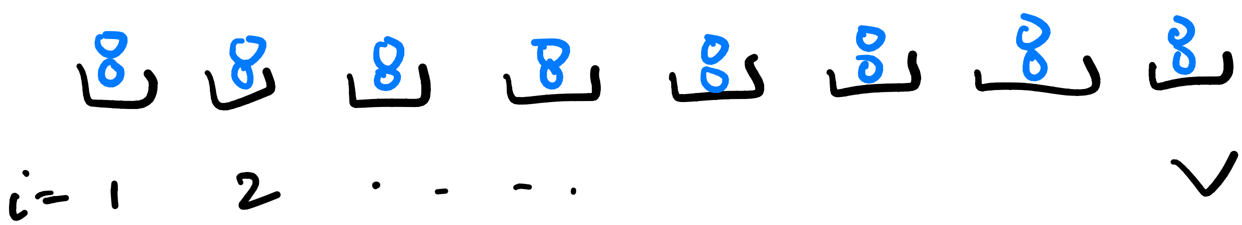
$\equiv$ integer closest to $\bar{n}$.

[assume $\bar{n} \neq N + \frac{1}{2}$

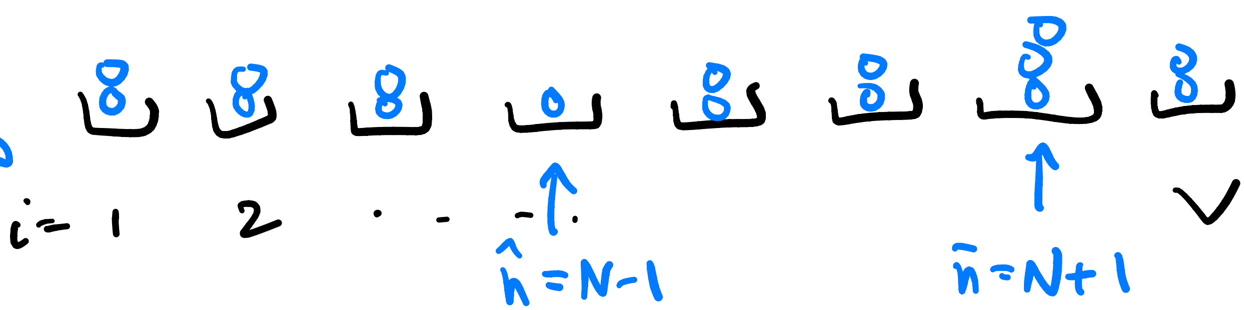
otherwise $N, N+1$ are degenerate]

$$|gs\rangle = \prod_i \frac{(b_i^\dagger)^N}{\sqrt{N!}} |0\rangle .$$

<u>unique!</u>



N=2



N=2

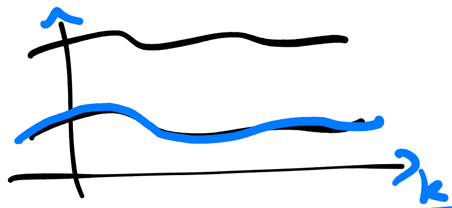
GAP

$$E = E_0 + O(U) \xrightarrow[\text{limit}]{\text{thermodynamic}} E_0 + O(U).$$

Consequence of gap: ① This is an insulator.

free fermions

vs band insulator



← quantum!

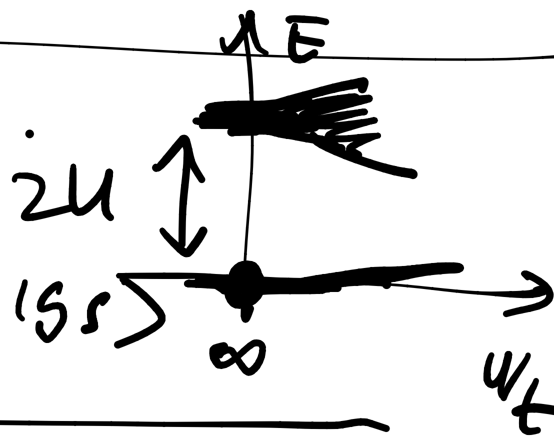
("Mott insulator".) traffic jam.

$|gs\rangle$ is an eigenstate of $\hat{n}_i$..

$[n_i, \varphi_j] = -i\hbar\delta_{ij} \Rightarrow \varphi_i$ is MAXIMALLY indefinite

$\langle b^\dagger \rangle = \langle n e^{i\varphi} \rangle = 0.$

② stable to finite U/t.



$\boxed{\text{for generic } \bar{n}}$:

SF

SSB of U(1)

→ Goldstone

⇒ gapless

?

MI

no SSB

large gap ~ U

If $\bar{n} = N + \frac{1}{2} \to 2^N$ degenerate groundstates

$|\downarrow\rangle_x = |N$ particles at $x\rangle$

$|\uparrow\rangle_x = |N+1$ particles at $x\rangle$

same

$U(\hat{n} - (N+\tfrac{1}{2}))^2$

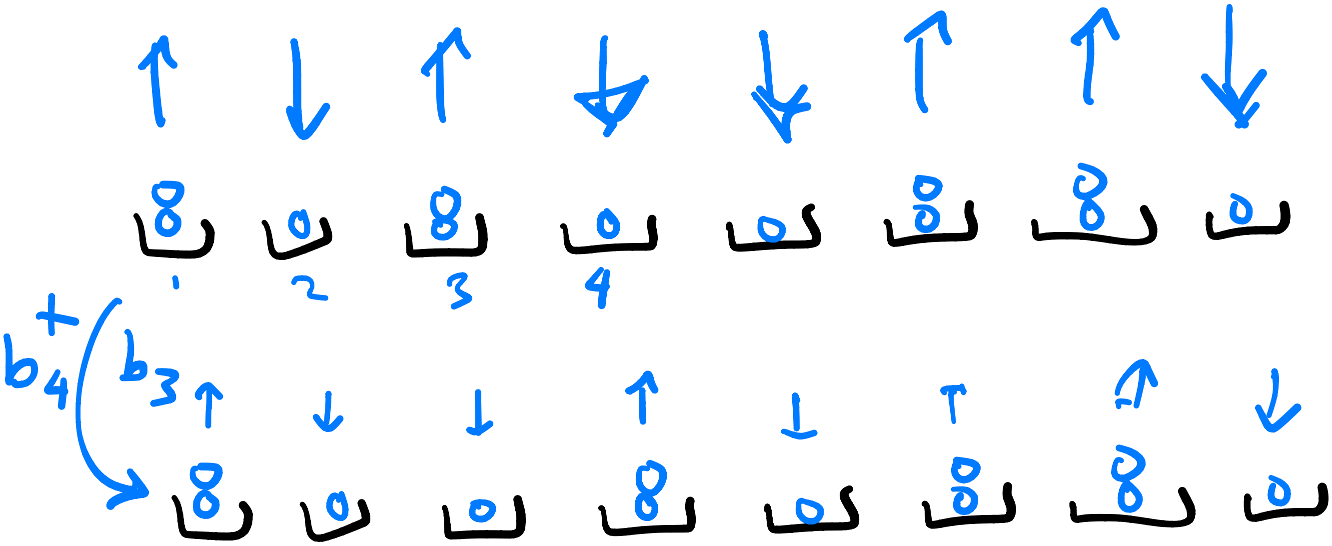
$$\begin{cases} S_i^+ = P\, b_i^+\, P \\ S_i^- = P\, b_i\, P \end{cases} \qquad \begin{cases} S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{cases}$$

$(S^+)^2 = 0.$

$P \equiv$ projector into degenerate subspace.

$\boxed{\text{1st-order degen. pert. thy}}$

$$H_{eff} = P H_t P = P\Big(-\sum_{\langle ij\rangle} b_i^\dagger b_j + h.c.\Big)P$$



$\bar{n} = 1\frac{1}{2}.$

$$H_{eff} = -\sum_{\langle ij\rangle} \left(S_i^+ S_j^- + h.c. \right)$$

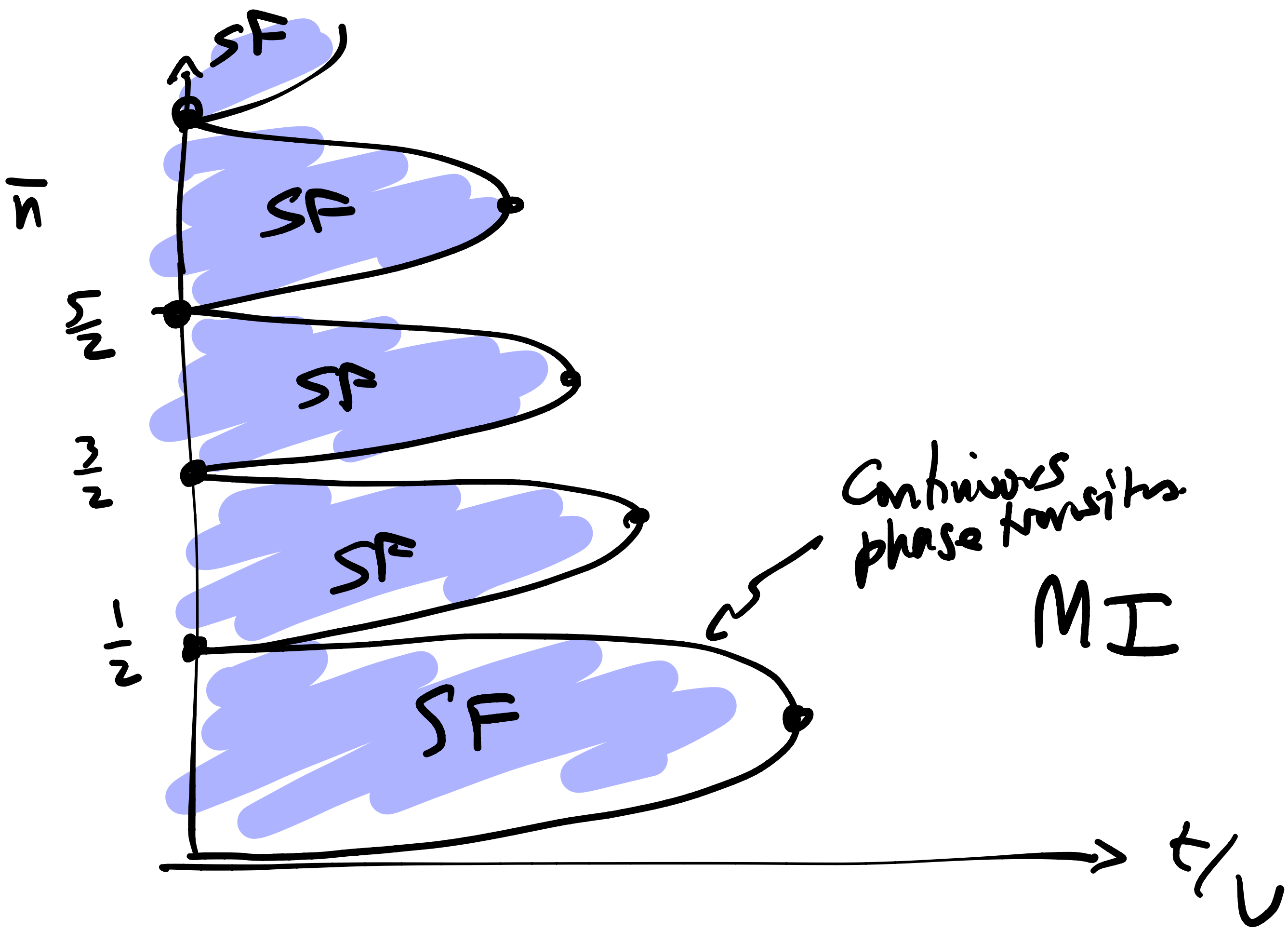
ferromagnetic. $U(1)$ symmetric

↑ rotating about z:

$$U_\alpha = e^{-i\alpha \sum_i S_i^z}.$$

$$|\psi, \alpha\rangle = U_\alpha \bigotimes_i |\rightarrow\rangle$$

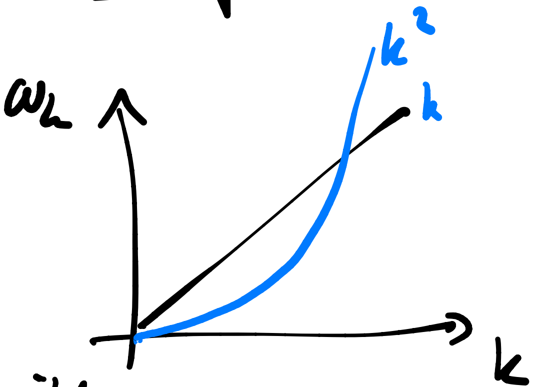
$$0 \neq \langle S_i^+ \rangle = \langle b_i^+ \rangle.$$ BEC!



Absence of low-lying modes

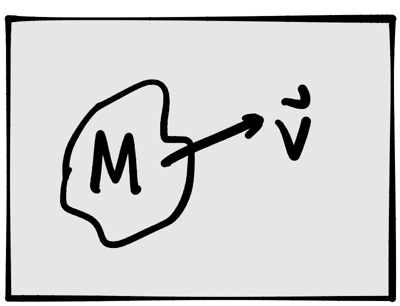
$\Rightarrow$ phenomenology of SF.

$\exists$ phonon mode $\omega_k \approx v_s|k|$. & $\boxed{\text{NOTHING ELSE.}}$



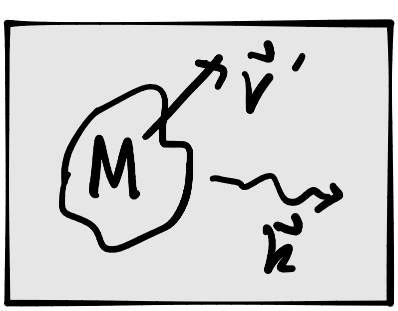
$$N(\omega) \propto k^{d-1} \frac{dk}{d\omega}$$

with only a linear-k mode $\exists$ critical velocity of chunk of fluid below which no excitations are created

① 

$$M\vec{v} = M\vec{v}\,' + \hbar\vec{k}.$$

$$\vec{v}\,' = \vec{v} - \frac{\hbar\vec{k}}{M}.$$

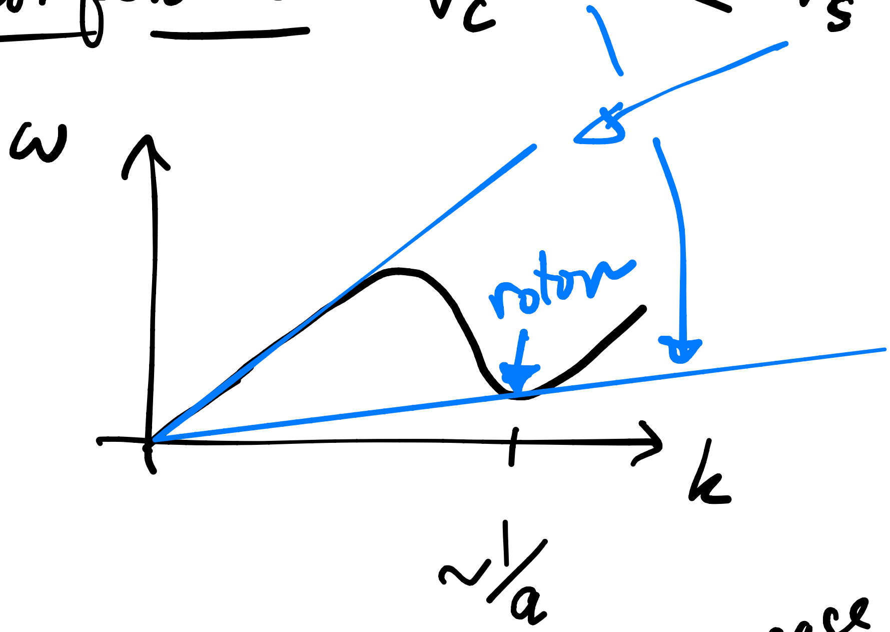
② 

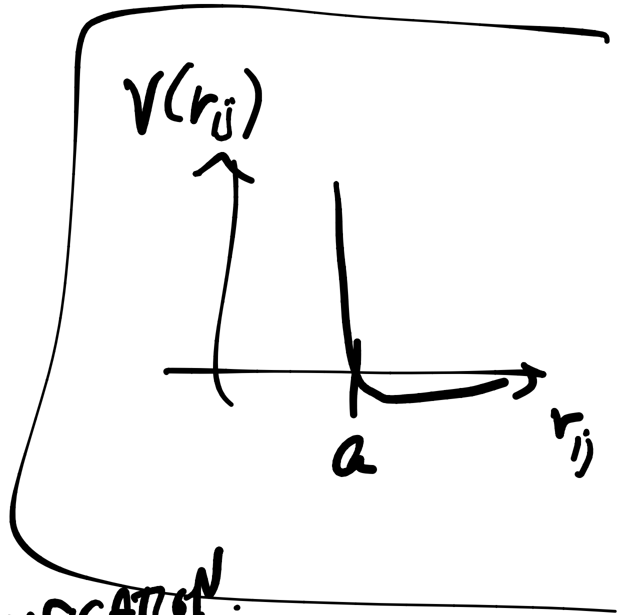
$$\exists\ 0 > \Delta E = \frac{1}{2}M(v')^2 + \hbar\omega(k) - \frac{1}{2}Mv^2$$

if $\omega(k) = v_c|k|$ $\Rightarrow$ $$= (-v+v_c)k + \frac{(\hbar k)^2}{2M}.$$

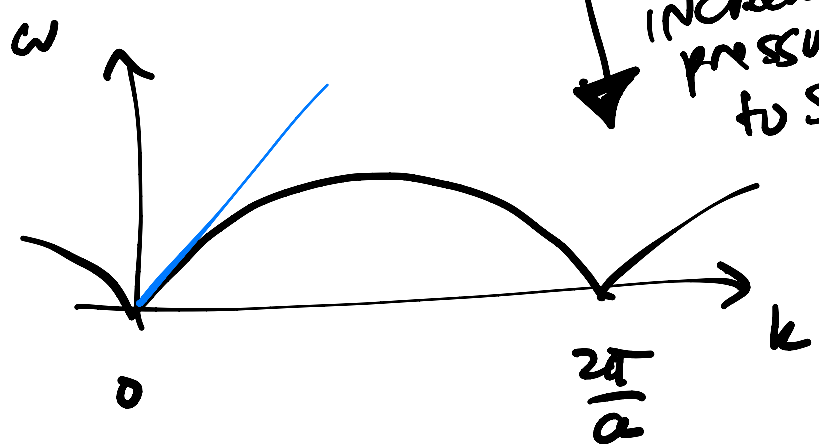
$\Longleftrightarrow v > v_c$.

<u>confession:</u> $V_c^{\text{actual}} < V_s \; . = V_c^{\text{Landau}}$.





INCREASE pressure to SOLIDIFICATION.



Bose statistics $\Longrightarrow$ paucity of low-lying modes:

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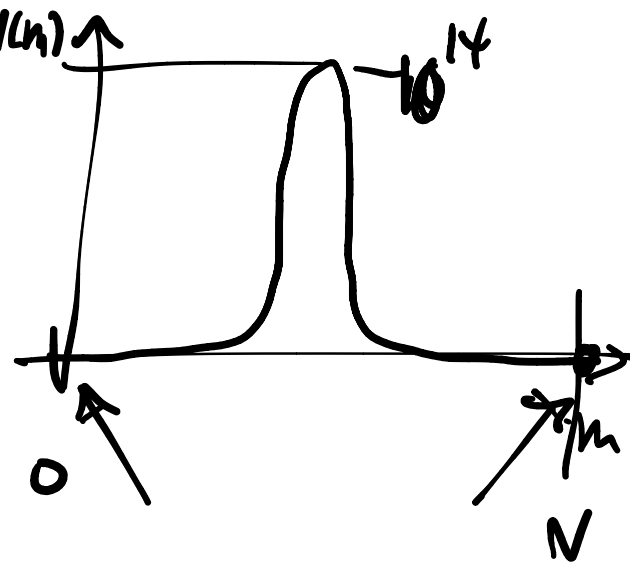
① [Leggett]: N particles in s orbitals.

$\boxed{\text{distinguishable particles} \quad s=2}$ $\quad 2^N = \sum_{m=0}^{N} \binom{N}{m} = \sum_{m=0}^{N} \underline{\underline{W(m)}}$

$W(m)$ = # of ways of putting m particles in box 1.

$$W(m) = \binom{N}{m} = \frac{N!}{m!(N-m)!}$$

eg $N=50$



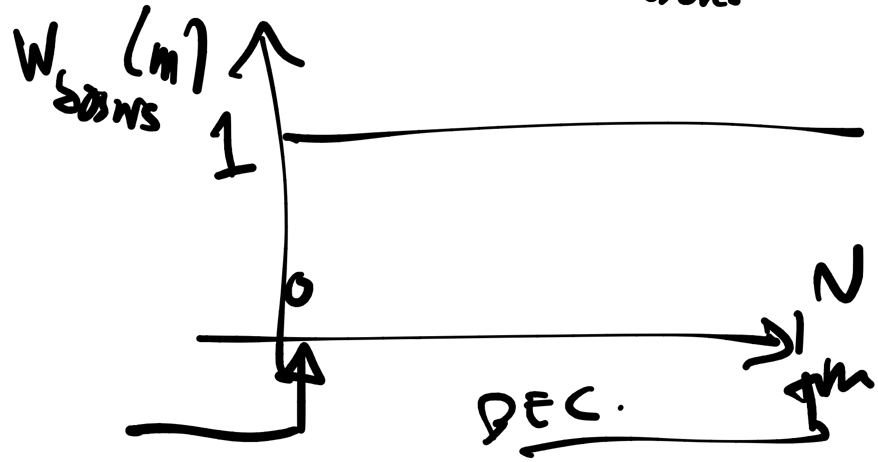
$$W(m\sim 0) \sim W(m \sim N) \sim 1$$
$$\ll W(m \sim N/2) \sim 10^{14}.$$

for s orbitals distinguishable : $$s^N = \sum_{\{m_i\}} \frac{N!}{\prod_{i=1}^{s} m_i!}$$

Bosons : $S=2$, $$W_{bosons}(m) = 1 \; !$$

$$|m, N-m) = \frac{(b_1^+)^m}{\sqrt{m!}} \frac{(b_2^+)^{N-m}}{\sqrt{(N-m)!}} |0\rangle$$

$S>2$: $W_{bosons}(\{m_i\}) = 1$



- $\Rightarrow$ small energetic preference for $m=N$.
  wins.
- Breaks down if $s \gg N$.

$s \simeq$ # of states w $\epsilon < kT$

$\Rightarrow T_c$ ✓.