$$H_{TFIM} = -J\left(\sum_x g X_x + \sum_{\underline{\underline{\langle xy\rangle}}} Z_x Z_y\right)$$

$$\text{MFT:} \quad |g_0\rangle \overset{?}{=} \bigotimes_x |\phi_x\rangle = \bigotimes_x |\phi\rangle$$

$$H_{TFIM} = -J \sum_x \left(g X_x + \underbrace{\left(\frac{1}{2}\sum_{\substack{\text{nbrs}\\ \text{of } x}} Z_y\right)}_{h} Z_x\right)$$

$$H_{boring} = -J \sum_x \left(g X_x + h Z_x\right)$$

$$H_{MFT} = -J \sum_x \left(g X_x + \underbrace{\left(\frac{1}{2}\sum_{\substack{\text{nbrs}\\ \text{of } x}} \langle Z_y\rangle\right)}_{h} Z_x\right)$$

$$H_{TFIM} \overset{?}{\simeq}$$

$$H_{boring} = \overset{J\sqrt{g^2+h^2}}{-}\sum_x \left(\sin\theta\, X_x + \cos\theta\, Z_x\right)$$

$$= -\sum_x \left(\underbrace{e^{-i\theta Y_x} Z_x e^{i\theta Y_x}}_{\substack{= e^{-i\theta\, \mathrm{ad}_Y} Z \\ = \cos\theta\, Z + \sin\theta\, X.}}\right)$$

$$\mathrm{ad}_Y Z = [Y, Z] = iX$$

$$\mathrm{ad}_Y^2 Z = [Y, iX] = Z.$$

$$\Rightarrow \underbrace{|\text{gs of } H_{\text{boring}}\rangle}_{\equiv \theta} = \underline{e^{-i\theta \sum_x Y_x} |\uparrow\uparrow\cdots\uparrow\rangle}.$$

$$= \bigotimes_x \left(\cos\theta\, |\uparrow\rangle_x - i \sin\theta\, |\downarrow\rangle_x\right)$$

$$H_{\text{boring}} = -U \sum Z\, U^{\dagger}$$

$$\because\; -\sum Z\, |\Uparrow\rangle = -N\, |\Uparrow\rangle.$$

$$\underbrace{-U \sum Z\, U^{\dagger}}_{H_{\text{boring}}}\; \underbrace{U |\Uparrow\rangle}_{|\text{gs of } H_{\text{boring}}\rangle} = -N\, U |\Uparrow\rangle.$$

$$U = e^{-i\theta \sum_x Y_x}$$

$$\sin\theta = \frac{g}{\sqrt{g^2+h^2}} \qquad \cos\theta = \frac{h}{\sqrt{g^2+h^2}}.$$

SELF-CONSISTENCY of

$$\langle\theta|\, Z_x\, |\theta\rangle :$$

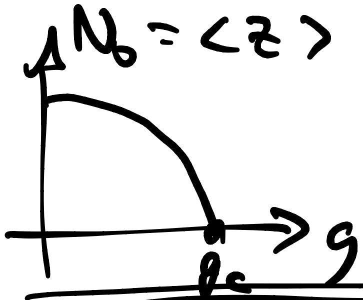
$$h \overset{!}{=} \frac{1}{2} \sum_{\text{nbrs}} \langle\theta|\, Z\, |\theta\rangle$$

$$= \frac{z}{2} \underbrace{\langle\theta|\, Z\, |\theta\rangle}_{\cos^2\theta - \sin^2\theta}.$$

$$m = \langle Z \rangle = \frac{zJm}{\sqrt{(zJm)^2 + (gJ)^2}} = \frac{zm}{\sqrt{(zm)^2 + g^2}}$$

$$\Longrightarrow \quad m = \pm\sqrt{1 - \frac{g^2}{z^2}} \quad \text{if } g < z.$$

$$= 0 \quad \text{if } g > z.$$



Correlation functions & Long-Range order

$$C(r, r') = \langle 0 | Z_r Z_{r'} | 0 \rangle$$

$$\overset{\text{transl inv't}}{=} C(r - r')$$

static structure factor: $\quad S(q) = \sum_r e^{-iqr} C(r).$

$$r \in a\{1 \ldots N\} \qquad q \in \frac{2\pi}{Na}\{1 \ldots N\}$$

TFIM: At $g = \infty$ $\qquad |gs\rangle = \bigotimes_x |\rightarrow\rangle_x$

$$\langle 0 | Z_j Z_l | 0 \rangle \Big|_{g=\infty} = \delta_{jl}.$$

$\boxed{g \gg 1}$ $\quad \langle 0 | Z_j Z_\ell | 0 \rangle \Big|_{g \gg 1} \overset{|j-\ell| \gg 1}{\sim} e^{-|j-\ell| a/\xi}$

idea: $\quad Z_j = \sum_k \frac{e^{-ikja}}{\sqrt{2\omega_k}} a_k + h.c.$

$[a_k, a_{k'}^\dagger] = \delta_{kk'}$

$a_k |0\rangle = 0$

$\langle 0 | Z_j Z_\ell | 0 \rangle$

$$= \sum_k \frac{e^{-ika(j-\ell)}}{2\omega_k}$$

$$\simeq \int d^d k \, \frac{e^{ik\cdot x}}{2\omega_k}$$

$\omega_k = \Delta + \frac{k^2}{2m} + \ldots$

$x \equiv a(j-\ell)$

$$\overset{x \gg a}{\sim} e^{-2m\Delta a|j-\ell|}$$

Mathematica

$\boxed{g \ll 1}$ $\quad \langle 0 | Z_j Z_\ell | 0 \rangle \Big|_{g \ll 1} \overset{|j-\ell| \gg a}{\sim} N_0^2(g)$

$$N_0^2(g) \xrightarrow{g \to 0} 1.$$

$$\langle 0 | Z_j | 0 \rangle = \pm N_0(g)$$

$$\lim_{N\to\infty} \lim_{h\to 0} \langle Z \rangle = 0. \qquad \lim_{h\to 0} \lim_{N\to\infty} \langle Z \rangle = N_0$$

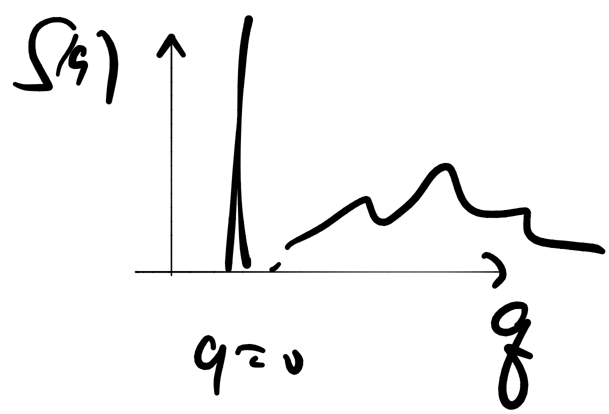
<u>"long-range order":</u>

<u>$N\to\infty$.</u> $q \in \frac{2\pi}{Na}\{1 \ldots N\} \longrightarrow (-\pi, \pi]$.

$$C(R) = \int_{-\pi}^{\pi} dq \, S(q) e^{iqR}.$$

$$\text{LRO}: \quad C(R) \overset{R\to\infty}{\sim} N_0^2.$$

$$\Longleftrightarrow \quad S(q) = 2\pi \delta(q) N_0^2 + \text{regular}$$



<u>spectral representation of $S(q)$:</u>

$$S(q) = \langle 0 | Z_q Z_{-q} | 0 \rangle$$

$$Z_q \equiv \frac{1}{\sqrt{N}} \sum_r e^{-iqr} Z_r = Z_{-q}^{\dagger}$$

$$S(q) = \langle 0| Z_q \underset{\mathbb{1} = \sum_n |n\rangle\langle n|}{\wedge} Z_{-q} |0\rangle = \sum_n \underbrace{|\langle n| Z_q |0\rangle|^2}_{\geq 0}.$$

(UNITARITY.)

$H|n\rangle = E_n|n\rangle.$

$$C(R=0) = \langle 0| \underbrace{Z_r Z_r}_{= Z_r^2 = \mathbb{1}} |0\rangle = \langle 0|0\rangle = 1.$$

$$\Longrightarrow 1 = C(R=0) = \int_{-\pi}^{\pi} dq\, S(q)$$

"SUM RULE".

Dynamical Structure factor

$$C(r,t) \equiv \langle 0| Z_r(t) Z_0(0) |0\rangle$$

$$Z_r(t) \equiv e^{iHt} Z_r e^{-iHt}.$$

$$\xrightarrow{t\to 0} Z_r. \Longrightarrow C(r,0) = C(r).$$

$$S(q,\omega) \equiv \sum_r e^{-iqr} \int_{-\infty}^{\infty} dt\, e^{i\omega t} C(r,t)$$

$$S(q) = \int d\omega\, S(q,\omega).$$

Spectral Rep :

$$S(q,\omega) = \sum_n |\langle n| Z_q |0\rangle|^2 2\pi \delta\underbrace{(\omega - (E_n - E_0))}$$

FINITE T : $\left(Z = \mathrm{tr}\, e^{-H/T} \right)$

$$C_T(x,t\,;x',t') \equiv \frac{1}{Z} \mathrm{tr}\, e^{-H/T} Z(x,t) Z(x',t')$$

$$S_T(q,\omega) = \int d^dx \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t') - i\vec{q}\cdot(x-x')} \cdot C_T(x,t\,;x',t').$$

$$= \frac{2\pi}{ZV} \sum_{nn'} e^{-E_{n'}/T} |\langle n| Z_q |n'\rangle|^2 \delta(\omega - (E_{n'} - E_n))$$

Fermi's Golden Rule:

$$\Gamma = \sum_F |\langle F|\Delta H|I\rangle|^2 \,\delta(\text{energy conservation})$$

$$\rho = \sum_I p_I \,|I\rangle\langle I| \qquad \left(\begin{array}{l}\sum_I p_I = 1 \\ p_I \geq 0\end{array}\right)$$

$$\Gamma = \sum_F \sum_I p_I \,|\langle F|\Delta H|I\rangle|^2 \,\delta(\;)$$

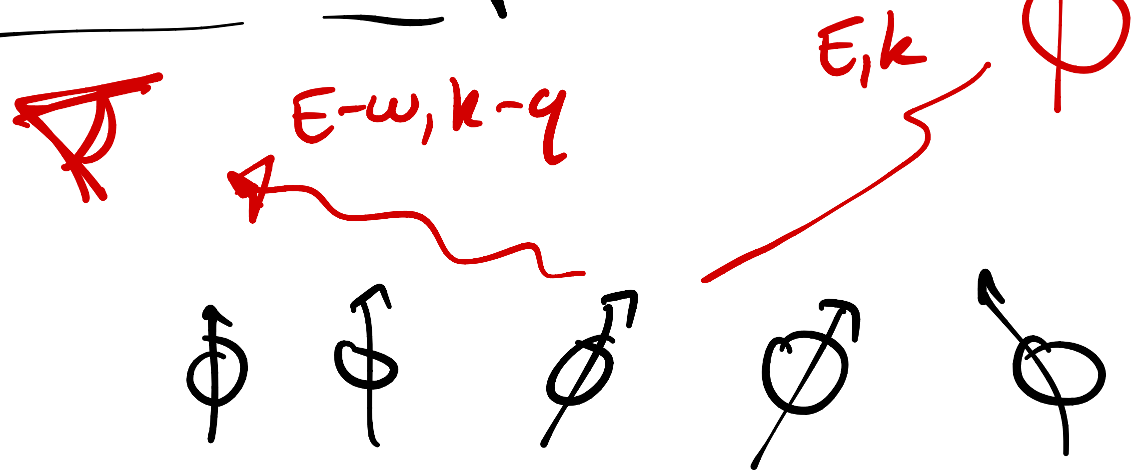
$$I: \quad |n'\rangle \text{ w/ } p_{n'} = \frac{e^{-E_{n'}/T}}{Z}.$$

$$\xrightarrow{T\to 0} |0\rangle.$$

$$\Delta H = Z_q$$

Neutron scattering:

$$\Delta H = \vec{S}_{\text{neutron}} \cdot \vec{\sigma}$$



I : $|n'\rangle \otimes |$ initial state of neutrons: $E, k, \hat{z} \rangle$

F : $|n\rangle \otimes | E-\omega, k-q, \hat{z} \rangle$

$$\Gamma = \Big| \langle n| \otimes \langle E-\omega, k-q\, \hat{z}| \underbrace{\vec{S}_{\text{hadron}} \cdot \vec{\sigma}}_{\propto Z_q} \times |n'\rangle \otimes |E, k, \hat{z}\rangle \Big|^2$$

$$\delta(\quad)$$

$$\propto S(q, \omega).$$

$$\sum_n |\rightarrow \cdots \underset{n}{\leftarrow} \rightarrow\rangle e^{iqn}$$

<u>Prediction</u> : $\boxed{g \gg 1}$

$$= Z_q |0\rangle.$$

$$\rightarrow S(q,\omega)$$

$$= \begin{cases} Z\, \delta(\omega-\omega_q) & \text{if } \omega < 3\Delta \\ \text{3-particle continuum.} \end{cases}$$

$\boxed{g \ll 1}$ $|\uparrow\uparrow\uparrow\uparrow\uparrow\rangle$ or $|\downarrow \ldots \downarrow\rangle$.

$|\uparrow \ldots \uparrow \bullet \downarrow \ldots \downarrow \bullet \uparrow \ldots \uparrow\rangle$

No 1-particle excitation.

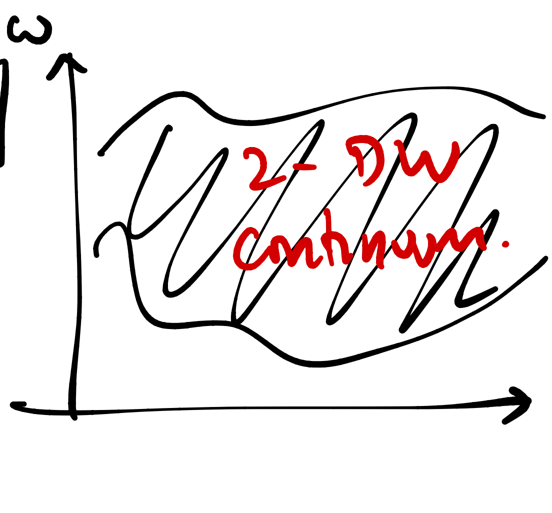
Z can't create one particle.

$\longrightarrow S(q,\omega) \sim$ big mess.

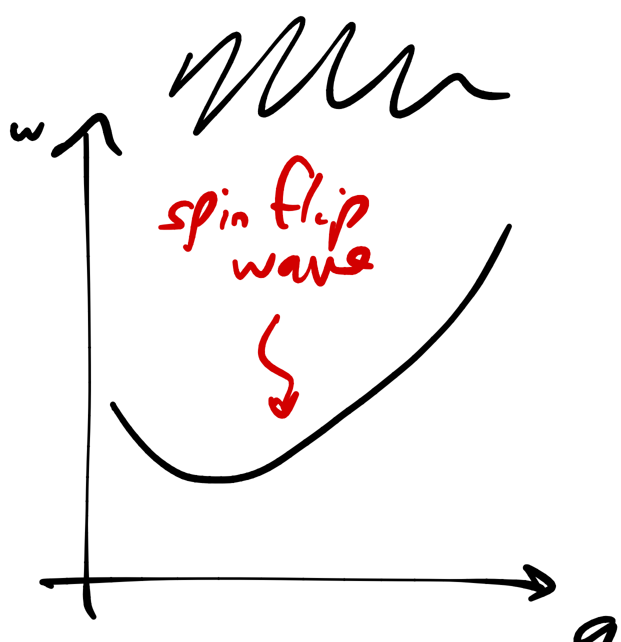
But: $S(q) \sim N_0^2 f(q)$.

$CoNb_2O_6$

$\boxed{g \ll 1}$



$\boxed{g \gg 1}$



ALSO: Apply $\Delta H = -hZ \longrightarrow U(\uparrow\uparrow\uparrow \bullet \downarrow\downarrow\downarrow\downarrow\downarrow \bullet \uparrow\uparrow\uparrow)$ $\longleftarrow r \longrightarrow$

$F_{DW} = -\sigma$. DWs are CONFINED. $= \sigma r$

<u>Confession</u> : $H_{TFIM} = \sum_{IJ} \gamma_I \gamma_J h_{IJ}$

γ_I = nonlocal(Z, X) $\quad \{\gamma_I, \gamma_J\} = 2\delta_{IJ}.$

is <u>gaussian</u>! [Jordan-Wigner]

$|0\rangle \leftrightarrow |\downarrow\rangle$

$c^\dagger|0\rangle \leftrightarrow |\uparrow\rangle$

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3. <u>Interacting Bosons.</u> <u>eg ^4He</u>

$|gs$ of N non-interacting bosons$\rangle = \frac{(b^\dagger_{p_0})^N}{\sqrt{N!}}|0\rangle$

$$\epsilon(p_0) = \min_p \epsilon(p)$$

MACROSCOPIC OCCUPATION OF <u>ONE</u> SINGLE-PARTICLE LEVEL.

$$\underline{\underline{n_{p_0} = N}}. \quad n_{p \neq p_0} = 0.$$

$\equiv$ Bose-Einstein Condensation (BEC).

$\underline{T>0}$. $0 \le T \le T_c$

$$N = N_0(T) + N\int_0^\infty \frac{\rho(\epsilon)\,d\epsilon}{e^{\beta\epsilon}-1}$$

$$\rho(\epsilon) = \sum_i \delta(\epsilon - \epsilon_i)$$ $\{\epsilon_i\}$ = spectrum of h.

T_c is when $N_0(T) \to 0$.

$\underline{\text{eg}}$: $\rho(\epsilon) \propto \epsilon^{\alpha-1} \Rightarrow N_0(T) = N\left(1-\left(\frac{T}{T_c}\right)^\alpha\right)$

$T < T_c$.

3d freespace : $\alpha = 3/2$

3d harmonic trap : $\alpha = 3$

2d free space : $\alpha = 1$.

$\underline{\text{Qs}}$: (1) Does BEC survive interactions?

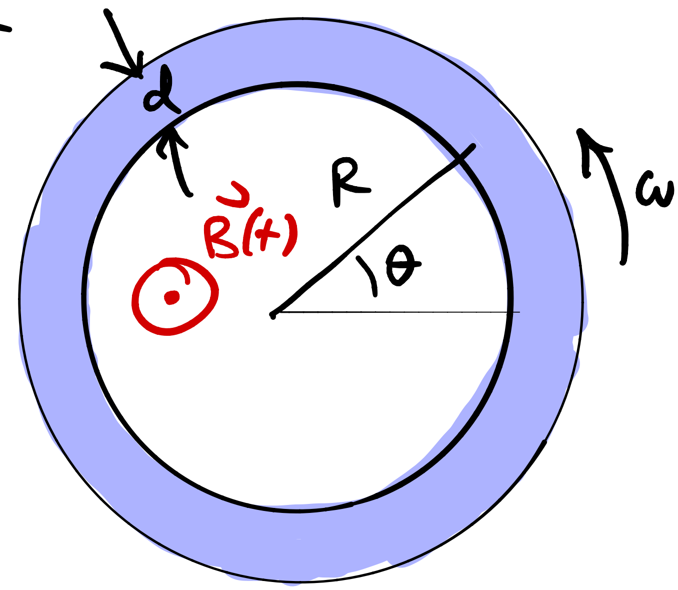
(2) Def?

(3) $\underline{\text{So what}}$?

① : A : BEC ⇒ Superfluidity. ← if the boson is neutral

Superconductivity. ← if the boson is charged under E&M.

Phenomenology : $d \ll R$.



$\Psi(\vec{r}) = U(r, z, \theta)$

$\rightarrow U(r, z, \theta - \omega t)$

$I_{cl} \equiv N m R^2$

$\omega_c \equiv \dfrac{\hbar}{mR^2}$ "quantum of angular frequency"

$L = I_{cl}\,\omega = N \hbar \dfrac{\omega}{\omega_c}$.

$-\partial_t \int \vec{B} \cdot da = \oint_{ring} \vec{E} \cdot dl$

2 expts :

① Rotate the fluid. start at $t < 0$ by rotating container at $\omega \gg \omega_c$ at $T > T_c$.

Then cool below T_c.

NORMAL FLUID: transient rotation stops.

Neutral: viscous drag w walls of container.
Charged: ohmic drag.

SF: It doesn't stop. (!)