Slater determinants & permanents :

$$|\alpha_1 \dots \alpha_n\rangle = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} s^{\sigma} \, |\alpha_{\sigma_1}\rangle \otimes \dots |\alpha_{\sigma_n}\rangle$$

($s = \pm$ for B/F)

$$\langle \alpha_1 \dots \alpha_n | \beta_1 \dots \beta_n \rangle = \frac{1}{n!} \sum_{\sigma,\pi} s^{\pi} s^{\sigma} \langle \alpha_{\sigma_1} | \beta_{\pi_1} \rangle \cdots \langle \alpha_{\sigma_n} | \beta_{\pi_n} \rangle$$

$$= \frac{1}{n!} \sum_{\sigma,\pi} \underbrace{s^{\pi} s^{\sigma}}_{= s^{\pi} s^{\sigma^{-1}} = s^{\pi\sigma^{-1}}} \langle \alpha_1 | \beta_{\pi\sigma^{-1}(1)} \rangle \cdots \langle \alpha_n | \beta_{\pi\sigma^{-1}(n)} \rangle$$

$$= \cancel{\left(\frac{1}{n!} \sum_{\sigma}\right)}^{1} \sum_{\rho = \pi\sigma^{-1}} s^{\rho} \langle \alpha_1 | \beta_{\rho_1} \rangle \cdots \langle \alpha_n | \beta_{\rho(n)} \rangle$$

$$= \begin{vmatrix} \langle \alpha_1 | \beta_1 \rangle & \cdots & \langle \alpha_1 | \beta_n \rangle \\ \vdots & & \\ \langle \alpha_n | \beta_1 \rangle & \cdots & \langle \alpha_n | \beta_n \rangle \end{vmatrix}_s$$

$$|A|_s \equiv \sum_{\pi} s^{\pi} A_{1\pi_1} \cdots A_{n\pi_n} .$$

Particles in free space. $V = L_x L_y L_z$

single-particle states: $u_p(r) \equiv \langle r | p \rangle = \frac{e^{i\vec{p}\cdot\vec{r}}}{\sqrt{V}}$

PBC $\Rightarrow$ $p_i = \frac{2\pi n_i}{L_i}$ $\quad i = x, y, z$
$\quad n_i \in \mathbb{Z}$.

$s = \uparrow, \downarrow$.

$a^\dagger_{ps} |0\rangle = |$ one particle in p and spin s $\rangle$.

Amplitude to find at r is $u_p(r)$.

a_{ps} removes such a particle.

$a_{ps} a^\dagger_{p's'} - \zeta a^\dagger_{p's'} a_{ps} = \delta_{pp'} \delta_{ss'} \mathbb{1}$

$\zeta = \pm 1$ for B/F.

$$\psi^\dagger_s(r) \equiv \sum_p \underbrace{u^*_p(r) a^\dagger_{ps}}_{} = a^\dagger_s(r).$$

$$\langle r', s' | \psi^\dagger_s(r) |0\rangle = \sum_p u^*_p(r) \underbrace{\langle r' s' | a^\dagger_{ps} |0\rangle}_{u_p(r')\delta_{ss'}}$$

$$= \sum_p u^*_p(r) u_p(r')$$

$$\mathbb{1}_1 = \sum_p |p\rangle\langle p|$$

$$\langle r|\mathbb{1}_1|r'\rangle = \sum_p u_p^*(r)\, u_p(r') = \delta(r-r')$$

$$\Rightarrow \langle r'|\, \psi^\dagger(r)\, |0\rangle = \delta(r-r').$$

$\psi^\dagger(r)$ "field operators":

$$\psi_s(r)\, \psi^\dagger_{s'}(r') - \mathcal{J}\, \psi^\dagger_{s'}(r')\, \psi_s(r) = \delta_{ss'}\, \delta(r-r')$$

$$\psi(r)\, \psi(r') - \mathcal{J}\, \psi(r')\, \psi(r) = 0. \star$$

position eigenstates:

$$|r_1 \ldots r_n\rangle = \frac{1}{\sqrt{n!}}\, \psi^\dagger(r_n) \cdots \psi^\dagger(r_2)\, \psi^\dagger(r_1)\, |0\rangle$$

$$= \mathcal{J}\, |r_2\, r_1 \ldots r_n\rangle$$

$$\psi^\dagger(r)\, |r_1 \ldots r_n\rangle = \sqrt{n+1}\, |r_1 \ldots r_n, r\rangle$$

$$\psi(r)\,|r_1\cdots r_n\rangle = \frac{1}{\sqrt{n!}}\,\underline{\psi(r)\,\psi^+(r_n)}\cdots\psi^+(r_1)|0\rangle$$

$$= \frac{1}{\sqrt{n!}}\Big(\underline{\delta_{r,r_n}} + \gamma\,\underline{\psi^+(r_n)\,\psi(r)}\Big)\,\psi^+(r_{n-1})\cdots\psi^+(r_1)\,|0\rangle$$

$$= \cdots$$

$$= \frac{1}{\sqrt{n}}\Big[\delta_{r,r_n}|r_1\cdots r_{n-1}\rangle + \gamma\,\delta_{r,r_{n-1}}|r_1\cdots r_{n-2}r_n\rangle + \cdots \gamma^{n-1}\delta_{r,n}|r_2\cdots r_n\rangle\Big]$$

$$\langle r_1'\cdots r_{n'}'|r_1\cdots r_n\rangle$$

$$= \frac{\delta_{nn'}}{n!}\sum_{\pi} s^{\pi}\,\underline{\delta_{r_1,r'_{\pi 1}}\cdots\delta_{r_n,r'_{\pi n}}}\,.$$

<u>Warning</u>: These states are NOT normalized.

$$\langle r_1 r_2|r_1 r_2\rangle = \left(\frac{1}{\sqrt{2!}}\right)^2\langle 0|\psi_{r_1}\underbrace{\psi_{r_2}\,\psi^+_{r_2}}_{1+\gamma\,\cancel{\psi^+_{r_2}\psi_{r_2}}\to 0}\psi^+_{r_1}|0\rangle$$

$n=2$, $r_1\neq r_2$.

$$= \langle 0|\psi_{r_1}\psi^+_{r_1}|0\rangle = 1\,.$$

$$= \frac{1}{2}\,.$$

$$|r_1 r_2\rangle = \mathcal{J}|r_2 r_1\rangle.$$

$$\mathbb{1}_2 = \sum_{r_1, r_2} |r_1 r_2\rangle\langle r_1 r_2|$$

all

$$= \sum_{r_1 = r_2} |r_1 r_1\rangle\langle r_1 r_1| + 2\sum_{r_1<r_2} |r_1 r_2\rangle\langle r_1 r_2|$$

$|\psi\rangle$ w/ wavefunction $\Psi(r_1 .. r_n)$.

$$|\psi\rangle = \sum_{r_1 .. r_n} \Psi(r_1 .. r_n)\, |r_1 .. r_n\rangle$$

$\in \mathcal{H}_{B/F}$ even if $\Psi(r_1 .. r_n)$ is not (anti) symm.

$$\langle r_1' ... r_n' | \Psi\rangle = \sum_{r_1 .. r_n} \Psi(r_1 .. r_n)\, \langle r_1' .. r_n' | r_1 .. r_n\rangle$$

$$= \frac{1}{n!} \sum_{\pi} S^{\pi}\, \Psi(r_{\pi_1}' .. r_{\pi_n}').$$

if Ψ is (anti) symmetrized $= \Psi(r_1' .. r_n')$.

$$\langle \Psi | \Psi \rangle = \sum_{r_1 \ldots r_n} | \Psi(r_1 \ldots r_n) |^2$$

Given n-particle states Φ, Ψ.

$$\langle \Phi | \Psi \rangle = \sum_{r_1 \ldots r_n} \Phi^*(r_1 \ldots r_n) \Psi(r_1 \ldots r_n)$$

$$= \sum_r \langle \Phi | r_1 \ldots r_n \rangle \langle r_1 \ldots r_n | \Psi \rangle$$

$\underline{\forall \Phi}$ $\quad |\Psi\rangle = \sum_r |r_1 \ldots r_n\rangle\langle r_1 \ldots r_n | \Psi \rangle$

$\underline{\forall \psi:}$ $\quad \mathbb{1}_n = \sum_r |r_1 \ldots r_n\rangle\langle r_1 \ldots r_n|$

$$\mathbb{1}_n \, | \Phi_{n'} \rangle = \delta_{nn'} | \Phi_{n'} \rangle$$

$$\mathbb{1} = \sum_n \mathbb{1}_n = |0\rangle\langle 0| + \sum_{n \geq 1}^{\infty} \mathbb{1}_n \, .$$

vacum.

Operators on Fock space:

claim: $\rho(r) = \psi^\dagger(r) \psi(r)$ density of particle at r.

$$\langle \Phi_n | \rho(r) | \Phi_n \rangle = \langle \Phi_n | \psi^\dagger(r) \underset{\mathbb{1} = \mathbb{1}_{n-1}}{\mathbb{1}} \psi(r) | \Phi_n \rangle$$

$$= \sum_{r_1 \dots r_{n-1}} \langle \Phi_n | \underbrace{\psi^\dagger(r) | r_1 \dots r_{n-1} \rangle}_{\sqrt{n}\, | r_1 \dots r_{n-1}, r \rangle} \underbrace{\langle r_1 \dots r_{n-1} | \psi(r)}_{\langle r_1 \dots r_{n-1} r | \sqrt{n}} | \Phi_n \rangle$$

$$= n \sum_{r_1 \dots r_{n-1}} \langle \Phi_n | r_1 \dots r_{n-1} r \rangle \langle r_1 \dots r_{n-1} r | \Phi_n \rangle$$

$$= \sum_{r_1 \dots r_n} \langle \Phi_n | r_1 \dots r_n \rangle \langle r_1 \dots r_n | \Phi \rangle \underbrace{\sum_{i=1}^{n} \delta(r - r_i)}_{\rho(r)}$$

counts particle at r.

$$\rho_s(r) = \psi_s^\dagger(r) \psi_s(r)$$

$$\sum_s \rho_s(r) = \rho(r).$$

$$N = \sum_r \rho(r) = N = \sum_{ps} a_{ps}^\dagger a_{ps}.$$

$$K = \sum_{ps} a^{\dagger}_{ps} a_{ps} \frac{p^2}{2m}$$

$$a_{ps} = \sum_{r} \frac{e^{ipr}}{\sqrt{V}} \psi_s(r)^{\dagger} \qquad a^{\dagger}_{ps} = \sum_{r} \frac{e^{-ipr}}{\sqrt{V}} \psi_s(r)$$

$$K = \frac{1}{2mV} \sum_{rr'} \sum_{ps} \left(\vec{\nabla} e^{ipr}\right)\left(\vec{\nabla}' e^{-ipr'}\right) \psi_s(r)^{\dagger} \psi_s(r')$$

$$\boxed{\vec{p}\, e^{i\vec{p}\cdot\vec{r}} = -i \vec{\nabla} e^{i\vec{p}\cdot\vec{r}}}$$

$$\overset{\text{IBP}\times 2}{=} \frac{1}{2m} \sum_{rr's} \underbrace{\left(\frac{1}{V} \sum_{p} e^{ip(r-r')}\right)}_{\delta_{rr'}} \vec{\nabla}\psi^{\dagger}_s(r) \cdot \vec{\nabla}\psi_s(r')$$

$$= \sum_{r,s} \frac{\vec{\nabla}\hat{\psi}^{\dagger}_s(r) \cdot \vec{\nabla}\hat{\psi}_s(r)}{2m} .$$

Particle current: $\dot{\rho} + \vec{\nabla}\cdot\vec{j} = 0.$

$$\vec{j}(r) = \frac{1}{2mi}\left(\psi^{+}(r)\vec{\nabla}\psi(r) - (\vec{\nabla}\psi^{+}(r))\psi(r) \right)$$

spin density:

Pauli matrices.

$$\vec{S}(r) = \sum_{ss'} \psi_s^{+}(r)\, \frac{\vec{\sigma}_{ss'}}{2}\, \psi_{s'}(r)$$

$$[S_i(r), S_j(r')] = i\,\varepsilon_{ijk} S_k(r)\,\delta_{rr'}.$$

$i,j,k = x,y,z.$

"second quantization":

$$\langle\psi|\hat{K}|\psi\rangle = \sum_r \langle\psi| \frac{\hat{P}^2}{2m} |\psi\rangle$$

1-particle state

$$= \sum_r \frac{\vec{\nabla}\psi^{+}(r)\cdot\vec{\nabla}\psi(r)}{2m}.$$

1-body wavefunction

eg:

$$H_{free} = \sum_r \left(\frac{\vec\nabla \psi^\dagger(r) \vec\nabla \psi(r)}{2m} + \underbrace{\psi^\dagger(r)\psi(r)V(r)}_{\rho(r)\,V(r)} \right)$$

$$\longrightarrow H_1 = \frac{P^2}{2m} + V(r).$$

INTERACTIONS: $V^{(2)}(x_i, x_j)$

on 2-particle state

$$V^2 = \frac{1}{2} \sum_{x \neq y} |xy\rangle\langle xy|\, V^{(2)}(x,y)$$

want: $\hat{V}|r_1 \dots r_n\rangle = \frac{1}{2}\sum_{i \neq j} V^{(2)}(r_i, r_j)|r_1 \dots r_n\rangle$

$$\hat{V}_{guess} \stackrel{?}{=} \frac{1}{2}\sum_{x\,y} V^{(2)}(x,y)\, \rho(x)\rho(y)$$

almost:

$$\rho(x)\rho(y) = \hat{\psi}_x^\dagger \hat{\psi}_x \hat{\psi}_y^\dagger \hat{\psi}_y$$

$$= \zeta\, \hat{\psi}_x^\dagger \hat{\psi}_y^\dagger \hat{\psi}_x \hat{\psi}_y + \delta_{xy} \hat{\psi}_x^\dagger \hat{\psi}_y$$

$$= \psi_x^\dagger \psi_y^\dagger \psi_y \psi_x + \delta_{xy} \underline{\underline{\rho(x)}}.$$

$$\hat{V} = \hat{V}_{\text{given}} - \underline{\underline{\frac{1}{2}\sum_x V^{(2)}(x,x)\,\rho(x)}}$$

"self-energy"

normal-ordered: (creation ops) × (annihilation ops)

Check: $\psi_y \psi_x |r_1 \ldots r_n\rangle$

absent

$$= \psi_y \sum_{i=1}^{n} \zeta^{i-1} \delta_{x,r_i} |r_1 \ldots \hat{r}_i \ldots r_n\rangle$$

$$= \sum_{i=1}^{n} \zeta^{i-1} \delta_{x,r_i} \sum_{j\neq i}^{n} \delta_{y,r_j} \eta_{ji} |r_1 \ldots \hat{r}_i \ldots \hat{r}_j \ldots r_n\rangle$$

$$\eta_{ji} = \begin{cases} \zeta^{j-1} & j < i \\ \zeta^{j} & j > i \end{cases}$$

$$\hat{\psi}^+_x \hat{\psi}^+_y \hat{\psi}_y \hat{\psi}_x |r_1 \ldots r_n\rangle$$

$$= \sum_{j\neq i} \int^{i-1} \eta_{ji}\, \underline{\delta_{x r_i}}\, \underline{\delta_{y r_j}}\, |r_1 \ldots \hat{r}_i \ldots \hat{r}_j \ldots r_n\, y x\rangle$$

$$= \quad \ldots \quad |r_1 \ldots \hat{r}_i\, \hat{r}_j \ldots r_n\, r_j\, r_i\rangle$$

$$= \sum_{j\neq i} \delta_{x r_i} \delta_{y r_j} |r_1 \ldots r_n\rangle$$

$$\Rightarrow \hat{V}|r_1 \ldots r_n\rangle = \frac{1}{2} \sum_{i\neq j} V^{(2)}(r_i, r_j)\, |r_1 \ldots r_n\rangle$$

<u>Interaction</u>: $\quad = 0$ if $n < 2$.

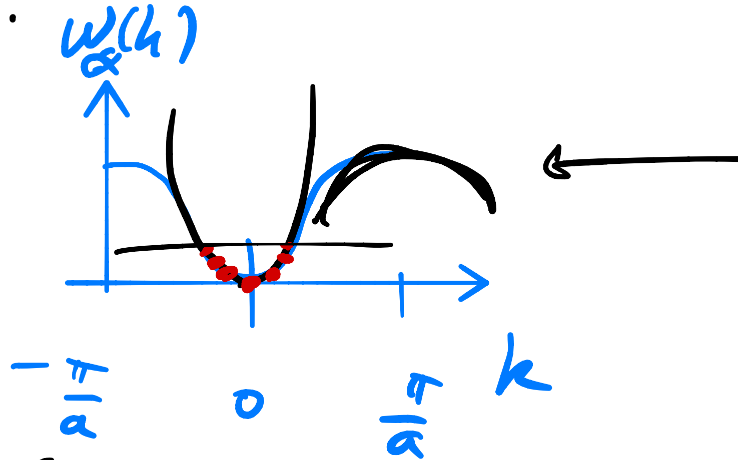
$$H = H_{\text{free}} + \lambda \hat{V} \,?$$

1.9 Many bosons vs Many fermions

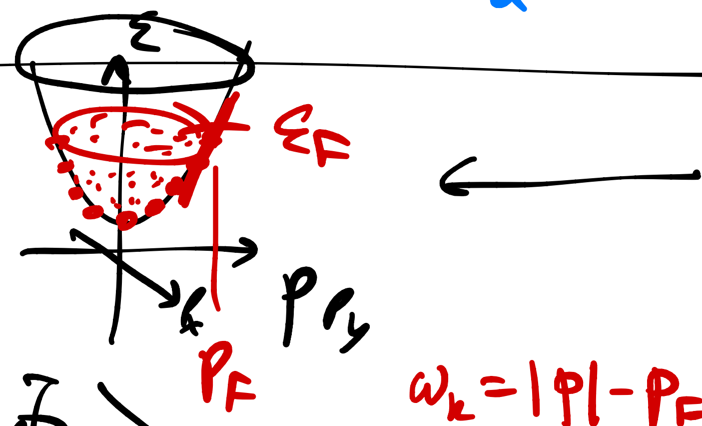
Groundstate of fermi gas

- $H = K$, $\zeta = -1$.
- $H = H_t + H_\epsilon$.

$$\omega(k) \overset{k \ll 1/a}{\simeq} \frac{k^2}{2m} + \mathcal{O}(k^4).$$



$|\Phi_0\rangle$



$$n_{ps} = \langle \Phi_0 | a^\dagger_{ps} a_{ps} | \Phi_0 \rangle$$

$$= \begin{cases} 1 & |p| < p_F \\ 0 & |p| > p_F \end{cases}.$$

$$N = \sum_{sp} n_{ps} = 2 \sum_{|p|<p_F} 1 \overset{V\to\infty}{=} 2V \int_0^{p_F} d^d p.$$

$$N \propto p_F^{\,d} V$$

$$N \overset{d=3}{=} \frac{p_F^3}{3\pi^2} \cdot V \implies p_F = 3\pi^2 \frac{N}{V} = 3\pi^2 n.$$

$$p_F \propto \left(\frac{N}{V}\right)^{1/d}.$$

THIS STATE IS WEIRD!

eg: fermi pressure.

$$E_0(V) = \langle \Phi_0 | H | \Phi_0 \rangle = \sum_{ps} \underbrace{\langle \Phi_0 | a^\dagger_{ps} a_{ps} | \Phi_0 \rangle}_{n_{ps}} \frac{p^2}{2m}$$

$$= \sum_s \sum_{p<p_F} \frac{p^2}{2m} \overset{V\to\infty}{=} 2V \int_{|p|<p_F} d^d p \, \frac{p^2}{2m}$$

$$\overset{d=3}{=} 2V \frac{4\pi}{(2\pi)^3} \int_0^{p_F} \frac{p^2}{2m} p^2 \, dp = \frac{p_F^2}{2m} \underbrace{\frac{p_F^3 V}{5\pi^2}}$$

$$\boxed{\varepsilon_F = p_F^2/2m.} \qquad = \frac{3}{5} \frac{p_F^2}{2m} N = \frac{3}{5} \varepsilon_F \cdot n$$

$$dE = T\,\cancelto{0}{dS} - P\,dV + \mu\,\cancelto{0}{dN}$$

<u>$T=0$, fixed N.</u> $\Rightarrow$ $P = -\left.\dfrac{\partial E_0}{\partial V}\right|_{N,\ T=0.}$

$$P = -\partial_V\Big|_N \left(\frac{3}{5} N \left(\frac{3\pi^2 N}{V}\right)^{2/3}\right)$$

$$= +\frac{3}{5}\cdot\frac{2}{3}\,(3\pi^2)^{2/3}\, N^{5/3}\cdot V^{-5/3}$$

$$= \frac{2}{3}\,\frac{E_0}{V}.$$

solid: