Physics 212C.

section: Jin-Long Huang
Th 4-5 pm.

my OH: Weds after Lecture.
or: send email anytime.

work: weekly HW
due Mondays.
(HW1 due April 6.)

submit via **canvas**.
pdf.
typesetting **encouraged**!

subject: • faster?
• **more**.

i.e. **many-body Q.M.**

Give Input!

1. An indefinite # of identical particles.

("second quantization".)
aka "quantum field theory"

($|n \text{ particle}\rangle + |n+1 \text{ particles}\rangle$ $\rightsquigarrow$ superfluids.)

1.1 SHO.

$$V(x) = V(x_0) + 0 + \frac{1}{2}V''(x_0)(x-x_0)^2 + \cancel{b(x-x_0)^3}$$

anharmonic terms.

WLOG $x_0 = 0$.

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

$$= \frac{\hbar\omega}{2}(P^2+Q^2)$$

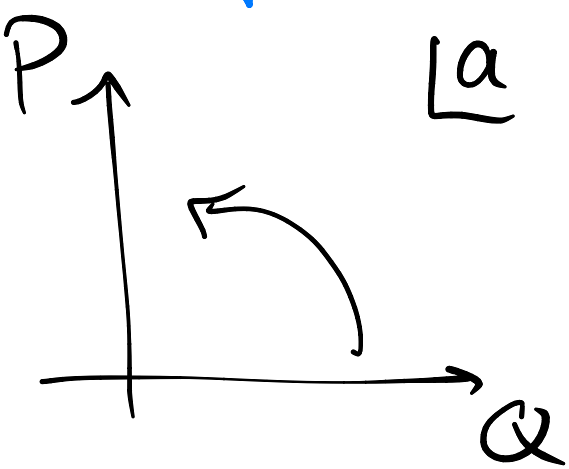
$$= \hbar\omega(a^\dagger a + 1/2)$$

$$Q \equiv \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$P \equiv \frac{1}{\sqrt{m\hbar\omega}}\, p$$

$$a = \frac{1}{\sqrt{2}}(Q+iP)$$

$$a^\dagger = \frac{1}{\sqrt{2}}(Q-iP).$$



$[x, p] = i\hbar \mathbb{1} \Rightarrow [a, a^\dagger] = \mathbb{1}$

$\hat{N} \equiv a^\dagger a \qquad H = \hbar\omega(\hat{N} + \frac{1}{2})$

$[N, a] = -a \qquad [N, a^\dagger] = a^\dagger .$

$a, a^\dagger$ are lowering & raising ops for N.

$\hat{N}|n\rangle = n|n\rangle. \qquad \hat{N}(a^\dagger|n\rangle) = (a^\dagger + a^\dagger\hat{N})|n\rangle$

$= (n+1)\, a^\dagger|n\rangle.$

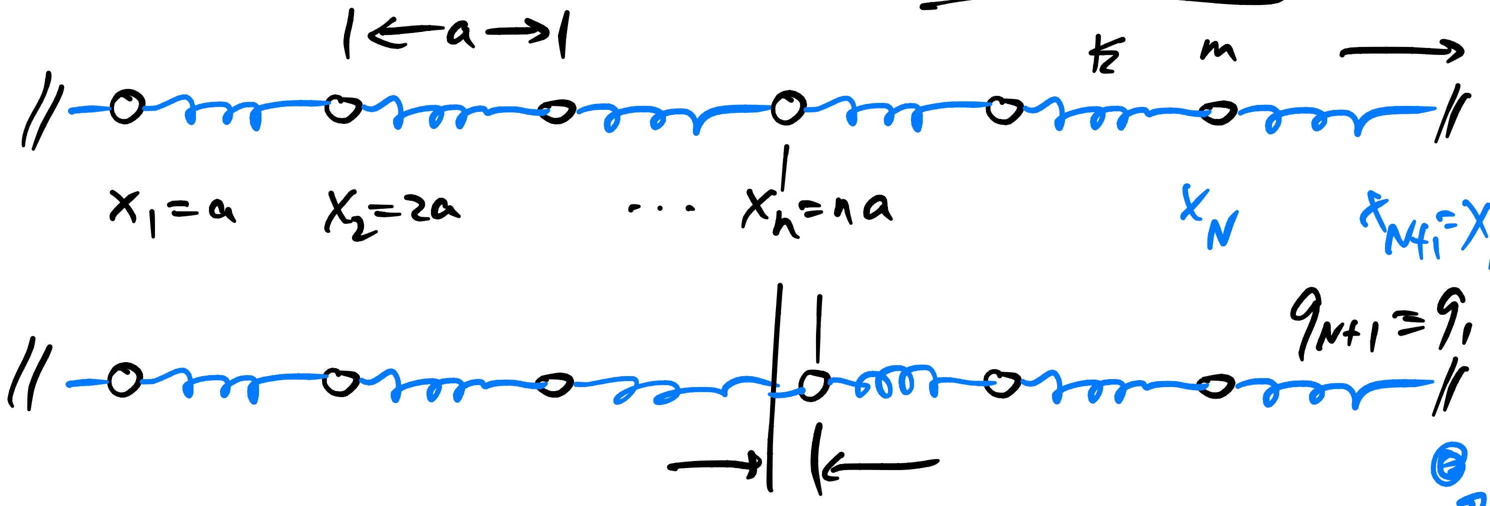
$$0 \le \| a|n\rangle \|^2 = \langle n|a^\dagger a|n\rangle = \langle n|N|n\rangle$$
$$= n \langle n|n\rangle$$

$\Rightarrow \underline{a|n=0\rangle = 0.}$

$|0\rangle, \; |1\rangle = a^\dagger|0\rangle, \; \cdots \; |n\rangle = c_n (a^\dagger)^n |0\rangle$

$1 = \langle n|n\rangle \Rightarrow c_n = \frac{1}{\sqrt{n!}}.$

1.2 Particles & fields. [& = and]



$$H = \sum_{n=1}^{N}\left[\frac{p_n^2}{2m} + \underbrace{\frac{k}{2}(q_n - q_{n-1})^2}_{k_{ab}\, q_a q_b}\right] + \cancel{\lambda q^4}$$

observation: $T : \begin{Bmatrix} q_1 q_2 \cdots q_{N-1} q_N \\ p_1 \cdots p_{N-1} p_N \end{Bmatrix} \longrightarrow \begin{Bmatrix} q_2 q_3 \cdots q_N q_1 \\ p_2 p_3 \cdots p_N p_1 \end{Bmatrix}$

is a symmetry.

$$[H, T] = 0.$$

$$[k, T] = 0.$$

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \\ & & & & \end{pmatrix}$$

$$k_{ab} T_{bc} = T_{ab} k_{bc}$$

$$T(x) = x + a.$$

$$T e^{ikx} = e^{ik(x+a)} = \underbrace{e^{ika}}\, e^{ikx}$$

Slogan: linear & transl. inv't

$\Longrightarrow$ solvable by Fourier.

Normal modes are: $x_n \equiv an$

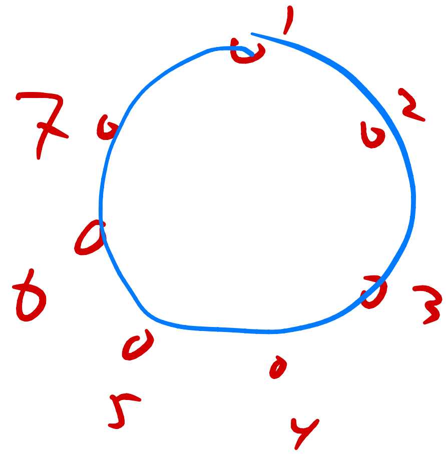
$$\begin{cases} \tilde{q}_k = \dfrac{1}{\sqrt{N}} \displaystyle\sum_{n=1}^{N} e^{ikx_n} q_n \equiv \sum_{n=1}^{N} \underbrace{U_{kn}} q_n \\ \tilde{p}_k = \dfrac{1}{\sqrt{N}} \displaystyle\sum_{n=1}^{N} e^{ikx_n} p_n \end{cases}$$

$N \times N$ unitary matrix.

auxiliary QM problem:

$$\mathcal{H} = \mathrm{span}\{ |n\rangle, \; |n+N\rangle = |n\rangle \}$$

pos'n of a particle on a discrete ring.



$$\hat{T}|n\rangle = |n+1\rangle$$

$$\hat{T} = \sum_{n=1}^{N} |n+1\rangle\langle n|$$

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{ikna} |n\rangle$$

$$\hat{T}|k\rangle = e^{ikan} |k\rangle.$$

momentum eigenstate.

Regulators: N is finite (IR)

$q_n \overset{!}{=} q_{n+N} \Rightarrow 1 = e^{ikNa} \Rightarrow k = \frac{2\pi}{Na} j, \quad j \in \mathbb{Z}.$

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a is finite (lattice) (UV)

only e^{ikan} appears $\Rightarrow k \simeq k + \frac{2\pi}{a}$.
$n \in \mathbb{Z}$.

Independent $k \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right)$.

(Brillouin zone)

$\Rightarrow$ N values of k. $\boxed{k_j = \frac{2\pi}{Na} j \quad j = 0 \ldots N-1}$

$$\boxed{U_{nk} = \frac{1}{\sqrt{N}} e^{ikna}}$$ is an $N \times N$ unitary.

$$\sum_k^N U_{nk} U^\dagger_{kn'} = \delta_{nn'} \qquad U_{kn} = \frac{1}{\sqrt{N}} e^{-ikna}$$

$$\sum_n^N U_{kn} U^\dagger_{nk'} = \delta_{kk'}.$$

$$= \sum_n U_{kn} U^*_{k'n} = \frac{1}{N} \sum_{n=1}^N e^{i(k'-k)na}$$

claim: $= \delta_{jj'}$

$$\underline{\underline{\sum_n p_n^2}} = \sum_n \sum_{kk'} U_{nk} p_k \, U_{nk'} p_{k'}$$

$$= \sum_{kk'} \underbrace{\sum_n (U^*_{k'n})^\dagger U_{nk}}_{\textstyle = \sum_n \left(u^+_{-k'n} \, u_{nk}\right) = \delta_{-k',k}} p_k p_{k'}$$

$$u^*_{k'n} = u_{-k'n}$$

$$= \underline{\underline{\sum_k p_k p_{-k}}} \, .$$

$$\sum_n (q_{\underline{\underline{n+1}}} - q_n)^2 = \sum_n \left((T-1)\underline{q_n}\right)^2$$

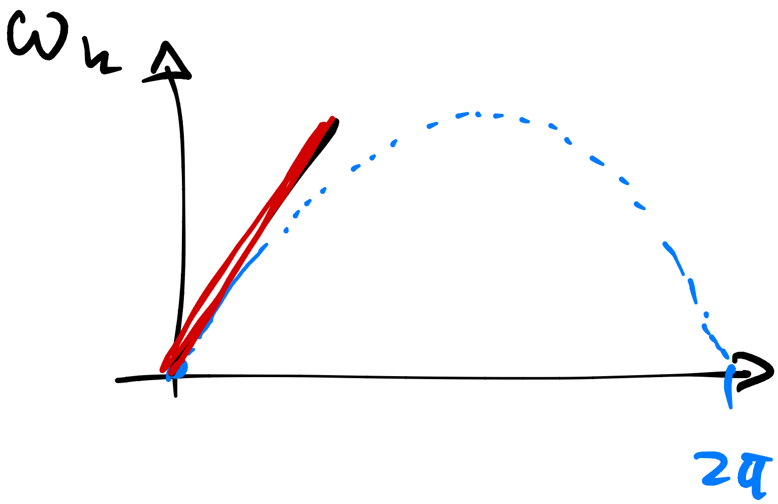
$$\underline{\sum_n} \sum_{kk'} \left(e^{-ikna} - 1\right)\left(e^{-ik'na} - 1\right) q_k q_{k'}$$

$$\underline{U_{-kn} \cdot U_{-k'n}} \rightarrow \delta_{k,-k'}$$

$$= \sum_k \underbrace{4 \sin^2 \frac{ka}{2}} \, q_k q_{-k} \, .$$

$$H = \sum_k \left[\frac{p_k p_{-k}}{2m} + \frac{1}{2} m \omega_k^2 q_k q_{-k} \right]$$

$$\omega_k = 2\sqrt{\frac{K}{m}} \sin\frac{|k|a}{2} \overset{k \ll 1/a}{\simeq} 2\sqrt{\frac{K}{m}} \frac{a}{2} |k| + O(k^3)$$



← dispersion rel'n.

$$i\partial_t q_n = [q_n, H] = \frac{p_n}{m} \qquad i\partial_t p_n = [p_n, H]$$

$$\Rightarrow m \ddot{q}_n = -K(2q_n - q_{n-1} - q_{n+1})$$

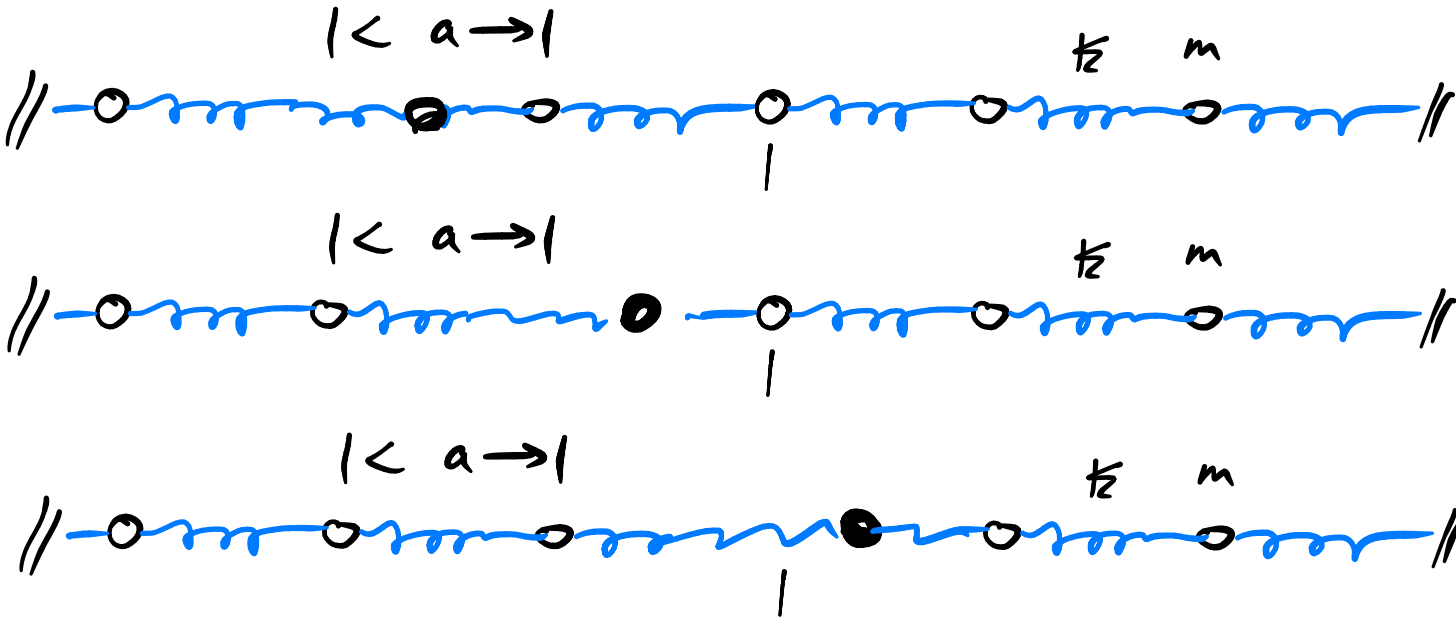
$$\Rightarrow m \ddot{q}_k = -K(2 - 2\cos ka) q_k \overset{\text{trig}}{=} -K 2 \sin |k|a/2 \; q_k .$$

$$q_k(t) = \sum_\omega e^{-i\omega t} q_{k\omega}$$

$$\Rightarrow 0 = (\omega^2 - \omega_k^2) q_{k\omega} \overset{k \ll 1/a}{\simeq} (\omega^2 - v_s^2 k^2) q_{k\omega} + \ldots$$

$$v_s = \frac{\partial \omega_k}{\partial k}\Big|_{k\to 0} = a\sqrt{\frac{k}{m}}\,.$$

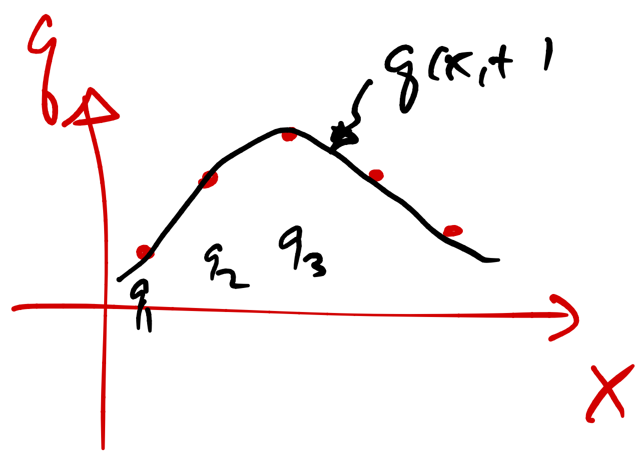
$$\Longleftrightarrow \quad (\partial_t^2 - v_s^2\,\partial_x^2)\, q(x,t) = 0. \qquad \text{wave } \underline{\text{eqn}}.$$

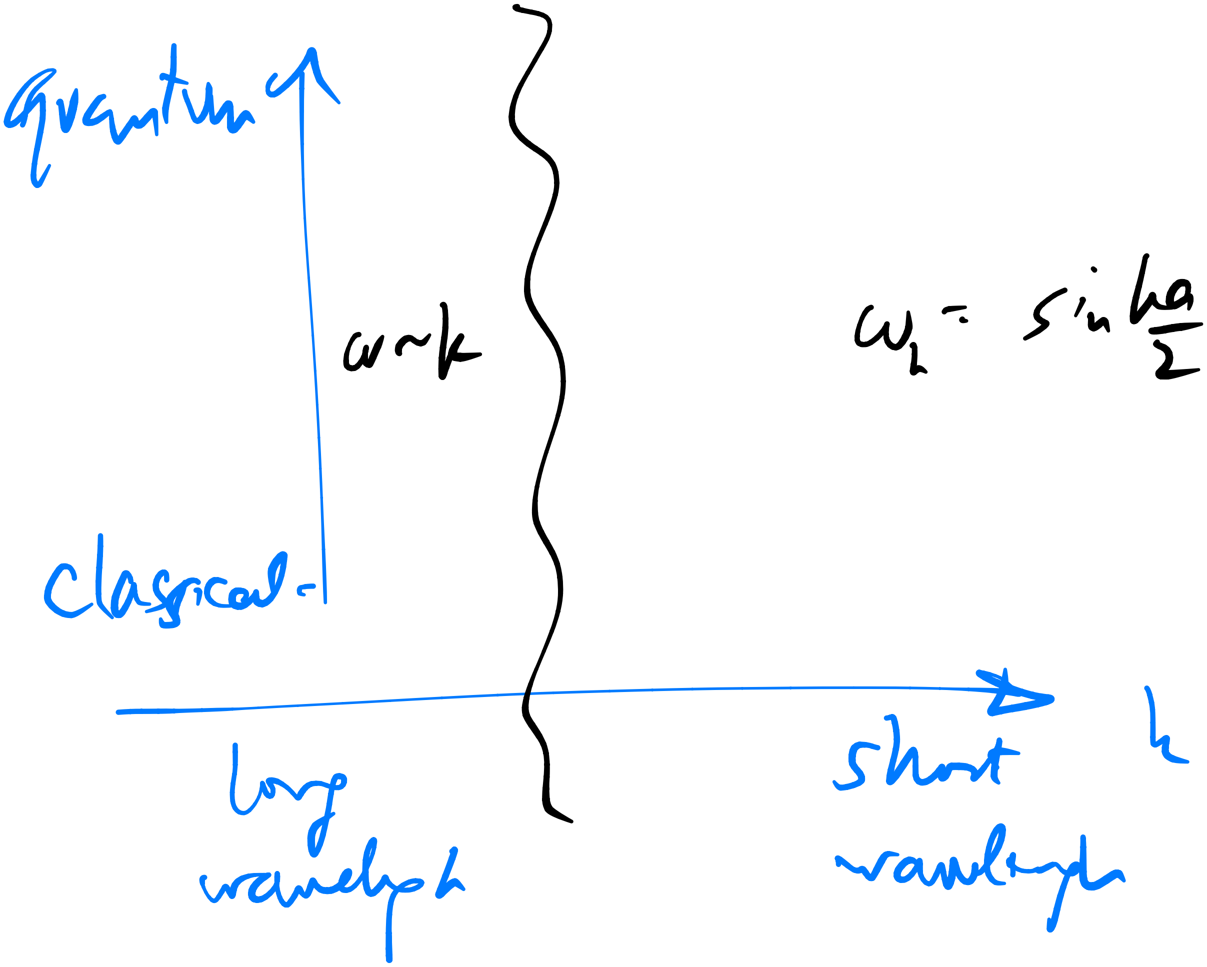


<u>Sound wave</u>. $\longrightarrow$ quantum sound wave $\equiv$ <u>phonon</u>.

<u>Who is $q(x,t)$?</u>

<u>a field</u>.




quantum
classical
$\omega \sim k$
$\omega_k = \sin\frac{ka}{2}$
k
long
wavelength
short
wavelength