University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2020
Assignment 9

Due 12:30pm Wednesday, June 3, 2020

Please refer to the first homework for submission format and procedures (and replace hw01 by hw09 in the relevant places, of course).

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1. **Peierls' instability.**

   On a previous homework, we studied a Hamiltonian describing (spinless) fermions hopping on a chain:

$$H = -t \sum_n (1 + u_n) c_n^\dagger c_{n+1} + h.c.$$

   Consider an extension of the model to include also *phonon* modes, *i.e.* degrees of freedom encoding the positions of the ions in the solid. (Again we ignore the spins of the electrons for simplicity.)

$$H = -t \sum_n (1 + u_n) c_n^\dagger c_{n+1} + h.c. + \sum_n K (u_n - u_{n+1})^2 \equiv H_F + H_E.$$

   Here u_n is the deviation of the nth ion from its equilibrium position (in the $+x$ direction), so the second term represents an elastic energy.

   (a) Consider a configuration

$$u_n = \phi(-1)^n \tag{1}$$

   where the ions move closer in pairs. Compute the single-particle electronic spectrum. (Hint: this enlarges the unit cell, making a fermionic analog of the problem of phonons in salt. Define $c_{2n} \equiv a_n, c_{2n+1} \equiv b_n$, and solve in Fourier space, $a_n \equiv \oint đk e^{2\mathbf{i}kn} a_k$ *etc.*) You should find that when $\phi \neq 0$ there is a gap in the electron spectrum (unlike $\phi = 0$).

   (b) Compute the many-body groundstate energy of H_F in the configuration (1), at half-filling (*i.e.* the number of electrons is half the number of available states).

   Compute H_E in this configuration, and minimize (graphically) the sum of the two as a function of ϕ.

(c) [Bonus problem: emergence of the Dirac equation] We can take a continuum limit of the above results. First, show that the low-energy excitations of $\mathbf{H}_0$ at a generic value of the filling are described the the massless Dirac lagrangian in 1+1 dimensions. Find an explicit choice of $1+1$-d gamma matrices which matches the answer from the lattice model. Show that the right-movers are right-handed $\gamma^5 \equiv \gamma^0\gamma^1 = 1$ and the left-movers are left-handed.

Next, include the coupling to phonons. Expand the spectrum near the minimum gap and include the effects of the field ϕ in the continuum theory.

(d) [Bonus problem] You should find that the energy is independent of the *sign* of ϕ. This means that there are two groundstates. We can consider a domain wall between a region of $+$ and a region of $-$. Show that this domain wall carries a fermion mode whose energy lies in the bandgap and has fermion number $\pm\frac{1}{2}$.

(e) [Bonus problem] Diagonalize the relevant tight-binding matrix and find the mid-gap fermion mode.

(f) Time-reversal played an important role here. If we allow complex hopping amplitudes, we can make a domain wall without midgap modes.

2. **Particle conservation and the f-sum rule.** Consider a collection of N particles (bosons or fermions) governed by a Hamiltonian of the form

$$\mathbf{H} = \sum_i \frac{\mathbf{p}_i^2}{2m} + \sum_{i<j} V(r_{ij}).$$

Recall that the operator

$$\rho_q = \sum_i e^{-\mathbf{i}q\cdot\mathbf{r_i}}$$

is the Fourier transform of the particle density $\rho(r) = \sum_i \delta^d(r - \mathbf{r_i})$, where $\mathbf{r_i}$ is the position of the ith particle.

(a) Find

$$\partial_t \rho_q = -\mathbf{i}[\rho_q, \mathbf{H}]$$

and show that it can be written in the form

$$[\rho_q, \mathbf{H}] = \vec{q} \cdot \vec{\mathbf{J}}_q$$

where

$$\vec{\mathbf{J}}_q = \frac{1}{2}\sum_i \left(\frac{\vec{\mathbf{p}}_i}{m} e^{-\mathbf{i}q\cdot\mathbf{r_i}} + e^{-\mathbf{i}q\cdot\mathbf{r_i}} \frac{\vec{\mathbf{p}}_i}{m} \right). \qquad (2)$$

Interpret this as a continuity equation.

(b) For later use, compute $[\vec{\mathbf{J}}_q, \rho_q^\dagger]$.

(c) Consider the object

$$\langle\Phi_0| [[\rho_q, \mathbf{H}], \rho_q^\dagger] |\Phi_0\rangle$$

where Φ_0 is the groundstate. Compute this by inserting a resolution of the identity $\mathbb{1} = \sum_n |\Phi_n\rangle\langle\Phi_n|$ in terms of energy eigenstates $\mathbf{H}|\Phi_n\rangle = (E_0 + \omega_n)|\Phi_n\rangle$ and show that it is equal to

$$\langle\Phi_0| [[\rho_q, \mathbf{H}], \rho_q^\dagger] |\Phi_0\rangle = 2\int d\omega S(q,\omega)$$

where

$$S(q,\omega) = \sum_n \delta(\omega - \omega_n) |\langle\Phi_n| \rho_q |\Phi_0\rangle|^2 \tag{3}$$

is the dynamical structure factor.

(d) Conclude (by combining the previous parts) that the f-sum rule

$$\int d\omega S(q,\omega) = \frac{N\hbar^2 q^2}{2m}.$$

is true.

3. **The role of the Fermi surface in the Cooper problem.** Redo the analysis of the Cooper problem, but without the Fermi surface. That is, set $k_F = 0$. Show that in this case the eigenvalue problem can be written as $-\frac{1}{v_0} = \Phi(E)$ with (in the thermodynamic limit, in $d = 3$ and at $K = 0$)

$$\Phi(E \equiv -2\Delta) = \frac{m}{\pi^2}\left(-k_a + 2\sqrt{m\Delta}\arctan\frac{k_a}{2\sqrt{m\Delta}}\right).$$

(Here k_a is the maximum k which participates in the interaction.) Show that a minimum interaction strength $|v_0|$ is required for a boundstate to form (in contrast to the case with $k_F \neq 0$).