University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2020
Assignment 7

Due 12:30pm Monday, May 18, 2020

Please refer to the first homework for submission format and procedures (and replace hw01 by hw07 in the relevant places, of course).

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1. **Brain-warmer.** Suppose we have a wavefunction Ψ of N bosons on a thin ring of radius R, governed by a Hamiltonian of the form

$$\mathbf{H} = \sum_i \frac{p_i^2}{2m} + \sum_{i<j} V(|r_{ij}|).$$

Let θ_i be the angular coordinate of the ith particle. Now suppose we make a new state $\Psi' = e^{-\mathbf{i}\sum_i \theta_i}\Psi$. Show that

$$\langle \Psi' | \mathbf{H} | \Psi' \rangle = \langle \Psi | \mathbf{H} | \Psi \rangle - \omega_c L + \frac{1}{2} I_{cl} \omega_c^2$$

where L is the expected angular momentum of the state $|\Psi\rangle$, $\omega_c \equiv \frac{\hbar}{mR^2}$, $I_{cl} = NmR^2$.

2. **Density matrix and correlation functions.** Recall the single-particle density matrix in the state $\rho = \sum_s p_s |\Psi_s\rangle\langle\Psi_s|$, as defined in lecture,

$$\rho_1(r, r') \equiv \sum_s p_s \Psi^\star(r, r_2, \cdots r_N)\Psi(r', r_2, \cdots r_N).$$

This can be defined for either bosons or fermions.

(a) Check that ρ_1 is proportional to the two-point correlation function

$$\rho_1(r, r') \propto \mathrm{tr}\rho\psi^\dagger(r)\psi(r') \equiv \left\langle \psi^\dagger(r)\psi(r') \right\rangle$$

and find the proportionality constant. Check that it works for both bosons and fermions.

(b) [Bonus question] Prove that for a fermionic state the eigenvalues of $\rho_1(r, r')$ are between 0 and 1.

[Hint: for fermions, the expectation value of the number operator $\psi^\dagger(r)\psi(r)$ in any state is ≤ 1.]

3. **Topological terms, particle on a ring.**

The purpose of this problem is to demonstrate that total derivative terms in the action do affect the physics quantum mechanically.

The euclidean path integral for a particle on a ring with magnetic flux $\theta = \int \vec{B} \cdot \mathrm{d}\vec{a}$ through the ring is given by

$$Z = \int [D\phi] e^{-\int_0^\beta \mathrm{d}\tau \left(\frac{m}{2}\dot{\phi}^2 - \mathbf{i}\frac{\theta}{2\pi}\dot{\phi} \right)} \ .$$

Here

$$\phi \equiv \phi + 2\pi \tag{1}$$

is a coordinate on the ring. Because of the identification (1), ϕ need not be a single-valued function of τ – it can wind around the ring. On the other hand, $\dot{\phi}$ is single-valued and periodic and hence has an ordinary Fourier decomposition. This means that we can expand the field as

$$\phi(\tau) = \frac{2\pi}{\beta} Q\tau + \sum_{\ell \in \mathbb{Z} \setminus 0} \phi_\ell e^{\mathbf{i}\frac{2\pi}{\beta}\ell\tau}. \tag{2}$$

(a) Show that the $\dot{\phi}$ term in the action does not affect the classical equations of motion. In this sense, it is a topological term.

(b) Using the decomposition (2), write the partition function as a sum over topological sectors labelled by the *winding number* $Q \in \mathbb{Z}$ and calculate it explicitly.

[Hint: use the Poisson resummation formula

$$\sum_n f(n) = \sum_l \hat{f}(2\pi l)$$

where $\hat{f}(p) = \int dx e^{-\mathbf{i}px} f(x)$ is the fourier transform of f.]

(c) Use the result from the previous part to determine the energy spectrum as a function of θ.

(d) Derive the canonical momentum and Hamiltonian from the action above and verify the spectrum.

4. **Excitations and energy of a weakly-interacting bose fluid.**

We can see the appearance of the sound mode in the continuum by the following treatment, where we regard the interaction strength as weak.

(a) Write the Hamiltonian

$$\mathbf{H} = \sum_p \mathbf{b}_p^\dagger \mathbf{b}_p \frac{p^2}{2m} + \int d^d r \int d^d r' V(|r - r'|) \mathbf{b}_r^\dagger \mathbf{b}_{r'}^\dagger \mathbf{b}_{r'} \mathbf{b}_r$$

entirely in terms of the momentum-space operators $\mathbf{b}_p, \mathbf{b}_p^\dagger$. After doing this, specialize to the case of contact interactions, *i.e.* $V(r_{ij}) = U\delta^d(r_{ij})$.

(b) Here is the trick. In the BEC state, $\mathbf{b}_0^\dagger \mathbf{b}_0 = N_0 \sim N$. We will approximate such a state as a coherent state for $\mathbf{b}_0$ with eigenvalue $b_0 \sim \sqrt{N} \gg 1$, a complex number. Since the commutator $[\mathbf{b}_0^\dagger, \mathbf{b}] = 1 \ll N$ we make a small error by this approximation. We will treat all the other creation and annihilation operators $\mathbf{b}_{p\neq 0}$ as operators, but as *small*. That is, $\mathbf{b}_0 \sim b_0 = \mathcal{O}(N), \mathbf{b}_p = \mathcal{O}(N^0)$.

Expand the Hamiltonian keeping only the terms of order N and larger.

To eliminate b_0, use the fact that the total number of particles is

$$N = |b_0|^2 + \sum_{p\neq 0} \mathbf{b}_p^\dagger \mathbf{b}_p$$

to write it in terms of $\mathbf{b}_{p\neq 0}, \mathbf{b}_{p\neq 0}^\dagger$.

(c) In the previous part you should find a hamiltonian which is quadratic in the $\mathbf{b}_{p\neq 0}, \mathbf{b}_{p\neq 0}^\dagger$. This is an Easy Problem (according to our classification). One way to solve it is to substitute

$$\mathbf{b}_p^\dagger = u_p \mathbf{a}_p + v_p \mathbf{a}_{-p}^\dagger, \quad \mathbf{b}_p^\dagger = u_p \mathbf{a}_p^\dagger + v_p \mathbf{a}_{-p}$$

where the coefficients u_p, v_p are to be determined and can be assumed to be real, and even functions of p.

Show that demanding that $\mathbf{a}_p, \mathbf{a}_p^\dagger$ satisfy canonical boson commutation relations implies $u_p^2 - v_p^2 = 1$. Find $\mathbf{a}_p$ in terms of $\mathbf{b}_p, \mathbf{b}_p^\dagger$.

(d) To determine u_p, v_p (or α_p in $u_p = \cosh \alpha_p$) demand that the off-diagonal terms drop out of the Hamiltonian

$$\mathbf{H} = \sum_{p\neq 0} \epsilon_p \mathbf{a}_p^\dagger \mathbf{a}_p + C.$$

[Hint: to solve this condition, I recommend making the substitution $u_p = \frac{1}{\sqrt{A_p^2 - 1}}$, and solving for A_p.]

Find the resulting energy spectrum ϵ_p and [bonus problem] the c-number term C.

(e) Expand ϵ_p about $p = 0$ and find a phonon mode. Determine the speed of sound. What does ϵ_p do at large p?

(f) [bonus problem] How does the groundstate energy density E/V depend on the density $n = N/V$ and the scattering length $a \equiv \frac{mU}{4\pi}$?