University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2020
Assignment 6

Due 12:30pm Monday, May 11, 2020

Please refer to the first homework for submission format and procedures (and replace hw01 by hw06 in the relevant places, of course).

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1. **Interacting particles on a very small lattice.**

   Consider the Hamiltonian

$$\mathbf{H} = -t \sum_{i=1}^{N} \left(\mathbf{a}_i^\dagger \mathbf{a}_{i+1} + \mathbf{a}_{i+1}^\dagger \mathbf{a}_i \right) + V \sum_i \mathbf{n}_i \mathbf{n}_{i+1}$$

   describing particles on a circular chain ($\mathbf{a}_{i+N} = \mathbf{a}_i$). Here $\mathbf{n}_i \equiv \mathbf{a}_i^\dagger \mathbf{a}_i$. Assume $t, V > 0$.

   (a) Suppose that the operators $\mathbf{a}$ are fermionic ($\{\mathbf{a}_i, \mathbf{a}_j\} = \delta_{ij}$). Suppose there are only three (N=3) sites. Write the matrix form of the Hamiltonian acting on the sector with exactly two fermions. Beware of signs. Find its eigenvalues and eigenvectors. Feel free to use some software (*e.g.* Mathematica or Sympy). Compare to the case with exactly one fermion.

   (b) Consider general N sites and exactly $N-1$ particles. Again compare to the case of a single particle.

   (c) Consider again $N = 3$ and exactly two particles, but now suppose that the particles are bosons. Write down the matrix representation of the Hamiltonian in this case. Plot the spectrum as a function of V/t.

2. **Brain-warmer: Spin rotations.** The goal of this problem is to check that we get the same result for mean field theory of the Transverse Field Ising Model as we did from the variational perspective.

   (a) Show that

$$\mathbf{H}(\theta) \equiv -K \sum_i (\sin\theta \mathbf{X}_i + \cos\theta \mathbf{Z}_i) = -K \mathbf{U} \sum_i \mathbf{Z}_i \mathbf{U}^\dagger$$

   where

$$\mathbf{U} = e^{-\mathbf{i}\theta \sum_i \mathbf{Y}_i}.$$

   This is a global rotation about the y-axis.

(b) Conclude that the groundstate of $\mathbf{H}(\theta)$ is

$$|\theta\rangle \equiv \mathbf{U} \otimes_i |\uparrow\rangle_i .$$

(c) Compute $m = \langle\theta| \mathbf{Z}_i |\theta\rangle$.

(d) Impose the self-consistency condition that m is the expectation value used to determine the mean field in

$$\mathbf{H}_{\text{TFIM}} \simeq \mathbf{H}_{\text{MFT}} = -J\sum_i g\mathbf{X}_i - \sum_i \mathbf{Z}_i \left(\frac{1}{2}\sum_{\text{neighbors } j \text{ of } i} \langle \mathbf{Z}_j\rangle\right) = -J\sum_i \left(g\mathbf{X}_i - \frac{1}{2}zm\mathbf{Z}_i\right).$$

3. **Two coupled spins.**

This is a very useful warmup for the next problem. Consider a four-state system consisting of two qbits,

$$\mathcal{H} = \text{span}\{|\epsilon_1\rangle \otimes |\epsilon_2\rangle \equiv |\epsilon_1\epsilon_2\rangle, \epsilon = \uparrow_z, \downarrow_z\}.$$

(a) For each qbit, define $\boldsymbol{\sigma}^\pm \equiv \frac{1}{2}(\boldsymbol{\sigma}^x \pm \mathbf{i}\boldsymbol{\sigma}^y)$. (These are raising and lowering operators for $\boldsymbol{\sigma}^z$: $[\boldsymbol{\sigma}^z, \boldsymbol{\sigma}^\pm] = \pm 2\boldsymbol{\sigma}^\pm$. Check this.)

Show that

$$\vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2 = 2\left(\boldsymbol{\sigma}_1^+\boldsymbol{\sigma}_2^- + \boldsymbol{\sigma}_1^-\boldsymbol{\sigma}_2^+\right) + \boldsymbol{\sigma}_1^z\boldsymbol{\sigma}_2^z.$$

Here, by for example $\boldsymbol{\sigma}_1^x$ I mean the operator $\boldsymbol{\sigma}^x \otimes \mathbb{1}$ which acts as

$$\boldsymbol{\sigma}^x \otimes \mathbb{1}\,|\uparrow \epsilon_2\rangle = |\downarrow \epsilon_2\rangle, \quad \boldsymbol{\sigma}^x \otimes \mathbb{1}\,|\downarrow \epsilon_2\rangle = |\uparrow \epsilon_2\rangle .$$

(b) Determine the action of the operator $\vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2$ on the basis states

$$|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle .$$

(c) Show that the four vectors

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right), \quad |1,1\rangle \equiv |\uparrow\uparrow\rangle, \quad |1,0\rangle \equiv \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right), \quad |1,-1\rangle \equiv |\downarrow\downarrow\rangle$$

are orthonormal and are eigenvectors of $\vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2$ with eigenvalues 1 or -3.

(d) Show that they are also eigenvectors of $\mathbf{J}^2 \equiv (\vec{\boldsymbol{\sigma}}_1 + \vec{\boldsymbol{\sigma}}_2)^2$ and $\mathbf{J}^z \equiv \boldsymbol{\sigma}_1^z + \boldsymbol{\sigma}_2^z$ and find their eigenvalues.

(e) Consider the operator

$$\mathcal{P}_{1,2} \equiv \frac{1}{2}\left(\mathbb{1} + \vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2\right)$$

acting on the two spins. Show that $\mathcal{P}_{1,2}$ acts by exchanging the states of the two spins:

$$\mathcal{P}_{1,2}\,|\epsilon_1\epsilon_2\rangle = |\epsilon_2\epsilon_1\rangle \ .$$

(f) Show that the operator

$$Q_{1,2} \equiv \frac{1}{4}\left(\mathbb{1} - \vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2\right)$$

acts as a projector onto the (singlet) state $|0,0\rangle$.

4. **Spin chains and spin waves.**

A one-dimensional ($\mathsf{SU}(2)$-symmetric) *ferromagnet* can be represented as a chain of N qbits (spin-1/2 particles) numbered $n = 0, \ldots N-1, N \gg 1$, fixed along a line with a spacing ℓ between each successive pair. It is convenient to use periodic boundary conditions, where the Nth spin is identified with the 0th spin: $n + N \equiv n$. Suppose that each spin interacts only with its two nearest neighbors, so the Hamiltonian can be written as

$$\mathbf{H} = \frac{1}{2} N J \mathbb{1} - \frac{1}{2} J \sum_{n=0}^{N-1} \vec{\boldsymbol{\sigma}}_n \cdot \vec{\boldsymbol{\sigma}}_{n+1} \ .$$

where J is a *coupling constant* determining the strength of the interactions.

(a) Show that all eigenvalues E of $\mathbf{H}$ are non-negative, and that the minimum energy E_0 (the *ground state*) is obtained in the state where all the spins point in the same direction. A possible choice for the ground state $|\Phi_0\rangle$ is then

$$|\Phi_0\rangle = |{\uparrow_z}\rangle_{n=0} \otimes |{\uparrow_z}\rangle_{n=1} \otimes \ldots \otimes |{\uparrow_z}\rangle_{N-1} \equiv |{\uparrow\uparrow} \ldots {\uparrow}\rangle \, .$$

(b) Show that any state obtained from $|\Phi_0\rangle$ by rotating each of the spins by the same angle is also a possible ground state.

[Hint: the generator of spin rotations $\vec{\mathbf{J}} \equiv \sum_n \vec{\boldsymbol{\sigma}}_n$ commutes with the Hamiltonian.]

[Cultural remark: the phenomenon of a ground state which does not preserve a symmetry of the Hamiltonian is called *spontaneous symmetry breaking.*]

(c) Now we wish to find the low-energy excitations above the ground state $|\Phi_0\rangle$. Show that $\mathbf{H}$ can be written

$$\mathbf{H} = N J \mathbb{1} - J \sum_{n=0}^{N-1} \mathcal{P}_{n,n+1} = J \sum_{n=0}^{N-1} \left(\mathbb{1} - \mathcal{P}_{n,n+1}\right).$$

where

$$\mathcal{P}_{n,n+1} \equiv \frac{1}{2}\left(\mathbb{1} + \vec{\boldsymbol{\sigma}}_n \cdot \vec{\boldsymbol{\sigma}}_{n+1}\right) \ .$$

Using the result of the problem 3, show that the eigenvectors of $\mathbf{H}$ are linear combinations of vectors in which the number of up spins minus the number of down spins is fixed. Let $|\Psi_n\rangle$ be the state in which the spin n is down with all the other spins up. What is the action of $\mathbf{H}$ on $|\Psi_n\rangle$?

(d) We are going to construct eigenvectors $|k_s\rangle$ of $\mathbf{H}$ out of linear combinations of the $|\Psi_n\rangle$. Let

$$|k_s\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{\mathbf{i} k_s n \ell} |\Psi_n\rangle$$

with

$$k_s = \frac{2\pi s}{N\ell}, \quad s = 0, 1, ...N-1 \ .$$

Show that $|k_s\rangle$ is an eigenvector of $\mathbf{H}$ and determine the energy eigenvalue E_k. Show that the energy is proportional to k_s^2 as $k_s \to 0$. This state describes an elementary excitation called a *spin wave* or *magnon* with wave-vector k_s.

5. **Jordan-Wigner solution of the TFIM chain.** [Bonus problem. If you want to hand in your solution to this problem with HW07, that's fine.]

Let's look at the TFIM again:

$$\mathbf{H}_{\text{TFIM}} = -J \sum_j \left(g\mathbf{X}_j + \mathbf{Z}_j \mathbf{Z}_{j+1} \right)$$

has a phase transition between large-g and small-g phases.

(a) Verify the following statements.

(Disordered) large g: excitations are created by $\mathbf{Z}_j$ – they are spin flips. The groundstate is a condensate of domain walls: $\langle \boldsymbol{\tau}^z \rangle \neq 0$. Here $\boldsymbol{\tau}^z_{\bar{j}} \equiv \prod_{j>\bar{j}} \mathbf{X}_j$ is the operator which creates a domain wall between sites j and $j+1$.

(Ordered) small g: excitations are created by the 'disorder' operator $\boldsymbol{\tau}^z_{\bar{j}}$ – they are domain walls. The groundstate is a condensate of spins $\langle \mathbf{Z}_j \rangle \neq 0$, *i.e.* a ferromagnet.

So we understand what are the 'correct variables' (in the sense that they create the elementary excitations above the groundstate) at large and small g. I claim that the Correct Variables *everywhere* in the phase diagram are obtained by "attaching a spin to a domain wall". These words mean the following: let

$$\begin{aligned} \boldsymbol{\chi}_j &\equiv \mathbf{Z}_j \boldsymbol{\tau}^z_{j+\frac{1}{2}} = \mathbf{Z}_j \prod_{j'>j} \mathbf{X}_{j'} \\ \tilde{\boldsymbol{\chi}}_j &\equiv \mathbf{Y}_j \boldsymbol{\tau}^z_{j+\frac{1}{2}} = -\mathbf{i} \mathbf{Z}_j \prod_{j' \geq j} \mathbf{X}_{j'} \end{aligned} \tag{1}$$

The first great virtue of this definition is that these operators agree with the creators of the elementary excitations in both regimes we've studied:

When $g \ll 1$, $\langle \mathbf{Z}_j \rangle \simeq 1$ and more strongly, $\mathbf{Z}_j = \langle \mathbf{Z}_j \rangle + \text{small}$, so $\boldsymbol{\chi}_j \simeq \langle \mathbf{Z}_j \rangle \boldsymbol{\tau}^z_{j+\frac{1}{2}} \simeq \boldsymbol{\tau}^z_{j+\frac{1}{2}}$, the domain wall creation operator. Similarly, when $g \gg 1$, $\boldsymbol{\tau}^z_j \simeq 1+$ small, so $\boldsymbol{\chi}_j \simeq \mathbf{Z}_j \left\langle \boldsymbol{\tau}^z_{j+\frac{1}{2}} \right\rangle \simeq \mathbf{Z}_j$, which is the spin flipper on the paramagnetic vacuum.

(b) Now let us consider the algebra of these $\boldsymbol{\chi}$s. Verify that

- They are *real*: $\boldsymbol{\chi}_j^\dagger = \boldsymbol{\chi}_j, \tilde{\boldsymbol{\chi}}_j^\dagger = \tilde{\boldsymbol{\chi}}_j$.

and

- They are *fermionic*:

$$\text{if } i \neq j,\ \boldsymbol{\chi}_j \boldsymbol{\chi}_i + \boldsymbol{\chi}_i \boldsymbol{\chi}_j \equiv \{\boldsymbol{\chi}_j, \boldsymbol{\chi}_i\} = 0, \quad \{\tilde{\boldsymbol{\chi}}_j, \tilde{\boldsymbol{\chi}}_i\} = 0, \quad \{\boldsymbol{\chi}_j, \tilde{\boldsymbol{\chi}}_i\} = 0. \quad (2)$$

When they are at the same site:

$$\boldsymbol{\chi}_j^2 = 1 = \tilde{\boldsymbol{\chi}}_j^2. \quad \text{In summary: } \boxed{\{\boldsymbol{\chi}_i, \boldsymbol{\chi}_j\} = 2\delta_{ij},\ \{\tilde{\boldsymbol{\chi}}_i, \tilde{\boldsymbol{\chi}}_j\} = 2\delta_{ij},}.$$

Notice that (2) means that $\boldsymbol{\chi}_i$ cares about $\boldsymbol{\chi}_j$ even if $|i-j| \gg 1$. Fermions are weird and non-local!

Recall from a previous homework that real fermion operators like this are called *Majorana* fermion operators. We can make more familiar-looking objects by making complex combinations:

$$\mathbf{c}_j \equiv \frac{1}{2}(\boldsymbol{\chi}_j - \mathbf{i}\tilde{\boldsymbol{\chi}}_j) \quad \Longrightarrow \quad \mathbf{c}_j^\dagger = \frac{1}{2}(\boldsymbol{\chi}_j + \mathbf{i}\tilde{\boldsymbol{\chi}}_j)$$

These satisfy the more familiar anticommutation relations:

$$\{\mathbf{c}_i, \mathbf{c}_j^\dagger\} = \delta_{ij}, \quad \{\mathbf{c}_i, \mathbf{c}_j\} = 0, \quad \{\mathbf{c}_i^\dagger, \mathbf{c}_j^\dagger\} = 0,$$

and in particular, $\left(\mathbf{c}_i^\dagger\right)^2 = 0$, like a good fermion creation operator should.

We can write $\mathbf{H}_{\text{TFIM}}$ in terms of the fermion operators. We need to know how to write $\mathbf{X}_j$ and $\mathbf{Z}_j\mathbf{Z}_{j+1}$.

(c) Show that the operator which *counts* spin flips in the paramagnetic phase is

$$\mathbf{X}_j = -\mathbf{i}\tilde{\boldsymbol{\chi}}_j \boldsymbol{\chi}_j = -2\mathbf{c}_j^\dagger \mathbf{c}_j + 1 = (-1)^{\mathbf{c}_j^\dagger \mathbf{c}_j}.$$

(d) Show that the operator which counts domain walls is

$$\mathbf{Z}_j \mathbf{Z}_{j+1} = \mathbf{i}\tilde{\boldsymbol{\chi}}_{j+1} \boldsymbol{\chi}_j.$$

(e) Conclude that

$$\mathbf{H}_{\text{TFIM}} = -J\sum_j \left(\mathbf{i}\tilde{\boldsymbol{\chi}}_{j+1}\boldsymbol{\chi}_j + g\mathbf{i}\boldsymbol{\chi}_j\tilde{\boldsymbol{\chi}}_j\right)$$

is quadratic in these variables, for any g! Free at last!

(f) The hamiltonian is quadratic in the $\mathbf{c}$s, too, since they are linear in the $\boldsymbol{\chi}$s. In terms of complex fermions, show that

$$\mathbf{X}_j = 1-2\mathbf{c}_j^\dagger\mathbf{c}_j, \quad \mathbf{Z}_j = -\prod_{i>j}(1-2\mathbf{c}_i^\dagger\mathbf{c}_i)\left(\mathbf{c}_j + \mathbf{c}_j^\dagger\right) = -\prod_{i>j}(-1)^{\mathbf{c}_i^\dagger\mathbf{c}_i}\left(\mathbf{c}_j + \mathbf{c}_j^\dagger\right).$$

In terms of their Fourier modes $\mathbf{c}_k \equiv \frac{1}{\sqrt{N}}\sum_j \mathbf{c}_j e^{-\mathbf{i}kx_j}$, show that the TFIM hamiltonian is

$$\mathbf{H}_{\text{TFIM}} = J\sum_k \left(2(g-\cos ka)\mathbf{c}_k^\dagger\mathbf{c}_k - \mathbf{i}\sin ka\left(\mathbf{c}_{-k}^\dagger\mathbf{c}_k^\dagger + \mathbf{c}_{-k}\mathbf{c}_k\right) - g\right)$$

(g) This Hamiltonian is quadratic in $\mathbf{c}_k$s, but not quite diagonal. The solution for the spectrum involves one more operation the fancy name for which is 'Bogoliubov transformation', which is the introduction of new (complex) mode operators which mix particles and holes:

$$\boldsymbol{\gamma}_k = u_k\mathbf{c}_k - \mathbf{i}v_k\mathbf{c}_{-k}^\dagger$$

Demanding that the new variables satisfy canonical commutators $\{\boldsymbol{\gamma}_k, \boldsymbol{\gamma}_{k'}^\dagger\} = \delta_{k,k'}$ requires $u_k = \cos(\phi_k/2), v_k = \sin(\phi_k/2)$. We fix the angles ϕ_k by demanding that the hamiltonian in terms of $\boldsymbol{\gamma}_k$ be diagonal – no $\boldsymbol{\gamma}_k\boldsymbol{\gamma}_{-k}$ terms. Show that the resulting condition is $\tan\phi_k = \frac{\epsilon_2(k)}{\epsilon_1(k)}$ with $\epsilon_1(k) = 2J(g - \cos ka), \epsilon_2(k) = -J\sin ka$, and $\mathbf{H} = \sum_k \epsilon_k\left(\boldsymbol{\gamma}_k^\dagger\boldsymbol{\gamma}_k - \frac{1}{2}\right)$, with $\epsilon_k = \sqrt{\epsilon_1^2 + \epsilon_2^2}$. The end result is that the exact single-particle (single $\boldsymbol{\gamma}$) dispersion is

$$\epsilon_k = 2J\sqrt{1 + g^2 - 2g\cos ka} \quad .$$

The argument of the sqrt is positive for $g \geq 0$. This is minimized at $k = 0$, which tells us the exact gap at all g:

$$\epsilon_k \geq \epsilon_0 = 2J|1-g| = \Delta(g)$$

which, ridiculously, is just what we got from 1st order perturbation theory on each side of the transition.