University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2020
Assignment 5

Due 12:30pm Monday, May 4, 2020

Please refer to the first homework for submission format and procedures (and replace hw01 by hw05 in the relevant places, of course).

---

1. **Brain-warmer: a beam of particles.** Suppose the occupation numbers for a state of bosons satisfy

$$n_{\vec{p}} = ce^{-\alpha(\vec{p}-\vec{p}_0)^2/2}.$$

   (a) Determine $c = c(n, \alpha, p_0)$ so that the average density is

$$n = \int đ^3 p n_{\vec{p}}.$$

   (b) Check that with this normalization, in the thermodynamic limit of $N \to \infty$ at fixed $n = N/V$, the pair correlation function is

$$g(x-y) = 1 + e^{-(x-y)^2/\alpha}.$$

2. **Further evidence for the clumping tendencies of bosons.**

   Consider again the model of a 1d crystalline solid that we discussed in class: It consists of N point masses, coupled to their neighbors:

$$\mathbf{H}_0 = \sum_{n=1}^{N} \left(\frac{\mathbf{p}^2}{2m} + \frac{1}{2}\kappa \left(\mathbf{q}_n - \mathbf{q}_{n-1}\right)^2 \right) = \sum_{\{k\}} \hbar\omega_k \left(\mathbf{a}_k^\dagger \mathbf{a}_k + \frac{1}{2} \right) \quad . \tag{1}$$

   Assume periodic boundary conditions $\mathbf{q}_n = \mathbf{q}_{n+N}$, so that the allowed wavenumbers are

$$\{k\} \equiv \{k_j = \frac{2\pi}{Na} j, \quad j = 1, 2...N\} \ .$$

   Consider a state with two phonons defined by

$$|k_1, k_2\rangle \equiv \mathbf{a}_{k_1}^\dagger \mathbf{a}_{k_2}^\dagger |0\rangle \ .$$

   (a) In the state $|k_1, k_2\rangle$, what is the probability of finding two phonons at the location x_1?

   Do this problem both using a first-quantized point of view and using the algebra of creation and annihilation operators. Make sure your answers agree!

   [Warning: the statement of this problem is deceptively simple.]

(b) Make sure your probabilities add up to one.

(c) Compare your result to the answer that would obtain if the particles were distinguishable (and occupied the same two single-particle states). Do bosons clump?

(d) Does the story change if $k_1 = k_2$?

(e) Bonus problem: for two fermions in the state $|k_1, k_2\rangle$, what is the probability of finding one at x_1 and one at x_2? Check that your probabilities add to one. (It is possible to do this part in parallel with the others.)

3. [Bonus problem] Describe the outcome of the intensity interferometry (Hanbury-Brown and Twiss) experiment for beams of fermions.

4. [Bonus problem] Consider free fermions with single-particle Hamiltonian

$$h = t \sum_n |n\rangle\langle n+1| + h.c. + \sum_n V_n |n\rangle\langle n|.$$

(a) For the case without an external potential, $V_n = 0$, numerically evaluate the single-particle Green's function

$$G(n, m) \equiv \langle \Phi_0 | \psi_n^\dagger \psi_m | \Phi_0 \rangle$$

in the groundstate. Plot it as a function of the separation between the two points.

(b) Now add a *random* potential V_n. Choose each V_n independently from a Gaussian distribution with width v. How does this affect the Green's function? What happens if you average $G(n, m)$ over v for fixed $n - m$.