University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2020
Assignment 2

Due 12:30pm Monday, April 13, 2020

Please refer to the first homework for submission format and procedures (and replace hw01 by hw02 in the relevant places, of course).

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1. **Brain-warmer: oscillator algebra.** Convince yourself that an operator $\mathcal{O}$ made of creation and annihilation operators $\mathbf{a}_k$ and $\mathbf{a}_k^\dagger$ for various k commutes with the number operator $\sum_k \mathbf{N}_k$ if and only if it has the same number of $\mathbf{a}$s as $\mathbf{a}^\dagger$s.

2. **Brain-warmer: time evolution.** Recall the expression for $\mathbf{q}_n$ in terms of creation and annihilation operators given in the lecture notes. Check that the expression for $\mathbf{p}_n$ in terms of creation and annihilation operators is consistent with the Heisenberg equations of motion

$$\mathbf{p}_n = m\dot{\mathbf{q}}_n = \frac{\mathbf{i}m}{\hbar}[\mathbf{H}, \mathbf{q}_n].$$

(That is, evaluate the right hand side of this expression using the algebra of $\mathbf{a}_k$ and $\mathbf{a}_k^\dagger$.

3. **Entropy and thermodynamics.** Consider a quantum system with hamiltonian $\mathbf{H}$ and Hilbert space $\mathcal{H}$ Its behavior in thermal equilibrium at temperature T can be described using the *thermal density matrix*

$$\boldsymbol{\rho}_\beta \equiv \frac{1}{Z} e^{-\beta \mathbf{H}}$$

where $\frac{\beta \equiv 1}{T}$ specifies the temperature and Z is a normalization factor. (We can think about this as the density matrix resulting from coupling the system to a heat bath and tracing out the Hilbert space of the heat bath.)

   (a) Find a formal expression for Z by demanding that $\boldsymbol{\rho}_\beta$ is normalized appropriately. This is called the *partition function*.

   (b) Recall that the von Neumann entropy of a density matrix is defined as

$$S[\rho] = -\text{tr}\rho \log \rho.$$

Show that the von Neumann entropy of $\boldsymbol{\rho}_\beta$ can be written as

$$S_\beta = E/T + \log Z$$

where $E \equiv \langle \mathbf{H} \rangle$ is the expectation value for the energy. Convince yourself that this is same as the thermal entropy.

(c) Evaluate Z and E and the heat capacity $C = \partial_T E$ for the case where the system is a simple harmonic oscillator

$$\mathcal{H} = \text{span}\{|n\rangle, n = 0, 1, 2...\}, \quad \mathbf{H} = \hbar\omega\left(\mathbf{n} + \frac{1}{2}\right)$$

with $\mathbf{n}|n\rangle = n|n\rangle$.

(d) Now evaluate the low-temperature equilibrium heat capacity for a harmonic mattress (the d-dimensional version of the harmonic chain). That is, find the heat capacity for a collection of harmonic oscillators labelled by wavenumber $\vec{k}$ in d dimensions,

$$\mathbf{H} = \sum_k \hbar\omega\left(a_k^\dagger a_k + \frac{1}{2}\right)$$

with dispersion relation $\omega_k = v_s|k|$.

4. **Momentum.** In this problem we consider a scalar field theory in d spatial dimensions. Consider the operator

$$\vec{\mathbf{P}} \equiv \int đ^d k \hbar \vec{k} a_k^\dagger a_k$$

where $\int đ^d k ... \equiv \int \frac{d^d k}{(2\pi)^d} ....$

(a) Find $[\vec{\mathbf{P}}, a_k^\dagger]$, and $[\vec{\mathbf{P}}, a_k]$.

(b) Show using 4a and the mode expansion of a scalar field that

$$[\vec{\mathbf{P}}, \phi(x)] = \mathbf{i}\hbar\vec{\nabla}\phi(x).$$

(c) Conclude (using Taylor's theorem) that

$$e^{-\mathbf{i}\vec{a}\cdot\vec{\mathbf{P}}/\hbar}\phi(x)e^{\mathbf{i}\vec{a}\cdot\vec{\mathbf{P}}/\hbar} = \phi(x + a)$$

and that therefore $\vec{\mathbf{P}}$ generates translations. Therefore $\vec{\mathbf{P}}$ is the operator representing the momentum carried by the field (like the Poynting vector for the electromagnetic field).

(d) Find $\vec{\mathbf{P}}\left|\vec{k}_1, \vec{k}_2 ... \vec{k}_n\right\rangle$, the action of this operator on a state of n phonons. Conclude that $\hbar\vec{k}$ is the momentum of the phonon labelled by wavenumber $\vec{k}$.

5. **Gaussian identity.** Show that for a gaussian quantum system

$$\left\langle e^{\mathbf{i}K\mathbf{q}}\right\rangle = e^{-A(K)\left\langle \mathbf{q}^2\right\rangle}$$

and determine $A(K)$. Here $\langle ... \rangle \equiv \langle 0| ... |0\rangle$. Here by 'gaussian' I mean that $\mathbf{H}$ contains only quadratic and linear terms in both $\mathbf{q}$ and its conjugate variable $\mathbf{p}$ (but for the formula to be exactly correct as stated you must assume $\mathbf{H}$ contains only terms quadratic in $\mathbf{q}$ and $\mathbf{p}$; for further entertainment fix the formula for the case with linear terms in $\mathbf{H}$).

I recommend using the path integral representation (with hints from the previous problem). Alternatively, you can use the harmonic oscillator operator algebra. Or, better, do it both ways.