University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2020
Assignment 1

Due 12:30pm Monday, April 6, 2020

- Homework will be handed in electronically. Please do not hand in photographs of hand-written work. The preferred option is to typeset your homework. It is easy to do and you need to do it anyway as a practicing scientist. A LaTeX template file with some relevant examples is provided here. If you need help getting set up or have any other questions please email me.

- To hand in your homework, please submit a pdf file through the course's canvas website, under the assignment labelled hw01. Please put the filename in the format:

  `212C-hw01-YourLastName-YourFirstName.pdf`

Thanks in advance for following these guidelines. Please ask me if you have any trouble.

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1. **Brain-warmer: oscillation of excited oscillator states.**

   Consider a 1d harmonic oscillator of frequency ω. Consider the initial state

   $$|\psi_{n,s}(0)\rangle \equiv \mathbf{T}(s)\,|n\rangle$$

   where $|n\rangle \equiv \frac{1}{\sqrt{n!}}\left(\mathbf{a}^\dagger\right)^n|0\rangle$ is the nth excited state and $\mathbf{T}(s) \equiv e^{-\mathbf{i}\mathbf{P}s}$ is the displacement operator.

   Describe (plot it as a function of q for some $n, t, s > 0$) the time evolution of the probability distribution: $\rho(q,t) = |\psi_{n,s}(q,t)|^2$ where $\psi_{n,s}(q,t) \equiv \left\langle q|e^{-\mathbf{i}\mathbf{H}t}|\psi_{n,s}(0)\right\rangle$. Does it keep its shape like it does for $n = 0$?

2. **Coherent states.**

   Consider a quantum harmonic oscillator with frequency ω. The creation and annihilation operators $\mathbf{a}^\dagger$ and $\mathbf{a}$ satisfy the algebra

   $$[\mathbf{a}, \mathbf{a}^\dagger] = 1$$

   and the vacuum state $|0\rangle$ satisfies $\mathbf{a}\,|0\rangle = 0$. Coherent states are eigenstates of the annihilation operator:

   $$\mathbf{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,.$$

(a) Show that

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha \mathbf{a}^\dagger} |0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

is an eigenstate of $\mathbf{a}$ with eigenvalue α. ($\mathbf{a}$ is not hermitian, so its eigenvalues need not be real.)

(b) Coherent states with different α are not orthogonal. ($\mathbf{a}$ is not hermitian, so its eigenstates need not be orthogonal.) Show that $|\langle\alpha_1|\alpha_2\rangle|^2 = e^{-|\alpha_1-\alpha_2|^2}$.

(c) Compute the expectation value of the number operator $\mathbf{n} = \mathbf{a}^\dagger\mathbf{a}$ in the coherent state $|\alpha\rangle$.

(d) Time evolution acts nicely on coherent states. The hamiltonian is $\mathbf{H} = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)$. Show that a coherent state evolves into a coherent state with an eigenvalue $\alpha(t)$:

$$e^{-\mathbf{i}\mathbf{H}t} |\alpha\rangle = e^{-\mathbf{i}\omega t/2} |\alpha(t)\rangle$$

where $\alpha(t) = e^{-\mathbf{i}\omega t}\alpha$.

(e) Show that the coherent states can be used to resolve the identity in the form

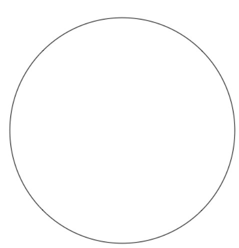
$$\mathbb{1} = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|$$

where $d^2\alpha \equiv d\alpha_1 d\alpha_2$ in terms of the real and imaginary parts of $\alpha = \alpha_1 + \mathbf{i}\alpha_2$. One way to do this is to relate this expression to $\mathbb{1} = \sum_{n=0}^{\infty} |n\rangle\langle n|$.

The following three problems form a triptych, on the subject of resolving the various infinities involved in the quantum mechanics of a particle on the real line. There are two such infinities: one is the fact that the real line goes on forever; this is resolved in problem 3. The other is the fact that in between any two points there are infinitely many points; this is resolved in problem 4. In problem 5 we resolve both to get a finite-dimensional Hilbert space.

3. **Particle on a circle.**

Consider a particle which lives on a circle:



That is, its coordinate x takes values in $[0, 2\pi R]$ and we identify $x \simeq x + 2\pi R$.

(a) Let's assume that the wavefunction of the particle is periodic in x:

$$\psi(x + 2\pi R) = \psi(x) \ .$$

What set of values can its momentum (that is, eigenvalues of the operator $\mathbf{p} = -i\hbar\partial_x$) take?

(b) Recall that the overall phase of the state vector is not physical data. This suggests the possibility that the wavefunction might not be periodic, but instead might acquire a phase when we go around the circle:

$$\psi(x + 2\pi R) = e^{i\varphi}\psi(x)$$

for some fixed φ. In this case what values does the momentum take?

4. **Particle on a lattice.**

Now consider a particle which lives on a lattice: its position can take only the discrete values $x = na, n \in \mathbb{Z}$ where a is some unit of length and n is an integer. We'll call the corresponding position eigenstates $|n\rangle$. The Hilbert space is still infinite-dimensional, but at least we have in our hands a countably infinite basis.

In this problem we will determine: what is the spectrum of the momentum operator $\mathbf{p}$ in this system?

(a) Consider the state

$$|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n\in\mathbb{Z}} e^{in\theta} |n\rangle \ .$$

Show that $|\theta\rangle$ is an eigenstate of the *translation operator* $\hat{T}$, defined by

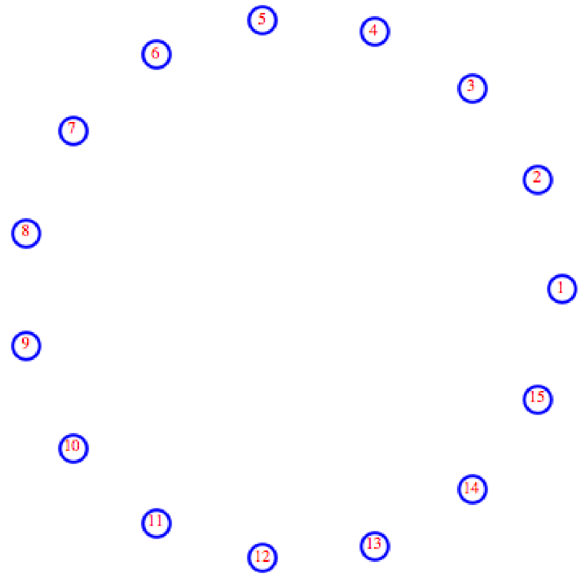
$$\hat{T} = \sum_{n\in\mathbb{Z}} |n+1\rangle \langle n| \ .$$

Why do I want to call θ momentum?

(b) What range of values of θ give different states $|\theta\rangle$? [Recall that n is an integer.]

5. **Discrete Laplacian.**

Consider again a particle which lives on a lattice, but now we'll wrap the lattice around a circle, in the following sense. Its position can take only the discrete values $x = a, 2a, 3a, ..., Na$ (where, again, a is some unit of length and again we'll call the corresponding position eigenstates $|n\rangle$). Suppose further that the particle lives on a circle, so that the site labelled $x = (N+1)a$ is the same as the site labelled $x = a$. We can visualize this as in the figure:



In this case, the Hilbert space has finite dimension N.

Consider the following $N \times N$ matrix representation of a Hamiltonian operator (a is a constant):

$$H = \frac{1}{a^2}\left(\left.\underbrace{\begin{matrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{matrix}}_{N}\right\} N \right)$$

(a) Convince yourself that this is equivalent to the following: Acting on an N-dimensional Hilbert space with orthonormal basis $\{|n\rangle, n = 1, \ldots, N\}$, $\hat{H}$ acts by

$$a^2 \hat{H}\,|n\rangle = 2\,|n\rangle - |n+1\rangle - |n-1\rangle\,, \qquad \text{with } |N+1\rangle \simeq |1\rangle$$

that is, we consider the arguments of the ket to be integers modulo N.

(b) Show that $\hat{H}$ and $\hat{T}$ (where $\hat{T}$ is the 'shift operator' defined by $\hat{T} : |n\rangle \mapsto |n+1\rangle$) can be simultaneously diagonalized.

Consider again the state

$$|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta} |n\rangle .$$

(c) Show that $|\theta\rangle$ is an eigenstate of $\hat{T}$, for values of θ that are consistent with the periodicity $n \simeq n + N$.

(d) What values of θ give different states $|\theta\rangle$? [Recall that n is an integer.]

(e) Find the matrix elements of the unitary operator $\mathbf{U}$ which relates position eigenstates $|n\rangle$ to momentum eigenstates $|\theta\rangle$: $U_{\theta n} \equiv \langle n|\theta\rangle$.

(f) Find the spectrum of $\hat{H}$.

Draw a picture of $\epsilon(\theta)$: plot the energy eigenvalues versus the 'momentum' θ.

(g) Show that the matrix above is an approximation to (minus) the 1-dimensional Laplacian $-\partial_x^2$. That is, show (using Taylor's theorem) that

$$a^2 \partial_x^2 f(x) = -2f(x) + (f(x+a) + f(x-a)) + \mathcal{O}(a)$$

(where "$\mathcal{O}(a)$" denotes terms proportional to the small quantity a).

(h) In the expression for the Hamiltonian, to restore units, I should have written:

$$\hat{H} |n\rangle = \frac{\hbar^2}{2m} \frac{1}{a^2} (2|n\rangle - |n+1\rangle - |n-1\rangle) , \quad \text{with } |N+1\rangle \simeq |1\rangle$$

where a is the distance between the sites, and m is the mass. Consider the limit where $a \to 0, N \to \infty$ and look at the lowest-energy states (near $p = 0$); show that we get the spectrum of a free particle on the line, $\epsilon = \frac{p^2}{2m}$.