Physics 215C: Particles and Fields
Spring 2019

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Last updated: 2021/04/20, 14:28:40

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0.1 Introductory remarks for the third quarter

Here are some goals for the quarter, both practical and philosophical:

- I would like to convince you that "non-renormalizable" does not mean "not worth your attention," and explain the incredibly useful notion of an Effective Field Theory.

- There is more to QFT than perturbation theory about free fields in a Fock vacuum. In particular, we will spend some time thinking about non-perturbative physics, effects of topology, solitons. Topology is one tool for making precise statements without perturbation theory (the basic idea: if we know something is an integer, it is easy to get many digits of precision!).

- There is more to QFT than the S-matrix. In a particle-physics QFT course (like 215 so far) you learn that the purpose in life of correlation functions or green's functions or off-shell amplitudes is that they have *poles* (at $p^\mu p_\mu - m^2 = 0$) whose residues are the S-matrix elements, which are what you measure (or better, are the distribution you sample) when you scatter the particles which are the quanta of the fields of the QFT. I want to make two extended points about this:

  1. In many physical contexts where QFT is relevant, you can actually measure the off-shell stuff. This is yet another reason why including condensed matter in our field of view will deepen our understanding of QFT.
  2. This is good, because the Green's functions don't always have simple poles! There are lots of interesting field theories where the Green's functions instead have power-law singularities, like $G(p) \sim \frac{1}{p^{2\Delta}}$. If you Fourier transform this, you don't get an exponentially-localized packet. The elementary excitations created by a field whose two point function does this *are not particles*. (Any conformal field theory (CFT) is an example of this.) The theory of particles (and their dance of creation and annihilation and so on) is an important but proper subset of QFT.

- In addition to its importance in high energy physics, I want to emphasize that QFT is also quite central in many aspects of condensed matter physics, and we will learn about this. From the point of view of someone interested in QFT, high energy particle physics has the severe drawback that it offers only one example! (OK, for some purposes we can think about QCD and the electroweak theory separately...)

  From the high-energy physics point of view, we could call this the study of *regulated* QFT, with a particular kind of lattice regulator. Why make a big deal about

'regulated'? Besides the fact that this is how QFT comes to us (when it does) in condensed matter physics, such a description is required if we want to know what we're talking about. For example, we need it if we want to know what we're talking about well enough to explain it to a computer. Many QFT problems are too hard for our brains. A related but less precise point is that I would like to do what I can to erase the problematic, theorist-centered perspective on QFT which 'begins from a classical lagrangian and quantizes it' etc, and leads to a term like 'anomaly'. (Anyway, we will talk about what is 'anomaly' next.)

- A central theme this quarter is coarse-graining in quantum systems with extensive degrees of freedom, aka the RG in QFT.

  I remind you that by 'extensive degrees of freedom' I mean that we are going to study models which, if we like, we can sprinkle over vast tracts of land, like *sod* (depicted in the figure at right). And also like sod, each little patch of degrees of freedom only interacts with its neighboring patches: this property of sod and of QFT is called *locality*.

  More precisely, in a quantum mechanical system, we specify the degrees of freedom by their Hilbert space; by an extensive system, I'll mean one in which the Hilbert space is of the form $\mathcal{H} = \otimes_{\text{patches of space}} \mathcal{H}_{\text{patch}}$ and the interactions are local $\mathbf{H} = \sum_{\text{patches}} \mathbf{H}(\text{nearby patches})$. (Actually, the Hilbert space of a gauge theory is *not* of this form; rather, it is a subspace of such a space which satisfies the Gauss law. This is a source of a lot of confusion, which I hope to dispel.)

  By 'coarse-graining' I mean ignoring things we don't care about, or rather only paying attention to them to the extent that they affect the things we do care about.

  To continue the sod example in 2+1 dimensions, a person laying the sod in the picture above cares that the sod doesn't fall apart, and rolls nicely onto the ground (as long as we don't do high-energy probes like bending it violently or trying to lay it down too quickly). These *long-wavelength* properties of *rigidity* and *elasticity* are collective, emergent properties of the microscopic constituents (sod molecules) – we can describe the dynamics involved in covering the Earth with sod (never mind whether this is a good idea in a desert climate) without knowing the microscopic theory of the sod molecules ('grass'). Our job is to think about the relationship between the microscopic model (grassodynamics) and its macroscopic counterpart (in this case, suburban landscaping). In my experience, learning to do this is approximately synonymous with *understanding*.

- The crux of many problems in physics is the correct choice of variables with which to label the degrees of freedom. Often the best choice is very different from the obvious choice; a name for this phenomenon is 'duality'. There are many examples of it and we will study some of them. This word is dangerous because it is *about* ambiguities in our (physics) language. I would like to reclaim it.

  An important bias in deciding what is meant by 'correct' or 'best' in the previous paragraph is: we will be interested in low-energy and long-wavelength physics, near the groundstate. For one thing, this is the aspect of the present subject which is like 'elementary particle physics'; the high-energy physics of these systems is of a very different nature and bears little resemblance to the field often called 'high-energy physics' (for example, there is volume-law entanglement).

- We'll be interested in models with a finite number of degrees of freedom *per unit volume*. This last is important, because we are going to be interested in the thermodynamic limit.

- An important goal for the course is demonstrating that many fancy phenomena precious to particle physicists can emerge from humble origins in the kinds of (completely well-defined) local quantum lattice models we will study. Here I have in mind: fermions, gauge theory, photons, anyons, strings, topological solitons, CFT, and many other sources of wonder I'm forgetting right now.

Topics which I hope to discuss this quarter include:

- effects of topology in QFT (this includes anomalies, topological solitons and defects, topological terms in the action)

- some illustrations of effective field theory (perhaps cleverly mixed in with the other subjects)

- the uses and limitations of path integrals of various kinds

- Large-N expansions

- more deep mysteries of gauge theory and its emergence in physical systems.

- duality.

I welcome your suggestions regarding which subjects in QFT we should study.

0.2 Sources and acknowledgement

The material in these notes is collected from many places, among which I should mention in particular the following:

Peskin and Schroeder, *An introduction to quantum field theory* (Wiley)

Zee, *Quantum Field Theory* (Princeton, 2d Edition)

Banks, *Modern Quantum Field Theory: A Concise Introduction* (Cambridge)

Schwartz, *Quantum field theory and the standard model* (Cambridge)

Xiao-Gang Wen, *Quantum Field Theory of Many-Body Systems*

Sidney Coleman, *Aspects of Symmetry*

Alexander Polyakov, *Gauge Fields and Strings*

Eduardo Fradkin, *Field Theories of Condensed Matter Systems*

Many other bits of wisdom come from the Berkeley QFT courses of Prof. L. Hall and Prof. M. Halpern.

0.3 Conventions

Following most QFT books, I am going to use the $+---$ signature convention for the Minkowski metric. I am used to the other convention, where time is the weird one, so I'll need your help checking my signs. More explicitly, denoting a small spacetime displacement as $dx^\mu \equiv (dt, d\vec{x})^\mu$, the Lorentz-invariant distance is:

$$ds^2 = +dt^2 - d\vec{x}\cdot d\vec{x} = \eta_{\mu\nu}dx^\mu dx^\nu \quad \text{with} \quad \eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} .$$

(spacelike is negative). We will also write $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\partial_t, \vec{\nabla}_x\right)^\mu$, and $\partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$. I'll use $\mu, \nu...$ for Lorentz indices, and $i, k, ...$ for spatial indices.

The convention that repeated indices are summed is always in effect unless otherwise indicated.

D is the number of spacetime dimensions, d is the number of space dimensions.

A consequence of the fact that english and math are written from left to right is that time goes to the left.

A useful generalization of the shorthand $\hbar \equiv \frac{h}{2\pi}$ is $đk \equiv \frac{dk}{2\pi}$. I will also write $\not{\delta}^d(q) \equiv (2\pi)^d\delta^{(d)}(q)$. I will try to be consistent about writing Fourier transforms as

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{ikx}\tilde{f}(k) \equiv \int đ^d k\; e^{ikx}\tilde{f}(k) \equiv f(x).$$

IFF $\equiv$ if and only if.
RHS $\equiv$ right-hand side. LHS $\equiv$ left-hand side. BHS $\equiv$ both-hand side.
IBP $\equiv$ integration by parts. WLOG $\equiv$ without loss of generality.
$+\mathcal{O}(x^n) \equiv$ plus terms which go like x^n (and higher powers) when x is small.
$+h.c. \equiv$ plus hermitian conjugate.
$\mathcal{L} \ni \mathcal{O}$ means the object $\mathcal{L}$ contains the term $\mathcal{O}$.

We work in units where $\hbar$ and the speed of light, c, are equal to one unless otherwise noted. When I say 'Peskin' I usually mean 'Peskin & Schroeder'.

Please tell me if you find typos or errors or violations of the rules above.

1 Anomalies

[Zee §IV.7; Polyakov, *Gauge Fields and Strings*, §6.3; K. Fujikawa, *Phys. Rev. Lett.* **42** (1979) 1195; Argyres, 1996 lectures on supersymmetry §14.3; Peskin, chapter 19]

Topology means the study of quantities which can't vary smoothly, but can only jump. Like quantities which must be integers. *Anomalies* are an example of a topological phenomenon in QFT, which is therefore robust against any change in the QFT which can be made continuously (like varying masses or couplings, or the cutoff or the resolution of our description, *i.e.* a renormalization group transformation).

Suppose we have in our hands a classical field theory in the continuum which has some symmetry. Is there a well-defined QFT whose classical limit produces this classical field theory and preserves that symmetry? The path integral construction of QFT offers some insight here. The path integral involves two ingredients: (1) an action, which is shared with the classical field theory, and (2) a path integral measure. It is possible that the action is invariant but the measure is not. This is called an anomaly. It means that the symmetry is broken, and its current conservation is violated by a known amount, and this often has many other consequences that can be understood by humans. It means that the theory *cannot* be regulated in a way that preserves the symmetry.

Notice that here I am speaking about actual, global symmetries. I am not talking about gauge redundancies. If you think that two field configurations are equivalent but the path integral tells you that they would give different contributions, you are doing something wrong. Such a 'gauge anomaly' means that the system has more degrees of freedom than you thought. (In particular, it does not mean that the world is inconsistent. For a clear discussion of this, please see Preskill, 1990.)

You could say that we have already seen a dramatic example of an anomaly: the violation of classical scale invariance (*e.g.* in massless ϕ^4 theory, or in massless QED) by quantum effects.

Notice that the name 'anomaly' betrays the bias that we imagine constructing a QFT by starting with a continuum action for a classical field theory; you would never imagine that *e.g.* scale invariance was an exact symmetry if you instead started from a well-defined quantum lattice model. Partly for this reason, the concept of 'anomaly' is not native to the condensed matter literature (but has recently been flourishing there).

The example we will focus on here is the *chiral anomaly*. This is an equation for the violation of the chiral (aka axial) current for fermions coupled to a background gauge field. The chiral anomaly was first discovered in perturbation theory, by computing a certain Feynman diagram with a triangle; the calculation was motivated by the

experimental observation of the process $\pi^0 \to \gamma\gamma$, which would not happen if the chiral current were conserved. (The relationship between the chiral current and the pion is explained in §3.5.)

I will outline a derivation of this effect (using the fermionic path integral) which is more illuminating than the triangle diagram. It shows that the one-loop result is exact – there are no other corrections. It shows that the quantity on the right hand side of the continuity equation for the would-be current integrates to an integer. It gives a proof of the *index theorem*, relating numbers of solutions of the Dirac equation in a background field configuration to a certain integral of field strengths. It butters your toast.

1.0.1 Chiral anomaly

Chiral symmetries. In even-dimensional spacetimes, the Dirac representation of $\mathrm{SO}(D-1,1)$ is reducible. This is because

$$\gamma^5 \equiv \prod_{\mu=0}^{D-1} \gamma^\mu \neq 1, \quad \text{satisfies} \quad \{\gamma^5, \gamma^\mu\} = 0, \forall \mu$$

which means that γ^5 commutes with the Lorentz generators

$$[\gamma^5, \Sigma^{\mu\nu}] = 0, \qquad \Sigma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu].$$

A left- or right-handed Weyl spinor is an irreducible representation of $\mathrm{SO}(D-1,1)$, $\psi_{L/R} \equiv \frac{1}{2}(1 \pm \gamma^5)\,\psi$. This allows the possibility that the L and R spinors can transform differently under a symmetry; such a symmetry is called a chiral symmetry.

Note that in $D = 4k$ dimensions, if ψ_L is a left-handed spinor in representation $\mathbf{r}$ of some group G, then its image under CPT, $\psi_L^{CPT}(t,\vec{x}) \equiv \mathbf{i}\gamma^0\,(\psi_L(-t,-\vec{x}))^\star$, is right-handed and transforms in representation $\bar{\mathbf{r}}$ of G. Therefore chiral symmetries arise when the Weyl fermions transform in *complex representations* of the symmetry group, where $\bar{\mathbf{r}} \neq \mathbf{r}$. (In $D = 4k+2$, CPT maps left-handed fields to left-handed fields. For more detail on discrete symmetries and Dirac fields, see Peskin §3.6.)

Some more explicit words (of review) about chiral fermions in $D = 3+1$, mostly notation. Recall Peskin's *Weyl* basis of gamma matrices in 3+1 dimensions, in which γ^5 is diagonal:

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\boldsymbol{\sigma}}^\mu \\ \boldsymbol{\sigma}^\mu & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^\mu \equiv (\mathbb{1}, \vec{\boldsymbol{\sigma}})^\mu, \quad \bar{\boldsymbol{\sigma}}^\mu \equiv (\mathbb{1}, -\vec{\boldsymbol{\sigma}})^\mu, \quad \gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

This makes the reducibility of the Dirac representation of $\mathrm{SO}(3,1)$ manifest, since the Lorentz generators are $\propto [\gamma^\mu, \gamma^\nu]$ block diagonal in this basis. The gammas are a map from the $(1, \mathbf{2_R})$ representation to the $(\mathbf{2_L}, \mathbf{1})$ representation. It is sometimes useful to denote the $\mathbf{2_R}$ indices by $\alpha, \beta = 1, 2$ and the $\mathbf{2_L}$ indices by $\dot{\alpha}, \dot{\beta} = 1, 2$. Then we can define two-component Weyl spinors $\psi_{L/R} = P_{L/R}\psi \equiv \frac{1}{2}(1 \pm \gamma^5)\psi$ by simply forgetting about the other two components. The conjugate of a L spinor $\chi = \psi_L$ (L means $\gamma^5\chi = \chi$) is right-handed:

$$\bar{\chi} = \chi^\dagger\gamma^0, \quad \bar{\chi}\gamma^5 = \chi^\dagger\gamma^0\gamma^5 = -\chi^\dagger\gamma^5\gamma^0 = -\chi^\dagger\gamma^0 = -\bar{\chi}.$$

We can represent any system of Dirac fermions in terms of a collection of twice as many Weyl fermions.

For a continuous symmetry G, we can be more explicit about the meaning of a complex representation. The statement that ψ is in representation $\mathbf{r}$ means that its transformation law is

$$\delta\psi_a = \mathbf{i}\epsilon^A \left(t^A_{\mathbf{r}}\right)_{ab} \psi_b$$

where $t^A, A = 1..\dim G$ are generators of G in representation $\mathbf{r}$; for a compact lie group G, we may take the t^A to be Hermitian. The conjugate representation, by definition, is one with which you can make a singlet of G – it's the way $\psi^{\star T}$ transforms:

$$\delta\psi^{\star T}_a = -\mathbf{i}\epsilon^A \left(t^A_{\mathbf{r}}\right)^T_{ab} \psi^{\star T}_b \ .$$

So:

$$t^A_{\bar{\mathbf{r}}} = -\left(t^A_{\mathbf{r}}\right)^T .$$

The condition for a complex representation is that this is different from $t^A_{\mathbf{r}}$ (actually we have to allow for relabelling of the generators and the basis). The simplest case is $G = \mathrm{U}(1)$, where t is just a number indicating the charge. In that case, any nonzero charge gives a complex representation.

Consider the effective action produced by integrating out Dirac fermions coupled to a *background* gauge field (the gauge field is just going to sit there for this whole calculation):

$$e^{\mathbf{i}S_{\mathrm{eff}}[A]} \equiv \int [D\psi D\bar{\psi}]\ e^{\mathbf{i}S[\psi,\bar{\psi},A]} \ .$$

We must specify how the fermions coupled to the gauge field. The simplest example is if A is a $U(1)$ gauge field and ψ is minimally coupled:

$$S[\psi, \bar{\psi}, A] = \int \mathrm{d}^D x \bar{\psi}\mathbf{i}\not{D}\psi, \qquad \not{D}\psi \equiv \gamma^\mu\left(\partial_\mu + \mathbf{i}A_\mu\right)\psi.$$

We will focus on this example, but you could imagine instead that A_μ is a non-Abelian gauge field for the group G, and ψ is in a representation R, with gauge generators $T^A(R)$ ($A = 1...\dim G$), so the coupling would be

$$\bar{\psi}\mathbf{i}\not{D}\psi = \bar{\psi}_a\gamma^\mu\left(\partial_\mu\delta_{ab} + \mathbf{i}A^A_\mu T^A(R)_{ab}\right)\psi_b\ . \tag{1.1}$$

Much of the discussion below applies for any even D.

In the absence of a mass term, the action (in the Weyl basis) involves no coupling between L and R:

$$S[\psi,\bar{\psi},A] = \int \mathrm{d}^D x\left(\psi_L^\dagger\mathbf{i}\sigma^\mu D_\mu\psi_L + \psi_R^\dagger\mathbf{i}\bar{\sigma}^\mu D_\mu\psi_R\right)$$

and therefore is invariant under the global chiral rotation

$$\psi \to e^{\mathbf{i}\alpha\gamma^5}\psi,\quad \psi^\dagger \to \psi^\dagger e^{-\mathbf{i}\alpha\gamma^5},\quad \bar{\psi}\to\bar{\psi}e^{+\mathbf{i}\alpha\gamma^5}.\quad \text{That is: } \psi_L\to e^{\mathbf{i}\alpha}\psi_L,\quad \psi_R\to e^{-\mathbf{i}\alpha}\psi_R.$$

(The mass term couples the two components

$$L_m = \bar{\psi}\left(\mathrm{Re}m + \mathrm{Im}m\gamma^5\right)\psi = m\psi_L^\dagger\psi_R + h.c.;$$

notice that the mass parameter is complex.) The associated Noether current is $j^5_\mu = \bar{\psi}\gamma^5\gamma_\mu\psi$, and it seems like we should have $\partial^\mu j^5_\mu \stackrel{?}{=} 0$. This follows from the massless (classical) Dirac equation $0 = \gamma^\mu\partial_\mu\psi$. (With the mass term, we would have instead $\partial^\mu j^5_\mu \stackrel{?}{=} 2\mathbf{i}\bar{\psi}\left(\mathrm{Re}m\gamma^5 + \mathrm{Im}m\right)\psi$.)

Notice that there is another current $j^\mu = \bar{\psi}\gamma^\mu\psi$. j^μ is the current which is coupled to the gauge field, $L \ni A_\mu j^\mu$. The conservation of this current is required for gauge invariance of the effective action

$$S_{\text{eff}}[A_\mu] \stackrel{!}{=} S_{\text{eff}}[A_\mu + \partial_\mu\lambda] \sim \log\left\langle e^{\mathbf{i}\int\lambda(x)\partial_\mu j^\mu}\right\rangle + S_{\text{eff}}[A_\mu].$$

No matter what happens we can't find an anomaly in j^μ. The anomalous one is the other one, the *axial current*.

To derive the conservation law we can use the Noether method. This amounts to substituting $\psi'(x) \equiv e^{\mathbf{i}\alpha(x)\gamma^5}\psi(x)$ into the action:

$$S_F[\psi'] = \int \mathrm{d}^D x\bar{\psi}e^{+\mathbf{i}\alpha\gamma^5}\mathbf{i}\not{D}e^{\mathbf{i}\alpha\gamma^5}\psi = \int \mathrm{d}^D x\left(\bar{\psi}\mathbf{i}\not{D}\psi + \bar{\psi}\mathbf{i}\gamma^5\left(\not{\partial}\alpha\right)\psi\right) \stackrel{\text{IBP}}{=} S_F[\psi] - \mathbf{i}\int\alpha(x)\partial^\mu\mathrm{tr}\bar{\psi}\gamma^5\gamma_\mu\psi.$$

Then we can completely get rid of $\alpha(x)$ if we can change integration variables, *i.e.* if $[D\psi'] \stackrel{?}{=} [D\psi]$. Usually this is true, but here we pick up an interesting Jacobian.

Claim:

$$e^{\mathbf{i}S_{\text{eff}}[A]} = \int [D\psi' D\bar{\psi}'] e^{\mathbf{i}S_F[\psi']} = \int [D\psi D\bar{\psi}] e^{\mathbf{i}S_F[\psi] + \mathbf{i}\int \mathrm{d}^D x \alpha(x)\left(\partial_\mu j_5^\mu - \mathcal{A}(x)\right)}$$

where $\mathcal{A}$ comes from the variation of the measure. That is,

$$[D\psi' D\bar{\psi}'] = [D\psi D\bar{\psi}] \det\left(e^{\mathbf{i}\alpha\gamma^5}\right) \equiv [D\psi D\bar{\psi}] e^{-\mathbf{i}\int \alpha\mathcal{A}}$$

so

$$e^{-\mathbf{i}\int \alpha\mathcal{A}} = e^{\mathrm{Tr}\log e^{\mathbf{i}\alpha\gamma^5}} = e^{\mathrm{Tr}\left(\mathbf{i}\alpha\gamma^5\right)}$$

or more explicitly

$$\mathcal{A}(x) = \sum_n \mathrm{tr}\bar{\xi}_n(x)\gamma^5\xi_n(x) \tag{1.2}$$

where ξ_n are a basis of eigenspinors of the Dirac operator.

The expression above for S_{eff} is actually independent of α, since the path integral is invariant under a change of variables. For a conserved current, α would multiply the divergence of the current and this demand would imply current conservation. Here this implies that instead of current conservation we have a specific violation of the current:

$$\partial^\mu j_\mu^5 = \mathcal{A}(x).$$

What is the anomaly $\mathcal{A}$? [Polyakov §6.3] An alternative useful (perhaps more efficient) perspective is that the anomaly arises from trying to define the axial current operator, which after all is a composite operator. Thus we should try to compute

$$\langle \partial_\mu j_5^\mu \rangle = \partial_\mu \left\langle \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x) \right\rangle$$

– the coincident operators on the RHS need to be regulated.

The classical (massless) Dirac equation immediately implies that the axial current is conserved (up to contact terms)

$$\partial_\mu \left(\bar{\psi}\gamma^\mu\gamma^5\psi\right) \stackrel{?}{=} 0.$$

Consider, on the other hand, the (Euclidean vacuum) expectation value

$$\begin{aligned} J_\mu^5 &\equiv \left\langle \bar{\psi}(x)\gamma_\mu\gamma^5\psi(x) \right\rangle \equiv Z^{-1}[A] \int [D\psi D\bar{\psi}] e^{-S_F[\psi]} j_\mu^5(x) \\ &= \quad + \quad + \quad + \cdots \\ &= -\mathrm{Tr}_{\ \gamma}\gamma_\mu\gamma^5 G^{[A]}(x,x) \end{aligned} \tag{1.3}$$

where G is the Green's function of the Dirac operator in the gauge field background (and the figure is from Polyakov's book). We can construct it out of eigenfunctions of $\mathbf{i}\not{D}$:

$$\mathbf{i}\not{D}\xi_n(x) = \epsilon_n \xi_n(x), \qquad \bar{\xi}_n(x)\mathbf{i}\gamma^\mu \left(-\overleftarrow{\partial}_\mu + \mathbf{i}A_\mu\right) = \epsilon_n \bar{\xi}_n \tag{1.4}$$

in terms of which[1]

$$G(x,x') = \sum_n \frac{1}{\epsilon_n}\xi_n(x)\bar{\xi}_n(x'). \tag{1.5}$$

(I am suppressing spinor indices all over the place, note that here we are taking the outer product of the spinors.)

We want to define the coincidence limit, as $x' \to x$. The problem with this limit arises from the large $|\epsilon_n|$ eigenvalues; the contributions of such short-wavelength modes are local and most of them can be absorbed in renormalization of couplings. It should not (and does not) matter how we regulate them, but we must pick a regulator. A convenient choice here is heat-kernel regulator:

$$G_s(x,x') \equiv \sum_n e^{-s\epsilon_n^2}\frac{1}{\epsilon_n}\xi_n(x)\bar{\xi}_n(x')$$

and

$$J^5_\mu(x) = \sum_n e^{-s\epsilon_n^2}\frac{1}{\epsilon_n}\bar{\xi}_n(x)\gamma^5\gamma_\mu\xi_n(x)\ .$$

The anomaly is

$$\partial^\mu J^5_\mu = \partial^\mu \left\langle j^5_\mu \right\rangle = \sum_n \mathbf{i}\partial^\mu\left(\bar{\xi}_n\gamma_\mu\gamma^5\xi_n\right)\frac{e^{-s\epsilon_n^2}}{\epsilon_n}.$$

The definition (1.4) says

$$\mathbf{i}\partial^\mu\left(\bar{\xi}_n\gamma^5\gamma_\mu\xi_n\right) = 2\epsilon_n\bar{\xi}_n\gamma_5\xi_n$$

using $\{\gamma^5,\gamma^\mu\}=0$. (Notice that the story would deviate dramatically here if we were studying the vector current which lacks the γ^5.) This gives

$$\partial^\mu J^5_\mu(x) = 2\mathrm{tr}_\gamma \left\langle x\right| \gamma^5 e^{-s\left(\mathbf{i}\not{D}\right)^2}\left|x\right\rangle$$

with

$$(\mathbf{i}\not{D})^2 = -\left(\gamma_\mu\left(\partial_\mu + \mathbf{i}A_\mu\right)\right)^2 = -\left(\partial_\mu + \mathbf{i}A_\mu\right)^2 - \frac{\mathbf{i}}{2}\Sigma_{\mu\nu}F^{\mu\nu}$$

where $\Sigma_{\mu\nu} \equiv \frac{1}{2}[\gamma_\mu,\gamma_\nu]$ is the spin Lorentz generator. This is (1.2), now better defined by the heat kernel regulator. [End of Lecture 1]

[1] Actually, this step is full of danger. (Polyakov has done it to me again. Thanks to Sridip Pal for discussions of this point.) See §1.0.2 below.

We've shown that in any even dimension,

$$\partial^\mu \left\langle j^5_\mu(x) \right\rangle = 2\mathrm{tr}_\gamma \left\langle x \right| \gamma^5 e^{s\not{D}^2} \left| x \right\rangle \tag{1.6}$$

This can now be expanded in small s, which amounts to an expansion in powers of A, F. If there is no background field, $A = 0$, we get

$$\left\langle x \right| e^{-s\left(\mathrm{i}\not{\partial}\right)^2} \left| x \right\rangle = \int \mathrm{đ}^D p \; e^{-sp^2} = \underbrace{K_D}_{=\frac{\Omega_{D-1}}{(2\pi)^D}} \frac{1}{s^{D/2}} \overset{D=4}{=} \frac{1}{16\pi^2 s^2}. \tag{1.7}$$

This term will renormalize the charge density

$$\rho(x) = \left\langle \psi^\dagger \psi(x) \right\rangle = \mathrm{tr}\gamma^0 G(x,x),$$

for which we must add a counterterm (in fact, it is accounted for by the counterterm for the gauge field kinetic term, *i.e.* the running of the gauge coupling). But it will not affect the axial current conservation which is proportional to

$$\mathrm{tr}\left(\gamma^5 G(x,x)\right)|_{A=0} \propto \mathrm{tr}\gamma^5 = 0.$$

Similarly, bringing down more powers of $(\partial + A)^2$ doesn't give something nonzero since the γ^5 remains.

In $D = 4$, the first term from expanding $\Sigma_{\mu\nu} F^{\mu\nu}$ is still zero from the spinor trace. (Not so in $D = 2$.) The first nonzero term comes from the next term:

$$\mathrm{tr}\left(\gamma_5 e^{-s\left(\mathrm{i}\not{D}\right)^2}\right)_{xx} = \underbrace{\left\langle x|e^{-s(\mathrm{i}D)^2}|x\right\rangle}_{\overset{(1.7)}{=}\frac{1}{16\pi^2 s^2}+\mathcal{O}(s^{-1})} \cdot \frac{s^2}{8} \cdot (\mathbf{i}^2) \underbrace{\mathrm{tr}\left(\gamma^5 \Sigma^{\mu\nu}\Sigma^{\rho\lambda}\right)}_{=4\epsilon^{\mu\nu\rho\lambda}} \cdot \underbrace{\mathrm{tr}_c}_{\text{color}} (F_{\mu\nu}F_{\rho\lambda}) + \mathcal{O}(s^1) \; .$$

In the abelian case, just ignore the trace over color indices, tr_c. The terms that go like positive powers of s go away in the continuum limit. Therefore

$$\partial_\mu J^\mu_5 = -2 \cdot \frac{1}{16\pi^2 s^2} \cdot \frac{s^2}{8} \cdot 4\epsilon^{\mu\nu\rho\lambda} \mathrm{tr}_c F_{\mu\nu} F_{\rho\lambda} + \mathcal{O}(s^1) = -\frac{1}{8\pi^2} \mathrm{tr} F_{\mu\nu} \left(\star F\right)^{\mu\nu} . \tag{1.8}$$

(Here $(\star F)^{\mu\nu} \equiv \frac{1}{8}\epsilon^{\mu\nu\rho\lambda} F_{\rho\lambda}$.) This is the chiral anomaly formula. It can also be usefully written as:

$$\partial_\mu J^\mu_5 = -\frac{1}{8\pi^2} \mathrm{tr} F \wedge F = -\frac{1}{32\pi^2} \vec{E} \cdot \vec{B}.$$

- This object on the RHS is a total derivative. In the abelian case it is

$$F \wedge F = \mathrm{d}\left(A \wedge F\right) \; .$$

Its integral over spacetime is a topological (in fact $16\pi^2$ times an integer) characterizing the gauge field configuration. How do I know it is an integer? The anomaly formula! The change in the number of left-handed fermions minus the number of right-handed fermions during some time interval is:

$$\Delta Q_A \equiv \Delta\left(N_L - N_R\right) = \int \mathrm{d}t \partial_t J_0^5 = \int_{M_4} \partial^\mu J_\mu^5 = 2\int_{M_4} \frac{F\wedge F}{16\pi^2}$$

where M_4 is the spacetime region under consideration. If nothing is going on at the boundaries of this spacetime region (*i.e.* the fields go to the vacuum, or there is no boundary, so that no fermions are entering or leaving), we can conclude that the RHS is an integer.

- Look back at the diagrams in (1.3). Which term in that expansion gave the nonzero contribution to the axial current violation? In $D = 4$ it is the diagram with three current insertions, the ABJ triangle diagram. So in fact we did end up computing the triangle diagram. But this calculation also shows that nothing else contributes, even non-perturbatively.

- We chose a particular regulator above. The answer we got did not depend on the cutoff; in fact, whatever regulator we used, we would get this answer.

- Consider what happens if we redo this calculation in other dimensions. We only consider even dimensions because in odd dimensions there is no analog of γ^5 – the Dirac spinor representation is irreducible. In $2n$ dimensions, we need n powers of F to soak up the indices on the epsilon tensor. Actually there is an analogous phenomenon in odd dimensions (sometimes called parity anomaly) of an effect that is independent of the masses of the fields which you'll study on the homework. Instead of F^n, the thing that appears is the Chern-Simons term.

- If we had kept the non-abelian structure in (1.1) through the whole calculation, the only difference is that the trace in (1.8) would have included a trace over representations of the gauge group. And we could have considered also a non-abelian flavor transformation in the chiral symmetry

$$\psi_I \to \left(e^{\mathrm{i}\gamma^5 g^a \tau^a}\right)_{IJ} \psi_J$$

for some flavor rotation generator τ^a. Then we would have found:

$$\partial^\mu j_\mu^{5a} = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\lambda} F_{\mu\nu}^A F_{\rho\lambda}^B \mathrm{tr}_{c,a}\left(T^A T^B \tau^a\right).$$

A similar statement applies to the case of multiple species of fermion fields: their contributions to the anomaly add. Sometimes they can cancel; the Electroweak gauge interactions are an example of this.

1.0.2 Zeromodes of the Dirac operator

Do you see why I said that the step involving the fermion Green's function was full of danger? The danger arises because the Dirac operator (whose inverse is the Green's function) can have *zeromodes*, eigenspinors with eigenvalue $\epsilon_n = 0$. In that case, $\mathbf{i}\not{D}$ is *not* invertible, and the expression (1.5) for G is ambiguous. This factor of ϵ_n is about to be cancelled when we compute the divergence of the current and arrive at (1.2). Usually this kind of thing is not a problem because we can lift the zeromodes a little and put them back at the end. But here it is actually hiding something important. The zeromodes cannot just be lifted. This is true because nonzero modes of $\mathbf{i}\not{D}$ must come in left-right pairs: this is because $\{\gamma^5, \mathbf{i}\not{D}\} = 0$, so $\mathbf{i}\not{D}$ and γ^5 cannot be simultaneously diagonalized in general. That is: if $\mathbf{i}\not{D}\xi = \epsilon\xi$ then $(\gamma^5\xi)$ is also an eigenvector of $\mathbf{i}\not{D}\xi$, with eigenvalue $-\epsilon$. Only for $\epsilon = 0$ does this fail, so zeromodes can come by themselves. So you can't just smoothly change the eigenvalue of some ξ_0 from zero unless it has a partner with whom to pair up. By taking linear combinations

$$\chi_n^{L/R} = \frac{1}{2}\left(1 \pm \gamma^5\right)\xi_n$$

these two partners can be arranged into a pair of simultaneous eigenvectors of $(\mathbf{i}\not{D})^2$ (with eigenvalue ϵ_n^2) and of γ^5 with $\gamma^5 = \pm$ respectively.

This leads us to a deep fact, called the (Atiyah-Singer) *index theorem*: *only* zeromodes can contribute to the anomaly. Any mode ξ_n with nonzero eigenvalue has a partner (with the same eigenvalue of $(\mathbf{i}\not{D})^2$) with the opposite sign of γ^5; hence they cancel exactly in

$$\mathrm{tr}\gamma^5 e^{-s(\mathbf{i}\not{D})^2} = \sum_{n,L/R} \bar{\chi}_n^{L/R}\gamma^5\chi_n^{L/R}e^{-s\epsilon_n^2} + \text{zeromodes} \quad .$$

So the anomaly equation tells us that the number of zeromodes of the Dirac operator, weighted by handedness (*i.e.* with a + for L and - for R) is equal to

$$n_L - n_R = \int d^D x \mathcal{A}(x) = \int \frac{1}{16\pi^2} F \wedge F.$$

A practical consequence for us is that it makes manifest that the result is independent of the regulator s.

Another consequence is that in the background of a gauge field configuration with nonzero $n_L - n_R = q$, the vacuum to vacuum amplitude

$$Z[A^q] = \int [D\psi D\bar{\psi}] e^{\mathbf{i}\int d^D x \bar{\psi}\mathbf{i}\not{D}\psi} = \det \mathbf{i}\not{D} = 0$$

vanishes, since it is the determinant of an operator with a kernel. Rather, only amplitudes for transitions which change the chiral charge by q are allowed.

1.0.3 The physics of the anomaly

[Polyakov, page 102; Kaplan 0912.2560 §2.1; Alvarez-Gaumé] Consider non-relativistic free (*i.e.* no 4-fermion interactions) fermions in 1+1 dimensions, *e.g.* with 1-particle dispersion $\omega_k = \frac{1}{2m}\vec{k}^2$. The groundstate of N such fermions is described by filling the N lowest-energy single particle levels, up the Fermi momentum: $|k| \leq k_F$ are filled. We must introduce an infrared regulator so that the levels are discrete – put them in a box of length L, so that $k_n = \frac{2\pi n}{L}$. (In Figure 1, the red circles are possible 1-particle states, and the green ones are the occupied ones.) The lowest-energy excitations of this groundstate come from taking a fermion just below the Fermi level $k = |k_F - k_1|$ and putting it just above, $k = |k_F + k_2|$; the energy cost is

$$E_{k_1-k_2} = \frac{1}{2m}(k_F + k_1)^2 - \frac{1}{2m}(k_F - k_2)^2 \simeq \frac{k_F}{m}(k_1 - k_2)$$

– we get relativistic dispersion with velocity $v_F = \frac{k_F}{m}$. The fields near these Fermi points in k-space satisfy the Dirac equation[2]:

$$(\omega - v_F\delta k)\,\psi_L = 0, \quad (\omega + v_F\delta k)\,\psi_R = 0$$

where $\delta k \equiv k - k_F$.

It would therefore seem to imply a conserved axial current – the number of left moving fermions minus the number of right moving fermions. But the fields ψ_L and ψ_R are not independent; with high-enough energy excitations, you reach the bottom of the band (near $k = 0$ here) and you can't tell the difference. This means that the numbers are *not* separately conserved.

We can do better in this 1+1d example and show that the amount by which the axial current is violated is given by the anomaly formula. Consider subjecting our poor 1+1d free fermions to an electric field $E_x(t)$ which is constant in space and slowly varies in time. Suppose we gradually turn

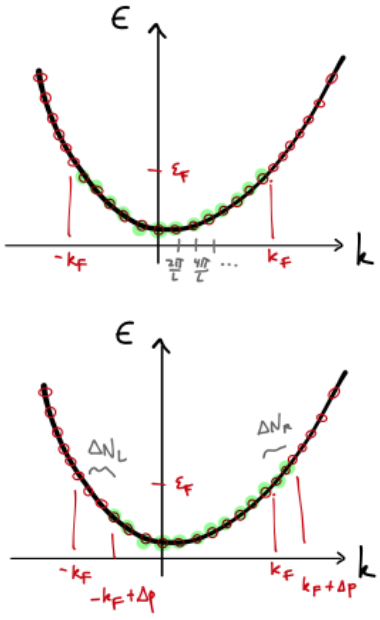


Figure 1: Green dots represent occupied 1-particle states. Top: In the groundstate. Bottom: After applying $E_x(t)$.

[2]This example is worthwhile for us also because we see the relativistic Dirac equation is emerging from a non-relativistic model; in fact we could have started from an even more distant starting point – *e.g.* from a lattice model, like

$$H = -t\sum_n c_n^\dagger c_{n+1} + h.c.$$

where the dispersion would be $\omega_k = -2t(\cos ka - 1) \sim \frac{1}{2m}k^2 + \mathcal{O}(k^4)$ with $\frac{1}{2m} = ta^2$.

it on and then turn it off; here gradually means slowly enough that the process is adiabatic. Then each particle experiences a force $\partial_t p = eE_x$ and its net change in momentum is

$$\Delta p = e \int \mathrm{d}t E_x(t).$$

This means that the electric field puts the fermions in a state where the Fermi surface $k = k_F$ has shifted to the right by Δp, as in the figure. Notice that the total number of fermions is of course the same – charge is conserved.

Now consider the point of view of the low-energy theory at the Fermi points. This theory has the action

$$S[\psi] = \int \mathrm{d}x\mathrm{d}t\bar{\psi}\left(\mathbf{i}\gamma^\mu\partial_\mu\right)\psi \ ,$$

where γ^μ are 2×2 and the upper/lower component of ψ creates fermions near the left/right Fermi point. In the process above, we have added N_R right-moving particles and taken away N_L left-moving particles, that is *added* N_L left-moving holes (aka anti-particles). The axial charge of the state has changed by

$$\Delta Q_A = \Delta(N_L - N_R) = 2\frac{\Delta p}{2\pi/L} = \frac{L}{\pi}\Delta p = \frac{L}{\pi}e\int \mathrm{d}t E_x(t) = \frac{e}{\pi}\int \mathrm{d}t\mathrm{d}x E_x = \frac{e}{2\pi}\int \epsilon_{\mu\nu}F^{\mu\nu}$$

On the other hand, the LHS is $\Delta Q_A = \int \partial^\mu J^A_\mu$. We can infer a local version of this equation by letting E vary slowly in space as well, and we conclude that

$$\partial_\mu J^\mu_A = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}.$$

This agrees exactly with the anomaly equation in $D = 1{+}1$ produced by the calculation above in (1.6) (see the homework).

2 Effective field theory

2.1 A parable on integrating out degrees of freedom

Here's another parable from QM which gives some useful perspective on renormalization in QFT and on the notion of effective field theory.

[Banks p. 138] Consider a system of two coupled harmonic oscillators. We will assume one of the springs is much stiffer than the other: let's call their natural frequencies ω_0, Ω, with $\omega_0 \ll \Omega$. The euclidean-time action is

$$S[Q,q] = \int dt \left[\frac{1}{2}\left(\dot{q}^2 + \omega_0^2 q^2\right) + \frac{1}{2}\left(\dot{Q}^2 + \Omega^2 Q^2\right) + gQq^2\right] \equiv S_{\omega_0}[q] + S_\Omega[Q] + S_{\rm int}[Q,q].$$

(The particular form of the q^2Q coupling is chosen for convenience. Don't take too seriously the physics at negative Q.) We can construct physical observables in this model by studying the path integral:

$$Z = \int [dQdq] e^{-S[Q,q]}.$$

Since I put a minus sign rather than an i in the exponent (and the potential terms in the action have + signs), this is a euclidean path integral.

Let's consider what happens if we do the path integral over the heavy mode Q, and postpone doing the path integral over q. This step, naturally, is called *integrating out* Q, and we will see below why this is a good idea. The result just depends on q; we can think of it as an *effective action* for q:

$$\begin{aligned} e^{-S_{\rm eff}[q]} &:= \int [dQ] e^{-S[q,Q]} \\ &= e^{-S_{\omega_0}[q]} \left\langle e^{-S_{\rm int}[Q,q]} \right\rangle_Q \end{aligned}$$

Here $\langle ... \rangle_Q$ indicates the expectation value of ... in the (free) theory of Q, with the action $S_\Omega[Q]$. It is a gaussian integral (because of our choice of $S_{\rm int}$:

$$\left\langle e^{-S_{\rm int}[Q,q]} \right\rangle_Q = \int [dQ] e^{-S_\Omega[Q] - \int dsJ(s)Q(s)} = \mathcal{N} e^{\frac{1}{4}\int dsdtJ(s)G(s,t)J(t)} .$$

This last equality is an application of the 'fundamental theorem of path integrals,' *i.e.* the gaussian integral. Here $J(s) \equiv gq(s)^2$. The normalization factor $\mathcal{N}$ is independent of J and hence of q. And $G(s,t)$ is the inverse of the linear operator appearing in S_Ω, the euclidean Green's function:

$$S_\Omega[Q] = \int dsdtQ(s)G^{-1}(s,t)Q(t).$$

More usefully, G satisfies

$$\left(-\partial_s^2 + \Omega^2\right) G(s,t) = \delta(s-t) \ .$$

The fact that our system is time-translation invariant means $G(s,t) = G(s-t)$. We can solve this equation in fourier space: $G(s) = \int đ\omega e^{-i\omega s} G_\omega$ makes it algebraic:

$$G_\omega = \frac{1}{\omega^2 + \Omega^2}$$

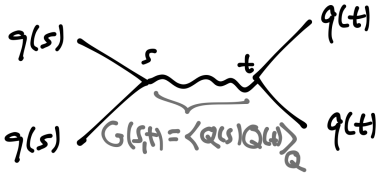
and we have

$$G(s) = \int đ\omega e^{-\mathbf{i}\omega s} \frac{1}{\omega^2 + \Omega^2}. \tag{2.1}$$

So we have:

$$e^{-S_{\text{eff}}[q]} = e^{-S_{\omega_0}[q]} e^{-\int dtds \frac{g^2}{2} q(s)^2 G(s,t) q(t)^2}$$

or taking logs



$$S_{\text{eff}}[q] = S_{\omega_0}[q] + \int dtds \frac{g^2}{2} q(s)^2 G(s,t) q(t)^2 \ . \tag{2.2}$$

Q mediates an interaction of four qs, an anharmonic term, a self-interaction of q. In Feynman diagrams, the leading interaction between q's mediated by Q comes from the diagram at left.

And the whole thing comes from exponentiating disconnected copies of this diagram. There are no other diagrams: once we make a Q from two qs what can it do besides turn back into two qs? Nothing. And no internal q lines are allowed, they are just sources, for the purposes of the Q integral.

But it is *non-local*: we have two integrals over the time in the new quartic term. This is unfamiliar, and *bad*: *e.g.* classically we don't know how to pose an initial value problem using this action.

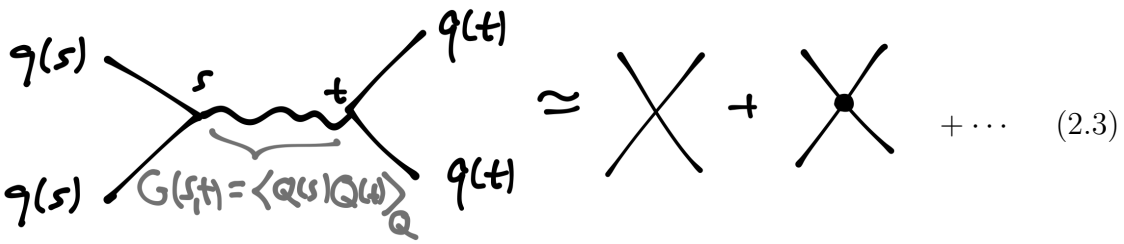
But now suppose we are interested in times much longer than $1/\Omega$, say times comparable to the period of oscillation of the less-stiff spring $2\pi/\omega$. We can accomplish this by Taylor expanding under the integrand in (2.1):

$$G(s) \overset{s \gg 1/\Omega}{\simeq} \int đ\omega e^{-\mathbf{i}\omega s} \frac{1}{\Omega^2} \underbrace{\frac{1}{1 + \frac{\omega^2}{\Omega^2}}}_{= \sum_n (-1)^n \left(\frac{\omega^2}{\Omega^2}\right)^n} \simeq \frac{1}{\Omega^2}\delta(s) + \frac{1}{\Omega^4}\partial_s^2 \delta(s) + ...$$

Plug this back into (2.2):

$$S_{\text{eff}}[q] = S_{\omega_0}[q] + \int dt \frac{g^2}{2\Omega^2} q(t)^4 + \int dt \frac{g^2}{2\Omega^4} \dot{q}^2 q^2 + ...$$

The effects of the heavy mode Q are now organized in a *derivative expansion*, with terms involving more derivatives suppressed by more powers of the high energy scale Ω.



(2.3)

A useful mnemonic for integrating out the effects of the heavy field in terms of Feynman diagrams: to picture Q as propagating for only a short time (compared to the external time $t-s$), we can contract its propagator to a point. The first term on the RHS shifts the q^4 term, the second shifts the kinetic term, the third involves four factors of $\dot{q}$...

On the RHS of this equation, we have various interactions involving four qs, which involve increasingly many derivatives. The first term is a quartic potential term for q: $\Delta V = \frac{g}{\Omega^2} q^4$; the leading effect of the fluctuations of Q is to *shift* the quartic self-coupling of q by a finite amount (note that we could have included a bare $\lambda_0 q^4$ potential term).

Notice that if we keep going in this expansion, we get terms with *more than two derivatives* of q. This is OK. We've just derived the right way to think about such terms: we treat them as a perturbation, and they are part of a never-ending series of terms which become less and less important for low-energy questions. If we want to ask questions about q at energies of order ω, we can get answers that are correct up to effects of order $\left(\frac{\omega}{\Omega}\right)^{2n}$ by keeping the nth term in this expansion.

Conversely if we are doing an experiment with precision Δ at energy ω, we can measure the effects of up to the nth term, with

$$\left(\frac{\omega}{\Omega}\right)^{2n} \sim \Delta.$$

Another important lesson: $S_{\text{eff}}[q]$ contains couplings with negative dimensions of energy

$$\sum_n c_n \left(\partial_t^n q\right)^2 q^2, \quad \text{with } c_n \sim \frac{1}{\Omega^{2n}},$$

exactly the situation where the S-matrix grows too fast at high energies that we discussed in 215B. In this case we know exactly where the probability is going: if we have enough energy to see the problem ($E \sim \Omega$), we have enough energy to kick the heavy mode Q out of its groundstate. [End of Lecture 2]

2.1.1 Attempt to consolidate understanding

We've just done some coarse graining: focusing on the dofs we care about (q), and actively ignoring the dofs we don't care about (Q), except to the extent that they affect those we do (e.g. the self-interactions of q).

Above, we did a calculation in a QM model with two SHOs. This is a paradigm of QFT in many ways. For one thing, free quantum fields are bunches of harmonic oscillators with natural frequency depending on k, $\Omega = \sqrt{\vec{k}^2 + m^2}$. Here we kept just two of these modes (one with large k, one with small k) for clarity. Perhaps more importantly, QM is just QFT in 0+1d. The more general QFT path integral just involves more integration variables. The idea of the Wilsonian RG (for continuum field theory) is essentially to do the integrals over the modes in descending order of wavenumber, and at each stage make the expansion described above to get a local action. And notice that basically all possible terms are generated, consistent with the symmetries (here for example, there is a $\mathbb{Z}_2$ symmetry under which $q \to -q$, so there are no odd powers of q). For more on this, see Physics 217.

The result of that calculation was that fluctuations of Q mediate various q^4 interactions. It adds to the action for q the following: $\Delta S_{\text{eff}}[q] \sim \int dtds q^2(t) G(t-s) q^2(s)$, as in Fig. 2.3.

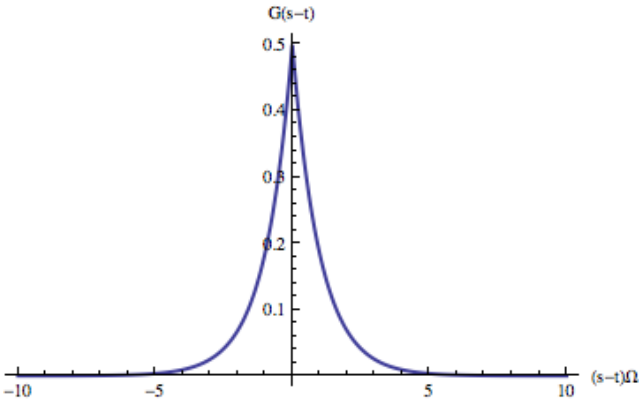
If we have the hubris to care about the exact answer, it's nonlocal in time. But if we want exact answers then we'll have to do the integral over q, too. On the other hand, the hierarchy of scales $\omega_0 \ll \Omega$ is useful if we ask questions about energies of order ω_0, e.g.

$$\langle q(t)q(0)\rangle \text{ with } t \sim \frac{1}{\omega_0} \gg \Omega$$

Then we can Taylor expand the function $G(t-s)$, and we find a series of corrections in powers of $\frac{1}{t\Omega}$ (or more accurately, powers of $\frac{\partial_t}{\Omega}$).

(Notice that it's not so useful to integrate out light degrees of freedom to get an action for the heavy degrees of freedom; that would necessarily be nonlocal and stay nonlocal and we wouldn't be able to treat it using ordinary techniques.)

The crucial point is that the scary non-locality of the effective action that we saw only extends a distance of order $\frac{1}{\Omega}$; the kernel $G(s-t)$ looks like this:



The mechanism we've just discussed is an essential ingredient in getting *any* physics done at all. *Why* can we do physics despite the fact that we do not understand the the-

ory of quantum gravity which governs Planckian distances? We happily do lots of physics without worrying about this! This is because the effect of those Planckian quantum gravity fluctuations – whatever they are, call them Q – on the degrees of freedom we do care about (*e.g.* the Standard Model, or an atom, or the sandwich you made this morning, call them collectively q) are encoded in terms in the effective action of q which are suppressed by powers of the high energy scale $M_{\rm Planck}$, whose role in the toy model is played by Ω. And the natural energy scale of your sandwich is much less than $M_{\rm Planck}$.

I picked the Planck scale as the scale to ignore here for rhetorical drama, and because we really are ignorant of what physics goes on there. But this idea is equally relevant for *e.g.* being able to describe water waves by hydrodynamics (a classical field theory) without worrying about atomic physics, or to understand the physics of atoms without needing to understand nuclear physics, or to understand the nuclear interactions without knowing about the Higgs boson, and so on deeper into the onion of physics.

This wonderful situation, which makes physics possible, has a price: since physics at low energies is so insensitive to high energy physics, it makes it hard to learn about high energy physics! People have been very clever and have learned a lot in spite of this vexing property of the RG called decoupling. We can hope that will continue. (Cosmological inflation plays a similar role in hiding the physics of the early universe. It's like whoever designed this game is *trying* to hide this stuff from us.)

The explicit functional form of $G(s)$ (the inverse of the (euclidean) kinetic operator for Q) is:

$$G(s) = \int đ\omega \frac{e^{-\mathbf{i}\omega s}}{\omega^2+\Omega^2} = e^{-\Omega|s|}\frac{1}{2\Omega}. \tag{2.4}$$

Do it by residues: the integrand has poles at $\omega = \pm\mathbf{i}\Omega$. The absolute value of $|s|$ is crucial, and comes from the fact that the contour at infinity converges in the upper (lower) half plane for $s<0$ ($s>0$).

Next, some comments about ingredients in this discussion, which provide a useful opportunity to review/introduce some important QFT technology:

- Please don't be confused by the formal similarity of the above manipulations with the construction of the generating functional of correlation functions of Q:

$$Z[J] \equiv \left\langle e^{\int dt Q(t) J(t)} \right\rangle_Q \ , \quad \langle Q(t_1)Q(t_2)...\rangle_Q = \frac{\delta}{\delta J(t_1)}\frac{\delta}{\delta J(t_2)}... \log Z[J]$$

It's true that what we did above amounts precisely to constructing $Z[J]$, and plugging in $J = g_0 q^2$. But the motivation is different: in the above q is also a dynamical variable, so we don't get to pick q and differentiate with respect to it; we are merely postponing doing the path integral over q until later.

- Having said that, what is this quantity $G(s)$ above? It is the (euclidean) two-point function of Q:

$$G(s,t) = \langle Q(s)Q(t)\rangle_Q = \frac{\delta}{\delta J(t)}\frac{\delta}{\delta J(s)}\log Z[J].$$

The middle expression makes it clearer that $G(s,t) = G(s-t)$ since nobody has chosen the origin of the time axis in this problem. This euclidean Green's function, the inverse of $-\partial_\tau^2 + \Omega^2$, is unique, once we demand that it falls off at large separation (unlike the real-time Green's function).

- **Adding more labels.** Quantum mechanics is quantum field theory in 0+1 dimensions. Except for our ability to do all the integrals, everything we are doing here generalizes to quantum field theory in more dimensions: quantum field theory *is* quantum mechanics (with infinitely many degrees of freedom). With more spatial dimensions, we'll want to use the variable x for the spatial coordinates (which are just labels on the fields!) and it was in anticipation of this step that I used q instead of x for my oscillator position variables.

2.2 Introduction to effective field theory

[Some nice lecture notes on effective field theory can be found here: J. Polchinski, A. Manohar, I. Rothstein, D. B. Kaplan, H. Georgi. A. Manohar]

Taking the example of the previous subsection to its logical conclusion, we are led to the idea of an *effective field theory* (EFT), or, *how to do physics without a theory of everything*. (You may notice that all the physics that has been done has been done without a theory of everything.) It is a weaponized version of selective inattention.

The basic idea is that the Hamiltonian (or the action) should contain all terms consistent with symmetries, organized according to an expansion in decreasing relevance to low energy physics. This is an implementation of the *totalitarian principle of physics*, that anything that can happen must happen. (This principle is best understood from the perspective of Wilson's work on the renormalization group, which is the subject of Physics 217.)

Diatribe about 'renormalizability'. There is no reason to demand that a field theory that we have found to describe physics in some regime should be a valid descrip-

tion of the world to arbitrarily short (or long!) distances. This is a happy statement: there can always be new physics that has been so far hidden from us. Rather, an EFT comes with a regime of validity, and with necessary cutoffs. As we will discuss, in a useful implementation of an EFT, the cutoff implies a small parameter in which we can expand (and hence compute). (In the example of $S_{\text{eff}}[q]$ of the previous subsection, the small parameter is ω/Ω.)

Caring about renormalizibility is pretending to know about physics at arbitrarily short distances. Which you don't.

Even when theories are renormalizable, this apparent victory is often false. For example, QED requires only two independent counterterms (for the mass and for the fine structure constant), and is therefore by the old-fashioned definition renormalizable, but it is superseded by the electroweak theory above 80GeV. Also: the coupling in QED actually increases logarithmically at shorter distances, and ultimately reaches a *Landau pole* at SOME RIDICULOUSLY HIGH ENERGY (of order $e^{+\frac{c}{\alpha}}$ where $\alpha \sim \frac{1}{137}$ is the fine structure constant (*e.g.* at the scale of atomic physics) and c is some numerical number. Plugging in numbers gives something like 10^{330} GeV, which is quite a bit larger than the Planck scale). This is of course completely irrelevant for physics and even in principle because of the previous remark about electroweak unification. And if not because of that, because of the Planck scale. A heartbreaking historical fact is that Landau and many other smart people gave up on QFT as a whole because of this silly fantasy about QED in an unphysical regime.

We will see below that even in QFTs which are non-renormalizable in the strict sense, there is a more useful notion of renormalizability: effective field theories come with a small parameter (often some ratio of mass scales), in which we may expand the action. A useful EFT requires a finite number of counterterms *at each order in the expansion*.

Furthermore, I claim that this is *always* the definition of renormalizability that we are using, even if we are using a theory which is renormalizable in the traditional sense, which allows us to pretend that there is no cutoff. That is, there could always be corrections of order $\left(\frac{E}{E_{\text{new}}}\right)^n$ where E is some energy scale of physics that we are doing and E_{new} is some UV scale where new physics might come in; for large enough n, this is too small for us to have seen. The property of renormalizibility that actually matters is that we need a finite number of counterterms at each order in the expansion in $\frac{E}{E_{\text{new}}}$.

Renormalizable QFTs are in some sense *less* powerful than non-renormalizable ones – the latter have the decency to tell us when they are giving the wrong answer! That is, they tell us at what energy new physics *must* come in; with a renormalizable theory

we may blithely pretend that it is valid in some ridiculously inappropriate regime like 10^{330} GeV.

Notions of EFT. There is a dichotomy in the way EFTs are used. Sometimes one knows a lot about the UV theory (*e.g.*

- electroweak gauge theory,
- QCD,
- electrons in a solid,
- water molecules

...) but it is complicated and unwieldy for the questions one wants to answer, so instead one develops an effective field theory involving just the appropriate and important dofs (*e.g.*, respectively,

- Fermi theory of weak interactions,
- chiral lagrangian (or HQET or SCET or ...),
- Landau Fermi liquid theory (or the Hubbard model or a topological field theory or ...),
- hydrodynamics (or some theory of phonons in ice or ...)

...). As you can see from the preceding lists of examples, even a single UV theory can have many different IR EFTs depending on what phase it is in, and depending on what question one wants to ask. The relationship between the pairs of theories above is always coarse-graining from the UV to the IR, though exactly what plays the role of the RG parameter can vary wildly. For example, in the case of the Fermi liquid theory, the scaling is $\omega \to 0$, and momenta scale towards the Fermi surface, not $\vec{k} = 0$.

A second situation is when one knows a description of some low-energy physics up to some UV scale, and wants to try to infer what the UV theory might be. This is a common situation in physics! Prominent examples include: the Standard Model, and quantized Einstein gravity. Occasionally we (humans) actually learn some physics and an example of an EFT from the second category moves to the first category.

Instructions for EFT. Answer the following questions:

1. what are the dofs?

2. what are the symmetries?

3. where is the cutoff on its validity?

Then write down all interactions between the dofs which preserve the symmetry in an expansion in derivatives, with higher-dimension operators suppressed by more powers of the UV scale.

I must also emphasize two distinct usages of the term 'effective field theory' which are common, and which the discussion above is guilty of conflating (this (often slippery) distinction is emphasized in the review article by Georgi linked at the beginning of this subsection). The Wilsonian perspective advocated above produces a low-energy description of the physics which is really just a way of solving (if you can) the original model; very reductively, it's just a physically well-motivated order for doing the integrals. If you really integrate out the high energy modes exactly, you will get a non-local action for the low energy modes. This is to be contrasted with the local actions one uses in practice, by truncating the derivative expansion. It is the latter which is really the action of the effective field theory, as opposed to the *full theory*, with some of the integrals done already. The latter will give correct answers for physics below the cutoff scale, and it will give them much more easily.

Some interesting and/or important examples where EFT has been useful (some of which we will discuss in more detail below) and where you can learn about them:

- Hydrodynamics [Kovtun]
- Fermi liquid theory [J. Polchinski, R. Shankar, *Rev. Mod. Phys.* **66** (1994) 129]
- chiral perturbation theory [D. B. Kaplan, §4]
- heavy quark effective theory [D. B. Kaplan, §1.3, Manohar and Wise, *Heavy Quark Physics*]
- random surface growth (KPZ) [Zee, chapter VI]
- color superconductors [D. B. Kaplan, §5]
- gravitational radiation from binary mergers [Goldberger, Rothstein, Porto]
- soft collinear effective theory [Becher, Stewart]
- magnets [Zee, chapter VI.5, hep-ph/9311264v1]

- effective field theory of cosmological inflation [Senatore et al, Cheung et al, Porto]
- effective field theory of dark matter direct detection [Fitzpatrick et al]

There are many others, the length of this list was limited by how long I was willing to spend digging up references.

2.3 The color of the sky

[from hep-ph/9606222 and nucl-th/0510023] Why is the sky blue? Basically, it's because the blue light from the sun scatters in the atmosphere more than the red light, and you (I hope) only look at the scattered light.

Here is an understanding of this fact using the EFT logic. Consider the scattering of photons off atoms (in a gas) at low energies. Low energy means that the photon does not have enough energy to probe the substructure of the atom – it can't excite the electrons or the nuclei. This means that the atom is just a particle, with some mass M.

The dofs are just the photon field and the field that creates an atom.

The symmetries are Lorentz invariance and charge conjugation invariance and parity. We'll use the usual redundant description of the photon which has also gauge invariance.

The cutoff is the energy ΔE that it takes to excite atomic energy levels we've left out of the discussion. We allow no inelastic scattering. This means we require

$$E_\gamma \ll \Delta E \sim \frac{\alpha}{a_0} \ll a_0^{-1} \ll M_{\text{atom}} \tag{2.5}$$

Because of this separation of scales, we can also ignore the recoil of the atom, and treat it as infinitely heavy.

Since there are no charged objects in sight – atoms are neutral – gauge invariance means the Lagrangian can depend on the field strength $F_{\mu\nu}$. Let's call the field which destroys an atom with velocity v ϕ_v. $v^\mu v_\mu = 1$ and $v_\mu = (1,0,0,0)_\mu$ in the atom's rest frame. The (Lorentz-singlet) Lagrangian can depend on v^μ. We can write a Lagrangian for the free atoms as

$$L_{\text{atom}} = \phi_v^\dagger \mathbf{i} v^\mu \partial_\mu \phi_v \quad .$$

This action is related by a boost to the statement that the atom at rest has zero energy – in the rest frame of the atom, the eom is just $\partial_t \phi_{v=(1,\vec{0})} = 0$. (If we didn't define the zero of energy to be at the rest mass, there would be an additional term $M_{\text{atom}} \phi_v^\dagger \phi_v$.) Notice that the kinetic term $\phi_v^\dagger \frac{\vec{\nabla}^2}{2M_{\text{atom}}} \phi_v$ is a very small correction given our hierarchy of scales (2.5).

So the Lagrangian density is

$$L_{\text{Maxwell}}[A] + L_{\text{atom}}[\phi_v] + L_{\text{int}}[A, \phi_v]$$

and we must determine L_{int}. It is made from local, Hermitian, gauge-invariant, Lorentz invariant operators we can construct out of $\phi_v, F_{\mu\nu}, v_\mu, \partial_\mu$ (it can only depend on $F_{\mu\nu} =$

$\partial_\mu A_\nu - \partial_\nu A_\mu$, and not A_μ directly, by gauge invariance, because the atom, and hence ϕ_v, is neutral.). It should actually only depend on the combination $\phi_v^\dagger \phi_v$ since we will not create and destroy atoms. (Notice that we didn't have to specify the statistics of the atoms or ϕ_v.) Therefore

$$L_{\text{int}} = c_1 \phi_v^\dagger \phi_v F_{\mu\nu} F^{\mu\nu} + c_2 \phi_v^\dagger \phi_v v^\sigma F_{\sigma\mu} v_\lambda F^{\lambda\mu} + c_3 \phi_v^\dagger \phi_v \left(v^\lambda \partial_\lambda\right) F_{\mu\nu} F^{\mu\nu} + \dots$$

$\dots$ indicates terms with more derivatives and more powers of velocity (*i.e.* an expansion in $\partial \cdot v$). Which are the most important terms at low energies? Demanding that the Maxwell term dominate, we get the power counting rules (so time and space should scale the same way):

$$[\partial_\mu] = 1, \quad [F_{\mu\nu}] = 2$$

This then implies $[\phi_v] = 3/2, [v] = 0$ and therefore

$$[c_1] = [c_2] = -3, [c_3] = -4 \ .$$

Terms with more partials are more irrelevant.

What makes up these dimensions? They must come from the length scales that we have integrated out to get this description – the size of the atom $a_0 \sim (\alpha m_e)^{-1}$ and the energy gap between the ground state and the electronic excited states $\Delta E \sim \alpha^2 m_e$. For $E_\gamma \ll \Delta E, a_0^{-1}$, we can just keep the two leading terms.

In the rest frame of the atom, these two leading terms $c_{1,2}$ represent just the scattering of E and B respectively. To determine their coefficients one would have to do a matching calculation to a more complete theory (compute transition rates in a theory that does include extra energy levels of the atom). But a reasonable guess is just that the scale of new physics (in this case atomic physics) makes up the dimensions: $c_1 \simeq c_2 \simeq a_0^3$. (In fact the magnetic term c_2 comes with extra factor of v/c which suppresses it.) The scattering cross section then goes like $\sigma \sim c_i^2 \sim a_0^6$; dimensional analysis ($[\sigma] = -2$ is an area, $[a_0^6] = -6$) then tells us that we have to make up four powers with the only other scale around:

$$\sigma \propto E_\gamma^4 a_0^6.$$

(The factor of E_γ^2 in the amplitude arises from $\vec{E} \propto \partial_t \vec{A}$.) Blue light, which has about twice the energy of red light, is therefore scattered 16 times as much.

The leading term that we left out is the one with coefficient c_3. The size of this coefficient determines when our approximations break down. We might expect this to come from the next smallest of our neglected scales, namely ΔE. That is, we expect

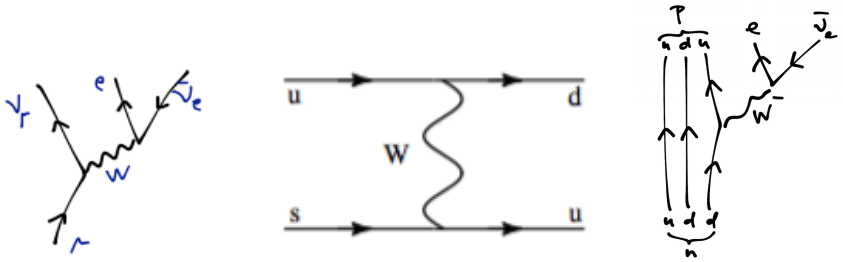
$$\sigma \propto E_\gamma^4 a_0^6 \left(1 + \mathcal{O}\left(\frac{E_\gamma}{\Delta E}\right)\right).$$

The ratio in the correction terms is appreciable for UV light.

2.4 Fermi theory of Weak Interactions

[from §5 of A. Manohar's EFT lectures] As another example of EFT, let's think about part of the Standard Model.

$$L_{EW} \ni -\frac{1}{2}\left(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu\right)\left(\partial^\mu W^{-\nu} - \partial^\nu W^{-\mu}\right) + M_W W^+_\mu W^{-\mu} \qquad (2.6)$$
$$-\frac{\mathbf{i}g}{\sqrt{2}}\bar{\psi}_i\gamma^\mu P_L\psi_j W^+_\mu V_{ij} \quad + \text{terms involving } Z \text{ bosons}$$



Some things intermediate, off-shell W bosons can do: μ decay, $\Delta S = 1$ processes, neutron decay

If we are asking questions with external momenta less than M_W, we can integrate out W and make our lives simpler:

$$\delta S_{eff} \sim \left(\frac{\mathbf{i}g}{\sqrt{2}}\right)^2 V_{ij}V^\star_{k\ell}\int \mathrm{d}^D p \frac{-\mathbf{i}g_{\mu\nu}}{p^2 - M_W^2}\left(\bar{\psi}_i\gamma^\mu P_L\psi_j\right)(p)\left(\bar{\psi}_k\gamma^\nu P_L\psi_\ell\right)(-p)$$

(I am lying a little bit about the W propagator in that I am not explicitly projecting out the fourth polarization with the negative residue. Also, the W carries electric charge, so the charges of $\bar{\psi}_i$ and ψ_j in (2.6) must differ by one.) This is non-local at scales $p \gtrsim M_W$ (recall the discussion of the subsection §2.1). But for $p^2 \ll M_W^2$,

$$\frac{1}{p^2 - M_W^2} \overset{p^2 \ll M_W^2}{\simeq} -\frac{1}{M_W^2}\left(1 + \underbrace{\frac{p^2}{M_W^2} + \frac{p^4}{M_W^4} + \ldots}_{\text{derivative couplings}}\right) \qquad (2.7)$$

$$S_F = -\frac{4G_F}{\sqrt{2}}V_{ij}V^\star_{kl}\int d^4x \left(\bar{\psi}_i\gamma^\mu P_L\psi_j\right)(x)\left(\bar{\psi}_k\gamma_\mu P_L\psi_\ell\right)(x) + \mathcal{O}\left(\frac{1}{M_W^2}\right) + \text{kinetic terms for fermions} \qquad (2.8)$$

where $G_F/\sqrt{2} \equiv \frac{g^2}{8M_W^2}$ is the Fermi coupling. We can use this (Fermi's) theory to compute the amplitudes above, and it is much simpler than the full electroweak theory (for example I don't have to lie about the form of the propagator of the W-boson like I

did above). It was discovered first and used quite effectively long before the existence of Ws was suspected. [End of Lecture 3]

On the other hand, this theory is *not the same* as the electroweak theory; for example it is not renormalizable, while the EW theory is. Its point in life is to help facilitate the expansion in $1/M_W$. There is something about the expression (2.8) that should make you nervous, namely the big red 1 in the $1/M_W^2$ corrections: what makes up the dimensions? This becomes an issue when we ask about ...

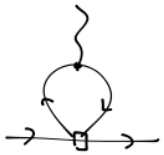
2.5 Loops in EFT

Suppose we try to define the Fermi theory S_F with a euclidean momentum cutoff $|k_E| < \Lambda$. We expect that we'll have to set $\Lambda \sim M_W$. A simple example which shows that this is problematic arises by asking about radiative corrections in the 4-Fermi theory to the coupling between the fermions and the photon (or the Z boson).

We are just trying to estimate the magnitude of this correction, so don't worry about the factors and the gamma matrices:

$$\sim I \equiv \underbrace{\frac{1}{M_W^2}}_{\propto G_F} \underbrace{\int^\Lambda \mathrm{d}^4k \frac{1}{k}\frac{1}{k} \mathrm{tr}\,(\gamma...)}_{\sim \int^\Lambda k\mathrm{d}k \sim \Lambda^2 \sim M_W^2} \sim \mathcal{O}(1).$$

Even worse, consider what happens if we use the vertex coming from the $\left(\frac{p^2}{M_W^2}\right)^\ell$ correction in (2.7)



$$\sim I_\ell \equiv \frac{1}{M_W^2} \int^\Lambda \mathrm{d}^4k \frac{1}{k^2} \left(\frac{k^2}{M_W^2}\right)^\ell \sim \mathcal{O}(1)$$

– it's also unsuppressed by powers of ... well, anything. This is a problem.

Fix: A way to fix this is to use a "mass-independent subtraction scheme", such as dimensional regularization and minimal subtraction ($\overline{\mathrm{MS}}$). The crucial feature is that the dimensionful cutoff parameter appears only inside logarithms ($\log \mu$), and not as free-standing powers (μ^2).

With such a scheme, we'd get instead

$$I \sim \frac{m^2}{M_W^2} \log \mu \qquad I_\ell \sim \left(\frac{m^2}{M_W^2} \right)^{\ell+1} \log \mu$$

where m is some mass scale *other* than the RG scale μ (like a fermion mass parameter, or an external momentum, or a dynamical scale like Λ_{QCD}).

We will give a more detailed example next. The point is that in a mass-independent scheme, the regulator doesn't produce new dimensionful things that can cancel out the factors of M_W in the denominator. It respects the 'power counting': if you see 2ℓ powers of $1/M_W$ in the coefficient of some term in the action, that's how many powers will suppress its contributions to amplitudes. This means that the EFT is like a renormalizable theory *at each order in the expansion* (here in $1/M_W$), in that there is only a finite number of allowed vertices that contribute at each order (counterterms for which need to be fixed by a renormalization condition). The insatiable appetite for counterterms is still insatiable, but it eats only a finite number at each order in the expansion. Eventually you'll get to an order in the expansion that's too small to care about, at which point the EFT will have eaten only a finite number of counterterms.

There is a price for these wonderful features of mass-independent schemes, which has two aspects:

- Heavy particles (of mass m) don't decouple when $\mu < m$. For example, in a mass-independent scheme for a gauge theory, heavy charged particles contribute to the beta function for the gauge coupling even at $\mu \ll m$.
- Perturbation theory will break down at *low* energies, when $\mu < m$; in the example just mentioned this happens because the coupling keeps running.

We will show both these properties very explicitly in the next subsection. The solution of both these problems is to integrate out the heavy particles by hand at $\mu = m$, and make a new EFT for $\mu < m$ which simply omits that field. Processes for which we should set $\mu < m$ don't have enough energy to make the heavy particles in external states anyway. (For some situations where you should still worry about them, see Aneesh Manohar's notes linked above.)

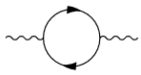
2.5.1 Comparison of schemes, case study

The case study we will make is the contribution of a charged fermion of mass m to the running of the QED gauge coupling.

Recall that the QED Lagrangian is

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}\left(\mathbf{i}\not{D} - m\right)\psi$$

with $D_\mu = \partial_\mu - \mathbf{i}eA_\mu$. By redefining the field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ by a constant factor we can move around where the e appears, *i.e.* by writing $\tilde{A} = eA$, we can make the gauge kinetic term look like $\frac{1}{4e^2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$. This means that the charge renormalization can be seen *either* in the vacuum polarization, the correction to the photon propagator:



. I will call this diagram $\mathbf{i}\Pi_{\mu\nu}$.

So the information about the running of the coupling is encoded in the gauge field two-point function:

$$\Pi_{\mu\nu} \equiv \langle A_\mu(p)A_\nu(q)\rangle = \left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\not{\delta}(p+q)\Pi(p^2) \ .$$

The factor $P_{\mu\nu} \equiv p_\mu p_\nu - p^2 g_{\mu\nu}$ is guaranteed to be the polarization structure by the gauge invariance Ward identity: $p^\mu \langle A_\mu(p)A_\nu(q)\rangle = 0$. That is: $p^\mu P_{\mu\nu} = 0$, and there is no other symmetric tensor made from p^μ which satisfies this. This determines the correlator up to a function of p^2, which we have called $\Pi(p^2)$.

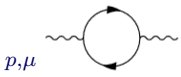
The choice of scheme shows up in our choice of renormalization condition to impose on $\Pi(p^2)$:

Mass-dependent scheme: subtract the value of the graph at $p^2 = -M^2$ (a very off-shell, euclidean, momentum). That is, we impose a renormalization condition which says

$$\Pi(p^2 = -M^2) \stackrel{!}{=} 1 \tag{2.9}$$

(which is the tree-level answer with the normalization above).

The contribution of a fermion of mass m and charge e is (factoring out the momentum-conserving delta function):



$$= -\int \mathrm{đ}^D k \,\mathrm{tr}\left((-\mathbf{i}e\gamma^\mu)\frac{-\mathbf{i}\left(\not{k}+m\right)}{k^2-m^2}(-\mathbf{i}e\gamma^\nu)\frac{-\mathbf{i}\left(\not{p}+\not{k}+m\right)}{(p+k)^2-m^2}\right)$$

The minus sign out front is from the fermion loop. Some boiling, which you can find in Peskin (page 247) or Zee (§III.7), reduces this to something manageable. The steps involved are: (1) a trick to combine the denominators, like the Feynman trick $\frac{1}{AB} = \int_0^1 dx \left(\frac{1}{(1-x)A+xB}\right)^2$. (2) some Dirac algebra, to turn the numerator into a polynomial

in k, p. As Zee says, our job in this course is not to train to be professional integrators. The result of this boiling can be written

$$\mathbf{i}\Pi^{\mu\nu} = -e^2 \int \text{đ}^D \ell \int_0^1 dx \frac{N^{\mu\nu}}{\left(\ell^2 - \Delta\right)^2}$$

with $\ell = k + xp$ is a new integration variable, $\Delta \equiv m^2 - x(1-x)p^2$, and the numerator is

$$N^{\mu\nu} = 2\ell^\mu \ell^\nu - g^{\mu\nu}\ell^2 - 2x(1-x)p^\mu p^\nu + g^{\mu\nu}\left(m^2 + x(1-x)p^2\right) + \text{terms linear in } \ell^\mu \quad .$$

In dim reg, the one-loop vacuum polarization correction satisfies the gauge invaraince Ward identity $\Pi^{\mu\nu} = P^{\mu\nu}\delta\Pi_2$ (unlike the euclidean momentum cutoff which is not gauge invariant). A peek at the tables of dim reg integrals shows that $\delta\Pi_2$ is:

$$\begin{aligned} \delta\Pi_2(p^2) &\overset{\text{Peskin p. 252}}{=} -\frac{8e^2}{(4\pi)^{D/2}} \int_0^1 dx x(1-x) \frac{\Gamma(2 - D/2)}{\Delta^{2-D/2}} \bar{\mu}^\epsilon \\ &\overset{D\to 4}{=} -\frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \left(\frac{2}{\epsilon} - \log\left(\frac{\Delta}{\mu^2}\right)\right) \end{aligned} \tag{2.10}$$

where we have introduced the heralded μ:

$$\mu^2 \equiv 4\pi\bar{\mu}^2 e^{-\gamma_E}$$

where γ_E is the Euler-Mascheroni constant. In the second line of (2.10), we expanded the Γ-function about $D = 4$; there are other singularities at other integer dimensions.

Mass-dependent scheme: Now back to our discussion of schemes. I remind you that in a mass-independent scheme, we demand that the counterterm cancels $\delta\Pi_2$ when we set the external momentum to $p^2 = -M^2$, so that the whole contribution at order e^2 is :

$$0 \overset{(2.9)!}{=} \Pi_2^{(M)}(p^2 = -M^2) = \underbrace{\delta_{F^2}^{(M)}}_{\text{counterterm coefficient for } \frac{1}{4}F_{\mu\nu}F^{\mu\nu}} + \delta\Pi_2$$

$$\implies \Pi_2^{(M)}(p^2) = \frac{e^2}{2\pi^2} \int dx x(1-x) \log\left(\frac{m^2 - x(1-x)p^2}{m^2 + x(1-x)M^2}\right).$$

Notice that the μs go away in this scheme.

Mass-Independent scheme: This is to be contrasted with what we get in a mass-independent scheme, such as $\overline{\text{MS}}$, in which Π is defined by the rule that we *subtract the* $1/\epsilon$ *pole*. This means that the counterterm is

$$\delta_{F^2}^{(\overline{\text{MS}})} = -\frac{e^2}{2\pi^2}\frac{2}{\epsilon} \underbrace{\int_0^1 dx x(1-x)}_{=1/6}.$$

(Confession: I don't know how to state this in terms of a simple renormalization condition on Π_2. Also: the bar in $\overline{\rm MS}$ refers to the (not so important) distinction between $\bar{\mu}$ and μ.) The resulting vacuum polarization function is

$$\Pi_2^{(\overline{\rm MS})}(p^2) = \frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\log\left(\frac{m^2 - x(1-x)p^2}{\mu^2}\right).$$

Next we will talk about beta functions, and verify the claim above about the failure of decoupling. First let me say some words about what is failing. What is failing – the price we are paying for our power counting – is the basic principle of the RG, namely that physics at low energies shouldn't care about physics at high energies, except for small corrections to couplings. An informal version of this statement is: you don't need to know about nuclear physics to make toast. A more formal version is the *Appelquist-Carazzone Decoupling Theorem*, which I will not state (Phys. Rev. D11, 28565 (1975)). So it's something we must and will fix.

Beta functions. $\boxed{M}$: First in the mass-dependent scheme. The fermion contribution to the beta function for the EM coupling is

$$\beta_e^{(M)} = \frac{e}{2}M\partial_M\Pi_2(p^2) = -\frac{1}{2}\left(\frac{e^3}{2\pi}\right)\int_0^1 dx x(1-x)\left(\frac{-2M^2x(1-x)}{m^2+M^2x(1-x)}\right) + \mathcal{O}(e^5)$$
$$\begin{cases} \overset{m\ll M}{\simeq} \frac{e^3}{2\pi^2}\int_0^1 dx x(1-x) = \frac{e^3}{12\pi^2} \\ \\ \overset{m\gg M}{\simeq} \frac{e^3}{2\pi^2}\int_0^1 dx x(1-x)\frac{M^2x(1-x)}{m^2} = \frac{e^3}{60\pi^2}\frac{M^2}{m^2} \end{cases}. \tag{2.11}$$

$$\boxed{\overline{\rm MS}}: \beta_e^{(\overline{\rm MS})} = \frac{e}{2}\mu\partial_\mu\Pi_2(p^2) = -\frac{1}{2}\frac{e^3}{2\pi^2}\underbrace{\int_0^1 dx x(1-x)}_{=1/6}\underbrace{\mu\partial_\mu\log\frac{m^2-p^2x(1-x)}{\mu^2}}_{=-2}$$
$$= \frac{e^3}{12\pi^2}. \tag{2.12}$$

Also, the $\overline{\rm MS}$ vacuum polarization behaves for small external momenta like

$$\Pi_2(p^2 \ll m^2) \simeq -\frac{e^3}{2\pi^2}\int_0^1 dx x(1-x) \quad \underbrace{\log\frac{m^2}{\mu^2}}_{\gg 1, \text{for } \mu\ll m! \text{ bad!}}$$

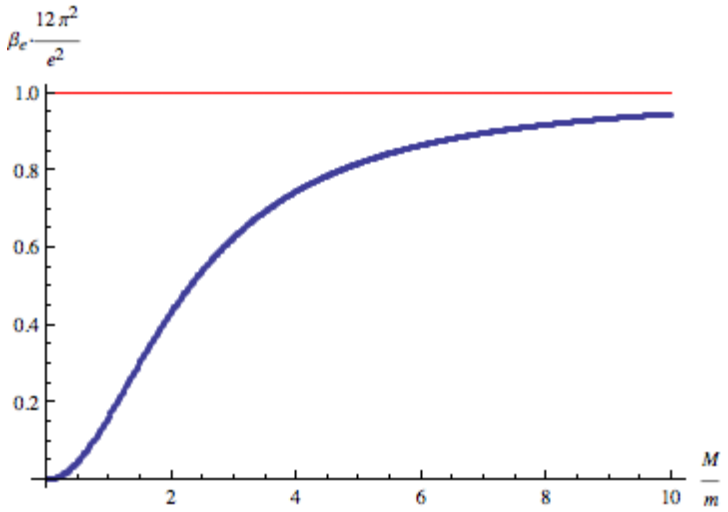
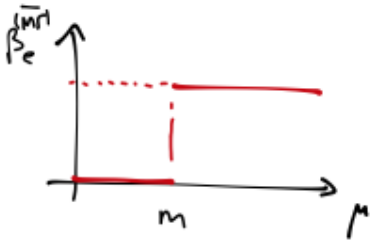


Figure 2: The blue curve is the mass-dependent-scheme beta function; at scales $M \ll m$, the mass of the heavy fermion, the fermion sensibly stops screening the charge. The red line is the $\overline{\text{MS}}$ beta function, which is just a constant, pinned at the UV value.

As I mentioned, the resolution of both these problems is simply to define a new EFT for $\mu < m$ which omits the heavy field. Then the strong coupling problem goes away and the heavy fields do decouple. The price is that we have to do this by hand, and the beta function jumps at $\mu = m$; the coupling is continuous, though.



2.6 The Standard Model as an EFT.

The Standard Model. [Schwartz, §29]

	$L = \begin{pmatrix}\nu_L \\ e_L\end{pmatrix}$	e_R	ν_R	$Q = \begin{pmatrix}u_L \\ d_L\end{pmatrix}$	u_R	d_R	H
$\mathsf{SU}(3)$	-	-	-	□	□	□	-
$\mathsf{SU}(2)$	□	-	-	□	-	-	□
$\mathsf{U}(1)_Y$	$-\frac{1}{2}$	-1	0	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{2}$

Table 1: The Standard Model fields and their quantum numbers under the gauge group. □ indicates fundamental representation, - indicates singlet. Except for the Higgs, each column is copied three times; each copy is called a *generation*. Except for the Higgs all the matter fields are Weyl fermions of the indicated handedness. Gauge fields as implied by the gauge groups. (Some people might leave out the right-handed neutrino, ν_R.)

Whence the values of the charges under the $\mathsf{U}(1)$ ("hypercharge")? The condition $Y_L + 3Y_Q = 0$ (where Y is the hypercharge) is required by anomaly cancellation. This implies that electrons and protons $p = \epsilon_{ijk} u_i u_j d_k$ have exactly opposite charges of the same magnitude.

The Lagrangian is just all the terms which are invariant under the gauge group $\mathsf{SU}(3) \times \mathsf{SU}(2) \times \mathsf{U}(1)$ with dimension less than or equal to four – all renormalizable terms. This includes a potential for the Higgs, $V(|H|) = m_H^2 |H|^2 + \lambda |H|^4$, where it turns out that $m_H^2 \leq 0$. The resulting Higgs vacuum expectation value breaks the Electroweak part of the gauge group

$$\mathsf{SU}(2) \times \mathsf{U}(1)_Y \overset{\langle H \rangle}{\rightsquigarrow} \mathsf{U}(1)_{EM}.$$

The broken gauge bosons get masses from the Higgs kinetic term

$$|D_\mu H|^2|_{H=\begin{pmatrix}0 \\ v/\sqrt{2}\end{pmatrix}} \quad \text{with} \quad D_\mu H = \left(\partial_\mu - \mathbf{i} g W_\mu^a \tau^a - \frac{1}{2}\mathbf{i} g' Y_\mu\right) H$$

where Y_μ is the hypercharge gauge boson, and $W^a, a = 1, 2, 3$ are the $\mathsf{SU}(2)$ gauge bosons. There are two massive W-bosons with electric charge ± 1 (as described in §2.4), with $M_W = \frac{vg}{2}$. The photon and Z boson are the linear combinations of Y and W^3 which diagonalize the remaining mass terms:

$$\begin{pmatrix}A_\mu \\ Z_\mu\end{pmatrix} = \begin{pmatrix}\cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w\end{pmatrix} \begin{pmatrix}W_\mu^3 \\ Y_\mu\end{pmatrix}.$$

Here $\tan\theta_w \equiv \frac{g'}{g}$ defines the Weinberg angle. The masses are $M_\gamma = 0$ and $M_Z = \frac{M_W}{\cos\theta_w} < M_W$.

Fermion masses come from (dimension-four) Yukawa couplings

$$\mathcal{L}_{\text{Yukawa}} = -Y^\ell_{ij}\bar{L}_i H e^j_R - Y^u_{ij}\bar{Q}^i H d^j_R - Y^d_{ij}\bar{Q}^i\left(\mathbf{i}\tau^2 H^\star\right)u^j_R + h.c.$$

The contortion with the τ^2 is required to make a hypercharge invariant. Plugging in the Higgs vev to *e.g.* the lepton terms gives $-m_e\bar{e}_L e_R + h.c.$ with $m_e = y_e v/\sqrt{2}$. There's lots of drama about the matrices Y which can mix the generations. The mass for the ν_R (which maybe could not exist – it doesn't have any charges at all) you can figure out on the homework later.

Here is a useful mnemonic for remembering the table of quantum numbers (possibly it is more than that): There are larger simple Lie groups that contain the SM gauge group as subgroups:

$$\begin{array}{ccccc} \mathsf{SU}(3)\times\mathsf{SU}(2)\times\mathsf{U}(1)_Y & \subset & \mathsf{SU}(5) & \subset & \mathsf{SO}(10) \\ \text{one generation} & = & 10\oplus\bar{5}\oplus 1 & = & 16 \end{array}$$

The singlet of $\mathsf{SU}(5)$ is the right-handed neutrino, but if we include it, one generation is an irreducible (spinor) representation of $\mathsf{SO}(10)$. This idea is called grand unification. It is easy to imagine that the gauge group is actually the larger groups on the right, and another instance of the Higgs mechanism accomplishes the breaking down to the Standard Model. (The running of the respective gauge couplings go in the right direction with approximately the right rate to unify to a single value at $M_{GUT} \sim 10^{16} GeV$.) Notice that this idea means leptons and quarks are in the same representations – they can turn into each other. This predicts that the proton should not be perfectly stable. Next we'll say more about this.

Beyond the Standard Model with EFT. At what energy does the Standard Model stop working? Because of the annoying feature of renormalizibility, it doesn't tell us. However, we have experimental evidence against a cutoff on the Standard Model (SM) at energies less than something like 10 TeV. The evidence I have in mind is the absence of interactions of the form

$$\delta L = \frac{1}{M^2}\left(\bar{\psi}A\psi\right)\cdot\left(\bar{\psi}B\psi\right)$$

(where ψ represent various SM fermion fields and A, B can be various gamma and flavor matrices) with $M \lesssim 10$ TeV. Notice that I am talking now about interactions *other* than the electroweak interactions, which as we've just discussed, for energies above $M_W \sim 80$GeV cannot be treated as contact interactions – you can see the Ws propagate!

If such operators were present, we would have found different answers for experiments at LEP. But such operators would be present if we consider new physics in addition to the Standard Model (in most ways of doing it) at energies less than 10 TeV. For example, many interesting ways of coupling in new particles with masses that make them accessible at the LHC would have generated such operators.

A little more explicitly: the Standard Model Lagrangian L_0 contains all the renormalizable (*i.e.* engineering dimension ≤ 4) operators that you can make from its fields (though the coefficients of the dimension 4 operators do vary through quite a large range, and the coefficients of the two relevant operators – namely the identity operator which has dimension zero, and the Higgs mass, which has engineering dimension two, are strangely small, and so is the QCD θ angle).

To understand what lies beyond the Standard Model, we can use our knowledge that whatever it is, it is probably heavy (it could also just be very weakly coupled, which is a different story), with some intrinsic scale $\Lambda_{\rm new}$, so we can integrate it out and include its effects by corrections to the Standard Model:

$$L = L_0 + \frac{1}{\Lambda_{\rm new}}\mathcal{O}^{(5)} + \frac{1}{\Lambda_{\rm new}^2}\sum_i c_i \mathcal{O}_i^{(6)} + \cdots$$

where the $\mathcal{O}$s are made of SM fields, and have the indicated engineering dimensions, and preserve the necessary symmetries of the SM (Lorentz symmetry and gauge invariance).

In fact there is only one kind of operator of dimension 5 meeting these demands:

$$\mathcal{O}^{(5)} = c_5 \epsilon_{ij} \left(\bar{L}^c\right)^i H^j \epsilon_{kl} L^k H^l$$

where $H^i = (h^+, h^0)^i$ is the $\mathrm{SU}(2)_{EW}$ Higgs doublet and $L^i = (\nu_L, e_L)^i$ is an $\mathrm{SU}(2)_{EW}$ doublet of left-handed leptons, and $\bar{L}^c \equiv L^T C$ where C is the charge conjugation matrix. (I say 'kind of operator' because we can have various flavor matrices in here.) On the problem set you get to see from whence such an operator might arise, and what it does if you plug in the higgs vev $\langle H \rangle = (0, v)$. This term violates lepton number symmetry ($L \to e^{\mathbf{i}\alpha_L} L, Q \to Q, H \to H$).

At dimension 6, there are operators that directly violate baryon number, such as

$$\epsilon_{\alpha\beta\gamma} (\bar{u}_R)^c_\alpha (u_R)_\beta \, (\bar{u}_R)^c_\gamma \, e_R.$$

You should read the above tangle of symbols as '$qqq\ell$' – it turns three quarks into a lepton. The epsilon tensor makes a color SU(3) singlet; this thing ϵqqq has the quantum numbers of a baryon, such as the proton and neutron. The long lifetime of the proton (you can feel it in your bones – see Zee p. 413) then directly constrains the scale of new physics appearing in front of this operator.

Two more comments about this:

- If we didn't know about the Standard Model, (but after we knew about QM and GR and EFT (the last of which people didn't know before the SM for some reason)) we should have made the estimate that dimension-5 Planck-scale-suppressed operators like $\frac{1}{M_{\rm Planck}} p\mathcal{O}$ would cause proton decay (into whatever $\mathcal{O}$ makes). This predicts $\Gamma_p \sim \frac{m_p^3}{M_{\rm Planck}^2} \sim 10^{-13} s^{-1}$ which is *not* consistent with our bodies not glowing. Actually it is a remarkable fact that there are no gauge-invariant operators made of SM fields of dimension less than 6 that violate baryon number symmetry ($L \to L, Q \to e^{\mathbf{i}\alpha_B} Q, H \to H$). This is an *emergent* symmetry, expected to be violated by the UV completion.

- Surely nothing can prevent $\Delta L \sim \left(\frac{1}{M_{\rm Planck}}\right)^2 qqq\ell$. Happily, this is consistent with the observed proton lifetime.

There are $\sim 10^2$ dimension 6 operators that preserve baryon number, and therefore are not as tightly constrained[3]. (Those that induce flavor-changing processes in the SM are more highly constrained and must have $\Lambda_{\rm new} > 10^4$ TeV.) Two such operators are considered equivalent if they differ by something which vanishes by the tree-level SM equations of motion. This is the right thing to do, even for off-shell calculations (like green's functions and for fields running in loops). You know this from a previous problem set: the EOM are true as operator equations – Ward identities resulting from being free to change integration variables in the path integral[4]. [End of Lecture 4]

2.7 Superconductors

Recall from 215B our effective (Landau-Ginzburg) description of superconductors which reproduces the Meissner effect, the Abelian Higgs model:

$$\mathcal{F} = \frac{1}{4} F_{ij} F_{ij} + |D_i \Phi|^2 + a|\Phi|^2 + \frac{1}{2} b|\Phi|^4 + ... \tag{2.13}$$

with $D_i \Phi \equiv (\partial_i - 2e\mathbf{i} A_i)\, \Phi$.

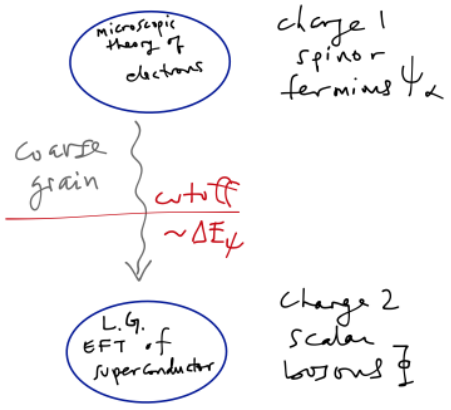
I want to make two more comments about this:

[3] Recently, humans have gotten better at counting these operators. See this paper.

[4] There are a few meaningful subtleties here, as you might expect if you recall that the Ward identity is only true up to contact terms. The measure in the path integral can produce a Jacobian which renormalizes some of the couplings; the changes in source terms will drop out of S-matrix elements (recall our discussion of changing field variables in the Consequences of Unitarity section.) but can change the form of Green's functions. For more information on the use of eom to eliminate redundant operators in EFT, see Arzt, hep-ph/9304230 and Georgi, "On-Shell EFT".

Symmetry breaking by fluctuations (Coleman-Weinberg). [Zee problem IV.6.9.] What happens near the transition, when $a = 0$ in (2.13)? Quantum fluctuations can lead to symmetry breaking.

New IR dofs. A feature of this example that I want you to notice: the microscopic description of real superconductor involves electrons – charge $1e$ spinor fermions, created by some fermionic operator ψ_α, $\alpha = \uparrow, \downarrow$.



We are describing the low-energy physics of a system of electrons in terms of a bosonic field, which (in simple 's-wave' superconductors) is roughly related to the electron field by

$$\Phi \sim \psi_\alpha \psi_\beta \epsilon^{\alpha\beta} \ ; \tag{2.14}$$

Φ is called a Cooper pair field. At least, the charges and the spins and the statistics work out. The details of this relationship are not the important point I wanted to emphasize. Rather I wanted to emphasize the dramatic difference in the correct choice of variables between the UV description (spinor fermions) and the IR description (scalar bosons). One reason that this is possible is that it costs a large energy to make a fermionic excitation of the superconductor. This can be understood roughly as follows: The microscopic theory of the electrons looks something like (ignoring the coupling to electromagnetism for now)

$$S[\psi] = S_2[\psi] + \int \mathrm{d}t \mathrm{d}^d x \ u \psi^\dagger \psi \psi^\dagger \psi + h.c. \tag{2.15}$$

where

$$S_2 = \int \mathrm{d}t \int \mathrm{đ}^d k \psi_k^\dagger \left(\mathbf{i}\partial_t - \epsilon(k) \right) \psi_k .$$

Spin is important here so that $\psi_\uparrow^\dagger \psi_\uparrow \psi_\downarrow^\dagger \psi_\downarrow$ is nonzero. A mean field theory description of the condensation of Cooper pairs (2.14) is obtained by replacing the quartic term in (2.15) by expectation values:

$$\begin{aligned} S_{MFT}[\psi] &= S_2[\psi] - \int \mathrm{d}t \mathrm{d}^d x \ u \left\langle \psi\psi \right\rangle \psi^\dagger \psi^\dagger + h.c. \\ &= S_2[\psi] - \int \mathrm{d}t \mathrm{d}^d x \ u \Phi \psi^\dagger \psi^\dagger + h.c. \end{aligned} \tag{2.16}$$

So an expectation value for Φ is a mass for the fermions. It is a funny kind of symmetry-breaking mass, but if you diagonalize the quadratic operator in (2.16) (actually it is done below) you will find that it costs an energy of order $\Delta E_\psi = u \left\langle \Phi \right\rangle$ to excite a fermion. That's the cutoff on the LG EFT.

A general lesson from this example is: the useful degrees of freedom at low energies can be very different from the microscopic dofs.

2.7.1 Lightning discussion of BCS.

I am sure that some of you are nervous about the step from $S[\psi]$ to $S_{MFT}[\psi]$ above. To make ourselves feel better about it, I will say a few more words about the steps from the microscopic model of electrons (2.15) to the LG theory of Cooper pairs (these steps were taken by Bardeen, Cooper and Schreiffer (BCS)).

First recall the *Hubbard-Stratonovich transformation* aka *completing the square*. In 0+0 dimensional field theory:

$$e^{-\mathbf{i}ux^4} = \frac{1}{\sqrt{\mathbf{i}\pi u}} \int_{-\infty}^{\infty} \mathrm{d}\sigma \; e^{-\frac{1}{\mathbf{i}u}\sigma^2 - 2\mathbf{i}x^2\sigma} \; . \tag{2.17}$$

At the cost of introducing an extra field σ, we turn a quartic term in x into a quadratic term in x. The RHS of (2.17) is gaussian in x and we know how to integrate it over x. (The version with $\mathbf{i}$ is relevant for the real-time integral.) Notice the weird extra factor of $\mathbf{i}$ lurking in (2.17). This can be understood as arising because we are trying to use a scalar field σ, to mediate a repulsive interaction (which it is, for positive u) (see Zee p. 193, 2nd Ed).

Actually, we'll need a complex H-S field:

$$e^{-\mathbf{i}ux^2\bar{x}^2} = \frac{1}{\mathbf{i}\pi u} \int_{\mathbb{C}} \mathrm{d}^2\sigma \; e^{-\frac{1}{\mathbf{i}u}|\sigma|^2 - \mathbf{i}x^2\bar{\sigma} - \mathbf{i}\bar{x}^2\sigma} \; , \tag{2.18}$$

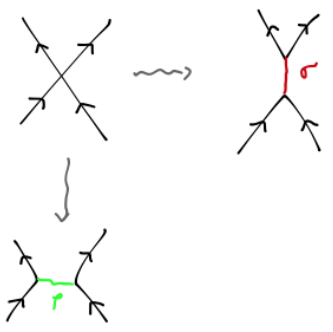
where $\int_{\mathbb{C}} \mathrm{d}^2\sigma ... \equiv \int_{-\infty}^{\infty} d\mathrm{Re}\,\sigma \int_{-\infty}^{\infty} d\mathrm{Im}\,\sigma ...$ (The field-independent prefactor is, as usual, not important for path integrals.)

We can use a field theory generalization of (2.18) to 'decouple' the 4-fermion interaction in (2.15):

$$Z = \int [D\psi D\psi^\dagger] e^{\mathbf{i}S[\psi]} = \int [D\psi D\psi^\dagger D\sigma D\sigma^\dagger] e^{\mathbf{i}S_2[\psi] + \mathbf{i}\int d^D x (\bar{\sigma}\psi\psi + h.c.) - \int \mathrm{d}^D x \frac{|\sigma|^2(x)}{\mathbf{i}u}} \; . \tag{2.19}$$

The point of this is that now the fermion integral is gaussian. At the saddle point of the σ integral (which is exact because it is gaussian), σ is the Cooper pair field, $\sigma_{\text{saddle}} = u\psi\psi$.

Notice that we made a choice here about in which 'channel' to make the decoupling – we could have instead introduces a different auxiliary field ρ and written $S[\rho, \psi] = \int \rho\psi^\dagger\psi + \int \frac{\rho^2}{2u}$, which would break up



the 4-fermion interaction in the t-channel (as an interaction of the fermion density $\psi^\dagger\psi$) instead of the s (BCS) channel (as an interaction of Cooper pairs ψ^2). At this stage both are correct, but they lead to different mean-field approximations below. That the BCS mean field theory wins is a consequence of the RG.

How can you resist doing the fermion integral in (2.19)? Let's study the case where the single-fermion dispersion is $\epsilon(k) = \frac{\vec{k}^2}{2m} - \mu$.

$$I_\psi[\sigma] \equiv \int [D\psi D\psi^\dagger] e^{\mathbf{i}\int \mathrm{d}t\mathrm{d}^d x \left(\psi^\dagger\left(\mathbf{i}\partial_t - \frac{\nabla^2}{2m} - \mu\right)\psi + \psi\bar{\sigma}\psi + \bar{\psi}\bar{\psi}\sigma\right)}$$

The action here can be written as the integral of

$$L = \begin{pmatrix}\bar{\psi} & \psi\end{pmatrix}\begin{pmatrix}\mathbf{i}\partial_t - \epsilon(-\mathbf{i}\nabla) & \sigma \\ \bar{\sigma} & -(\mathbf{i}\partial_t - \epsilon(-\mathbf{i}\nabla))\end{pmatrix}\begin{pmatrix}\psi \\ \bar{\psi}\end{pmatrix} \equiv \begin{pmatrix}\bar{\psi} & \psi\end{pmatrix} M \begin{pmatrix}\psi \\ \bar{\psi}\end{pmatrix}$$

so the functional integral is

$$I_\psi[\sigma] = \det M = e^{\mathrm{tr}\log M(\sigma)}.$$

If σ is constant (which will lower the energy), the matrix M is diagonal in momentum space, and the integral remaining to be done is

$$\int [D\sigma D\sigma^\dagger] e^{-\int \mathrm{d}^D x \frac{|\sigma(x)|^2}{2\mathbf{i}u} + \int đ^D k \log\left(\omega^2 - \epsilon_k^2 - |\sigma|^2\right)}.$$

It is often possible to do this integral by saddle point. This can justified, for example, by the largeness of the volume of the Fermi surface, $\{k|\epsilon(k) = \mu\}$, or by large N number of species of fermions. The result is an equation which determines σ, which as we saw earlier determines the fermion gap.

$$0 = \frac{\delta \text{exponent}}{\delta\bar{\sigma}} = \mathbf{i}\frac{\sigma}{2u} + \int đ\omega đ^d k \frac{2\sigma}{\omega^2 - \epsilon_k^2 - |\sigma|^2 + \mathbf{i}\epsilon}\ .$$

We can do the frequency integral by residues:

$$\int đ\omega \frac{1}{\omega^2 - \epsilon_k^2 - |\sigma|^2 + \mathbf{i}\epsilon} = \frac{1}{2\pi} 2\pi\mathbf{i} \frac{1}{2\sqrt{\epsilon_k^2 + |\sigma|^2}}.$$

The resulting equation is naturally called the *gap equation*:

$$1 = -2u \int đ^d p' \frac{1}{\sqrt{\epsilon(p')^2 + |\sigma|^2}} \tag{2.20}$$

which you can imagine solving self-consistently for σ[5]. Plugging back into the action (2.19) says that σ determines the energy cost to have electrons around; more precisely, σ is the energy required to break a Cooper pair.

Comments:

- Notice that a solution of (2.20) requires $u < 0$, an *attractive* interaction. Superconductivity happens because the u that appears here is not the bare interaction between electrons, which is certainly repulsive (and long-ranged). This is where the phonons come in in the BCS discussion.

- If we hadn't restricted to a delta-function 4-fermion interaction $u(p, p') = u_0$ at the outset, we would have found a more general equation like

$$\sigma(\vec{p}) = -\frac{1}{2}\int \mathrm{d}\!\!^{-d}p' \frac{u(p,p')\sigma(\vec{p}')}{\sqrt{\epsilon(p')^2 + |\sigma(p')|^2}} \ .$$

- A conservative perspective on the preceding calculation is that we have made a variational ansatz for the groundstate wavefunction, and the equation we solve for σ is minimizing the variational energy – finding the best wavefunction within the ansatz.

- I haven't included here effects of the fluctuations of the fermions. In fact, they make the four-fermion interaction which leads to Cooper pairing marginally *relevant*. This breaks the degeneracy in deciding how to split up the $\psi\psi\psi^\dagger\psi^\dagger$ into *e.g.* $\psi\psi\sigma$ or $\psi^\dagger\psi\rho$. BCS wins. This is explained beautifully in Polchinski, lecture 2, and R. Shankar. I will summarize the EFT framework for understanding this in §2.8.

- I've tried to give the most efficient introduction I could here. I left out any possibility of k-dependence or spin dependence of the interactions or the pair

[5] I should have said: and in fact one can solve it. As we will learn in the next section, the integral is dominated by the behavior near the Fermi surface, near which $\epsilon(p') \simeq v_F \ell$; this approximation is valid for $|\epsilon| < E_D$, some UV cutoff on this description. The result is

$$1 = -2u\int \frac{\mathrm{d}\!\!^{-d}p'}{\sqrt{\epsilon(p')^2+|\sigma|^2}} \simeq -2u\int_{FS}\frac{\mathrm{d}\!\!^{-d+1}k}{v_F}\int_{-E_D}^{E_D}\frac{d\ell}{\sqrt{v_F^2\ell^2+|\sigma|^2}} = N\log\left(\frac{E_D+\sqrt{E_D^2+|\sigma|^2}}{|\sigma|}\right)$$

where $N \equiv \int_{FS}\frac{\mathrm{d}\!\!^{-d+1}k}{2\pi v_F}$ is the density of states at the Fermi surface. The solution for σ is

$$|\sigma| = \frac{2E_D e^{\frac{1}{2NV}}}{e^{\frac{1}{NV}}-1} \overset{NV\ll 1}{\simeq} 2E_D e^{-\frac{1}{NV}}.$$

Notice that this is non-perturbative in the coupling strength V.

field, and I've conflated the pair field with the gap. In particular, I've been sloppy about the dependence on k of σ above.

- You can study a very closely related manipulation on the problem set, in an example (the Gross-Neveu model) where the saddle point is justified by large N.

2.8 Effective field theory of Fermi surfaces

[Polchinski, lecture 2 (I recommend these notes very strongly), and R. Shankar] Electrically conducting solids are a remarkable phenomenon. An arbitrarily small electric field $\vec{E}$ leads to a nonzero current $\vec{j} = \sigma\vec{E}$. This means that there must be gapless modes with energies much less than the natural cutoff scale in the problem.

Scales involved: The Planck scale of solid state physics (made by the logic by which Planck made his quantum gravity energy scale, namely by making a quantity with dimensions of energy out of the available constants) is

$$E_0 = \frac{1}{2}\frac{e^4 m}{\hbar^2} = \frac{1}{2}\frac{e^2}{a_0} \sim 13\text{eV}$$

(where $m \equiv m_e$ is the electron mass and the factor of 2 is an abuse of outside information) which is the energy scale of *chemistry*. Chemistry is to solids as the melting of spacetime is to particle physics. There are other scales involved however. In particular a solid involves a lattice of nuclei, each with $M \gg m$ (approximately the proton mass). So m/M is a useful small parameter which controls the coupling between the electrons and the lattice vibrations. Also, the actual speed of light $c \gg v_F$ can generally also be treated as ∞ to first approximation. v_F/c suppresses spin orbit couplings (though large atomic numbers enhance them: $\lambda_{\text{SO}} \propto Zv_F/c$).

Let us attempt to construct a Wilsonian-natural effective field theory of this phenomenon. The answer is called Landau Fermi Liquid Theory. What are the right low-energy degrees of freedom? Let's make a guess that they are like electrons – fermions with spin and electric charge. They will not have exactly the properties of free electrons, since they must incorporate the effects of interactions with all their friends. The 'dressed' electrons are called quasielectrons, or more generally quasiparticles.

Given the strong interactions between so many particles, why should the dofs have anything at all to do with electrons? Landau's motivation for this description (which is not always correct) is that we can imagine starting from the free theory and adiabatically turning up the interactions. If we don't encounter any phase transition along

the way, we can follow each state of the free theory, and use the same labels in the interacting theory.

We will show that there is a nearly-RG-stable fixed point describing gapless quasi-electrons. Notice that we are not trying to match this description directly to some microscopic lattice model of a solid; rather we will do bottom-up effective field theory.

Having guessed the necessary dofs, let's try to write an action for them consistent with the symmetries. A good starting point is the free theory:

$$S_{\text{free}}[\psi] = \int dt\, \mathrm{d}^d p \left(\mathbf{i}\psi^\dagger_\sigma(p)\partial_t \psi_\sigma(p) - (\epsilon(p) - \epsilon_F)\, \psi^\dagger_\sigma(p)\psi_\sigma(p) \right)$$

where σ is a spin index, ϵ_F is the Fermi energy (zero-temperature chemical potential), and $\epsilon(p)$ is the single-particle dispersion relation. For non-interacting non-relativistic electrons in free space, we have $\epsilon(p) = \frac{p^2}{2m}$. It will be useful to leave this as a general function of p. [6] [7]

The groundstate of the free theory is the filled Fermi sea:

$$|\text{gs}\rangle = \prod_{p|\epsilon(p)<\epsilon_F} \psi^\dagger_p |0\rangle\,, \quad \psi_p |0\rangle = 0, \quad \forall p.$$

(If you don't like continuous products, put the system in a box so that p is a discrete label.) The Fermi surface is the set of points in momentum space at the boundary of the filled states:

$$\text{FS} \equiv \{p|\epsilon(p) = \epsilon_F\}.$$

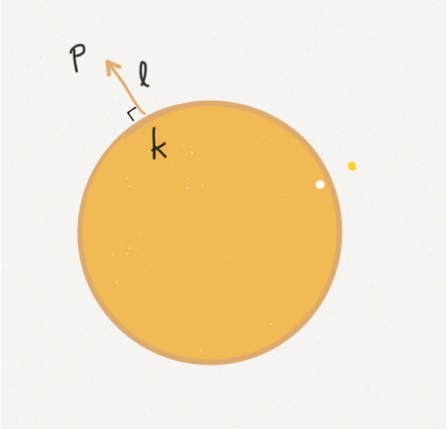
The low-lying excitations are made by adding an electron just above the FS or removing an electron (creating a hole) just below.

In order to define the power-counting rules for our EFT, we would like to define a scaling transformation which focuses on the low-energy excitations. We scale energies by a factor $E \to bE, b < 1$. In relativistic QFT, $\vec{p}$ scales like E, toward zero, $\vec{p} \to b\vec{p}$, since all the low-energy stuff is near the single special point $\vec{p} = 0$. Here the situation is much more interesting because there is a whole surface of low-energy stuff on the FS. This will lead to what's called *hyperscaling violation* – we can't just count powers of momentum.

[6] Notice that we are assuming translation invariance. I am not saying anything at the moment about whether translation invariance is discrete (the ions make a periodic potential) or continuous.

[7] We have chosen the normalization of ψ to fix the coefficient of the ∂_t term (this rescaling may depend on p).

One way to implement this is to introduce a hierarchical labeling of points in momentum space, by breaking the momentum space into *patches* around the FS. (An analogous strategy of labeling is also used in heavy quark EFT and in SCET.)



We'll use a slightly different strategy, following Polchinski. To specify a point $\vec{p}$, we pick the nearest point $\vec{k}$ on the FS, $\epsilon(\vec{k}) = \epsilon_F$ (draw a line perpendicular to the FS from $\vec{p}$), and let

$$\vec{p} = \vec{k} + \vec{\ell}.$$

So $d-1$ of the components are determined by $\vec{k}$ and one is determined by ℓ. (Clearly there are some exceptional cases if the FS gets too wiggly. Ignore these for now.)

$$\epsilon(p) - \epsilon_F = \ell v_F(\vec{k}) + \mathcal{O}(\ell^2), \qquad v_F \equiv \partial_p \epsilon|_{p=k}.$$

So a scaling rule which accomplishes our goal of focusing on the FS is

$$E \to bE, \quad \vec{k} \to \vec{k}, \quad \vec{l} \to b\vec{\ell}.$$

This implies

$$dt \to b^{-1}dt, \quad d^{d-1}\vec{k} \to d^{d-1}\vec{k}, \quad d\vec{\ell} \to bd\vec{\ell}, \quad \partial_t \to b\partial_t$$

$$S_{\text{free}} = \int \underbrace{dt\ d^{d-1}\vec{k}\ d\vec{\ell}}_{\sim b^0} \left(\mathbf{i}\psi^\dagger(p) \underbrace{\partial_t}_{\sim b^1} \psi(p) - \underbrace{\ell v_F(k)}_{\sim b^1} \psi^\dagger(p)\psi(p) \right)$$

In order to make this go like b^0 we require $\psi \to b^{-\frac{1}{2}}\psi$ near the free fixed point.

Next we will play the EFT game. To do so we must enumerate the symmetries we demand of our EFT:

1. Particle number, $\psi \to e^{\mathbf{i}\theta}\psi$

2. Spatial symmetries: time-translation invariance, and either (a) continuous translation invariance and rotation invariance (as for *e.g.* liquid ^{3}He) or (b) lattice symmetries. This means that momentum space is periodically identified, roughly $p \simeq p + 2\pi/a$ where a is the lattice spacing (the set of independent momenta is called the Brillouin zone (BZ)) and p is only conserved modulo an inverse lattice vector $2\pi/a$; the momentum There can also be some remnant of rotation invariance preserved by the lattice. Case (b) reduces to case (a) if the Fermi surface does not go near the edges of the BZ.

3. Spin rotation symmetry, $\mathsf{SU}(n)$ if $\sigma = 1..n$. In the limit with $c \to \infty$, this is an internal symmetry, independent of rotations.

4. Let's assume that $\epsilon(p) = \epsilon(-p)$, which is a consequence of *e.g.* parity invariance.

Now we enumerate all terms analytic in ψ (since we are assuming that there are no other low-energy dofs integrating out which is the only way to get non-analytic terms in ψ) and consistent with the symmetries; we can order them by the number of fermion operators involved. Particle number symmetry means every ψ comes with a $\psi^\dagger$. The possible **quadratic terms** are:

$$\int \underbrace{dt\ d^{d-1}\vec{k}\ d\vec{\ell}}_{\sim b^0}\ \mu(k)\ \underbrace{\psi^\dagger_\sigma(p)\psi_\sigma(p)}_{\sim b^{-1}} \sim b^{-1}$$

is relevant. This is like a mass term. But don't panic: it just shifts the FS around. The *existence* of a Fermi surface is Wilson-natural (*i.e.* a stable assumption given generic coefficients of all possible terms in the action); any precise location or shape (modulo something enforced by symmetries, like roundness) is not.

Adding one extra ∂_t or factor of ℓ costs a b^1 and makes the operator marginal; those terms are already present in S_{free}. Adding more than one makes it irrelevant.

Quartic terms:

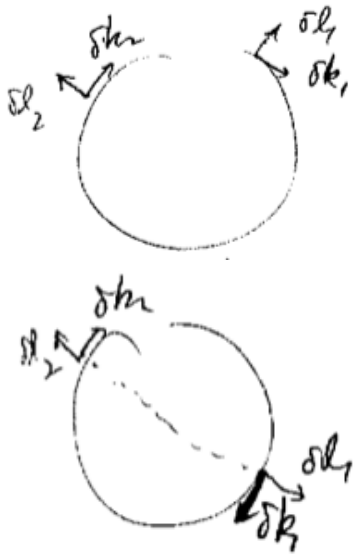
$$S_4 = \int \underbrace{dt\ \prod_{i=1}^{4} d^{d-1}\vec{k}_i d\vec{\ell}_i}_{\sim b^{-1+4-4/2}}\ u(4\cdots 1)\psi^\dagger_\sigma(p_1)\psi_\sigma(p_3)\psi^\dagger_{\sigma'}(p_2)\psi_{\sigma'}(p_4)\delta^d(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$$

The minus signs on $p_{3,4}$ is because $\psi(p)$ *removes* a particle with momentum p. We assume u depends only on k, σ, so does not scale – this will give the most relevant piece. How does the delta function scale?

$$\delta^d\left(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4\right) = \delta^d\left(k_1+k_2-k_3-k_4+\ell_1+\ell_2-\ell_3-\ell_4\right) \overset{?}{\simeq} \delta^d\left(k_1+k_2-k_3-k_4\right)$$

In the last (questioned) step, we used the fact that $\ell \ll k$ to ignore the contributions of the ℓs. If this is correct then the delta function does not scale (since ks do not), and $S_4 \sim b^1$ is irrelevant (and quartic interactions with derivatives are moreso). If this were correct, the free-fixed point would be exactly stable. [End of Lecture 5]

There are two important subtleties: (1) there exist phonons. (2) the questioned equality above is questionable because of kinematics of the Fermi surface. We will address these two issues in reverse order.



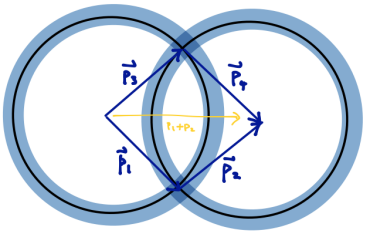
The **kinematic subtlety** in the treatment of the scaling of $\delta(p_1 + p_2 - p_3 - p_4)$ arises because of the geometry of the Fermi surface. Consider scattering between two points on the FS, where (in the labeling convention above)

$$p_3 = p_1 + \delta k_1 + \delta \ell_1, \quad p_4 = p_2 + \delta k_2 + \delta \ell_2,$$

in which case the momentum delta function is

$$\delta^d(p_1 + p_2 - p_3 - p_4) = \delta^d(\delta k_1 + \delta \ell_1 + \delta k_2 + \delta \ell_2).$$

For generic choices of the two points $p_{1,2}$ (top figure at left), δk_1 and δk_2 are linearly independent and the $\delta\ell$s can indeed be ignored as we did above. However, for two points with $p_1 = -p_2$ (they are called *nested*, as depicted in the bottom figure at left), then one component of $\delta k_1 + \delta k_2$ is automatically zero, revealing the tiny $\delta\ell$s to the force of (one component of) the delta function. In this case, $\delta(\ell)$ scales like b^{-1}, and for this particular kinematic configuration the four-fermion interaction is (classically) marginal. Classically marginal means quantum mechanics has a chance to make a big difference.



A useful visualization is at right ($d = 2$ with a round FS is shown; this is what's depicted on the cover of the famous book by Abrikosov-Gorkov-Dzyaloshinski): the blue circles have radius k_F; the yellow vector is the sum of the two initial momenta $p_1 + p_2$, both of which are on the FS; the condition that $p_3 + p_4$, each also on the FS, add up to the same vector means that p_3 must lie on the intersection of the two circles (spheres in $d > 2$). But when $p_1 + p_2 = 0$, the two circles are on top of each other so they intersect everywhere! Comments:

1. We assumed that both p_1 and $-p_2$ were actually *on* the FS. This is automatic if $\epsilon(p) = \epsilon(-p)$, *i.e.* if ϵ is only a function of p^2.

2. This discussion works for any $d > 1$.

3. **Forward scattering.** There is a similar phenomenon for the case where $p_1 = p_3$ (and hence $p_2 = p_4$). This is called *forward scattering* because the final momenta are the same as the initial momenta. (We could just as well take $p_1 = p_4$ (and hence $p_2 = p_3$).) In this case too the delta function will constrain the ℓs and will therefore scale.

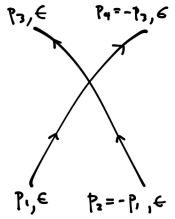
The tree-level-marginal 4-Fermi interactions at special kinematics leads to a *family* of fixed points labelled by 'Landau parameters'. In fact there is whole *function*'s worth of fixed points. In 2d, the fixed point manifold is parametrized by the forward-scattering function

$$F(\theta_1, \theta_2) \equiv u(\theta_4 = \theta_2, \theta_3 = \theta_1, \theta_2, \theta_1)$$

(Fermi statistics implies that $u(\theta_4 = \theta_1, \theta_3 = \theta_2, \theta_2, \theta_1) = -F(\theta_1, \theta_2)$.) and the BCS-channel (nesting) interaction:

$$V(\theta_1, \theta_3) = u(\theta_4 = -\theta_3, \theta_3, \theta_2 = -\theta_1, \theta_1).$$

Now let's think about what decision the fluctuations make about the fate of the nested interactions. The most interesting bit is the renormalization of the BCS interaction:



$= -\mathbf{i}V$

The electron propagator, obtained by inverting the kinetic operator S_{free}, is

$$G(\epsilon, p = k + l) = \frac{\mathbf{i}}{\epsilon(1 + \mathbf{i}\eta) - v_F(k)\ell + \mathcal{O}(\ell)^2}$$

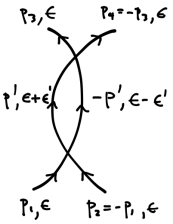
where I used $\eta \equiv 0^+$ for the infinitesimal specifying the contour prescription. (To understand the contour prescription for the *hole* propagator, it is useful to begin with

$$G(t, p) = \langle \epsilon_F | \, c_p^\dagger(t) c_p(0) \, | \epsilon_F \rangle \, , \quad c_p^\dagger(t) \equiv e^{-\mathbf{i}\mathbf{H}t} c_p^\dagger e^{\mathbf{i}\mathbf{H}t}$$

and use the free-fermion fact $[\mathbf{H}, c_p^\dagger] = \epsilon_p c_p^\dagger$.)

Let's assume rotation invariance. Then $V(\theta_3, \theta_1) = V(\theta_3 - \theta_1)$, $V_l = \int đ\theta e^{\mathbf{i}l\theta} V(\theta)$. Different angular momentum sectors decouple from each other at one loop.

We will focus on the s-wave bit of the interaction, so V is independent of momentum. We will integrate out just a shell in energy (depicted by the blue shaded shell in the Fermi surface figures). The interesting contribution comes from the following diagram:

$-\mathbf{i}\delta^{(1)}V =$ 

$$= (-\mathbf{i}V)^2 \int_{b\epsilon_0}^{\epsilon_0} \frac{d\epsilon' d^{d-1}k' d\ell'}{(2\pi)^{d+1}} \frac{\mathbf{i}^2}{(\epsilon + \epsilon' - v_F(k')\ell')\,(\epsilon - \epsilon' - v_F(k')\ell')}$$

do $\int d\epsilon'$ by residues

$$= V^2 \int \frac{d\epsilon' d^{d-1}k'}{(2\pi)^{d+1}} \frac{2\pi\mathbf{i}}{v_F(k')} \left(\underbrace{\epsilon - \epsilon' - (\epsilon + \epsilon')}_{=-2\epsilon'} \right)^{-1}$$

$$= +\mathbf{i}\frac{V^2}{2}\underbrace{\int_{b\epsilon_0}^{\epsilon_0}\frac{d\epsilon'}{\epsilon'}}_{=\log(1/b)}\underbrace{\int\frac{d^{d-1}k'}{(2\pi)^d v_F(k')}}_{\text{dos at FS}} \tag{2.21}$$

Don't forget the fermion loop minus sign. Between the first and second lines, we did the ℓ' integral by residues. The crucial point is that we are interested in external energies $\epsilon \sim 0$, but we are integrating out a shell near the cutoff, so $|\epsilon'| > |\epsilon|$ and the sign of $\epsilon + \epsilon'$ is opposite that of $\epsilon - \epsilon'$; therefore there is a pole on either side of the real ℓ axis and we get the same answer by closing the contour either way. On one side the pole is at $\ell' = \frac{1}{v_F(k')}(\epsilon + \epsilon')$. (In the t-channel diagram (what Shankar calls ZS), the poles are on the *same* side and it therefore does not renormalize the four-fermion interaction.)

The result to one-loop is then

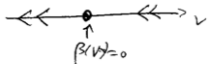
$$V(b) = V - V^2 N \log(1/b) + \mathcal{O}(V^3)$$

with $N \equiv \int \frac{d^{d-1}k'}{(2\pi)^d v_F(k')}$ is the density of states at the Fermi surface. From this we derive the beta function (recall that $b \to 0$ in the IR)

$$b\frac{d}{db}V(b) = \beta_V = NV^2(b) + \mathcal{O}(V^3)$$

and the solution of the flow equation at $E = bE_1$ is

$$V(E) = \frac{V_1}{1 + NV_1 \log(E_1/E)} \begin{cases} \to 0 & \text{in IR for } V_1 > 0 \text{ (repulsive)} \\ \to -\infty & \text{in IR for } V_1 < 0 \text{ (attractive)} \end{cases} \tag{2.22}$$

There is therefore a very significant dichotomy depending on the sign of the coupling at the microscopic scale E_1, as in this phase diagram: 

The conclusion is that if the interaction starts attractive at some scale it flows to large attractive values. The thing that is decided by our perturbative analysis is that (if $V(E_1) > 0$) the decoupling we did with σ ('the BCS channel') wins over the decoupling with ρ ('the particle-hole channel'). What happens at $V \to -\infty$? Here we need non-perturbative physics.

The non-perturbative physics is in general hard, but we've already done what we can in §2.7.1.

The remaining question is: Who is V_1 and why would it be attractive (given that Coulomb interactions between electrons, while screened and therefore short-ranged, are repulsive)? The answer is:

Phonons. The lattice of positions taken by the ions making up a crystalline solid spontaneously break many spacetime symmetries of their governing Hamiltonian. This

implies a collection of gapless Goldstone modes in any low-energy effective theory of such a solid[8]. The Goldstone theorem is satisfied by including a field

$$\vec{D} \propto \text{(local) displacement } \delta\vec{r} \text{ of ions from their equilibrium positions}$$

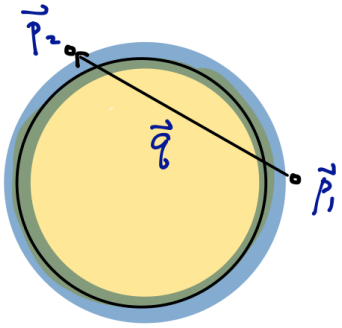
Most microscopically we have a bunch of coupled springs:

$$L_{\text{ions}} \sim \frac{1}{2} M \left(\delta\dot{\vec{r}}\right)^2 - k_{ij}\delta r^i \delta r^j + ...$$

with spring constants k independent of the nuclear mass M. It is useful to introduce a canonically normalized field in terms of which the action is

$$S[\vec{D} = (M)^{1/2}\,\delta\vec{r}] = \frac{1}{2}\int dt d^d q \left(\partial_t D_i(q)\partial_t D_i(-q) - \omega_{ij}^2(q) D_i(q) D_j(-q)\right).$$

Here $\omega^2 \propto M^{-1}$. Their status as Goldstones means that the eigenvalues of $\omega_{ij}^2(q) \sim |q|^2$ at small q: moving everyone by the same amount does not change the energy. This also constrains the coupling of these modes to the electrons: they can only couple through derivative interactions.



For purposes of their interactions with the electrons, a nonzero q which keeps the e^- on the FS must scale like $q \sim b^0$. Therefore

$$dt d^d q \left(\partial_t D\right)^2 \sim b^{+1+2[D]} \quad \Longrightarrow \quad D \sim b^{-\frac{1}{2}}$$

and the restoring force $dtdq D^2 \omega^2(q) \sim b^{-2}$ is relevant, and dominates over the ∂_t^2 term for

$$E < E_D = \sqrt{\frac{m}{M}} E_0 \quad \text{the Debye energy.}$$

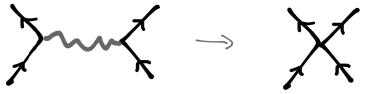
This means that phonons mediate *static* interactions below E_D – we can ignore retardation effects, and their effects on the electrons can be fully incorporated by the four-fermion interaction we used above (with some $\vec{k}$ dependence). How do they couple to the electrons?

$$S_{\text{int}}[D,\psi] = \int dt q^3 q d^2 k_1 d\ell_1 d^2 k_2 d\ell_2 \; M^{-\frac{1}{2}} g_i(q,k_1,k_2) D_i(q) \psi_\sigma^\dagger(p_1)\psi_\sigma(p_2)\delta^3(p_1 - p_2 - q)$$

[8]Note that there is a subtlety in counting Goldstone modes from spontaneously broken spacetime symmetries: there are more symmetry generators than Goldstones. Basically it's because the associated currents differ only by functions of spacetime; but a localized Goldstone particle is anyway made by a current times a function of spacetime, so you can't sharply distinguish the resulting particles. Some useful references on this subject are Low-Manohar and most recently Watanabe-Murayama.

$$\sim b^{-1+1+1-3/2} = b^{-1/2} \tag{2.23}$$

– here we took the delta function to scale like b^0 as above. This is relevant when we use the $\dot{D}^2$ scaling for the phonons; when the restoring force dominates we should scale D differently and this is irrelevant for generic kinematics. This is consistent with our previous analysis of the four-fermion interaction.

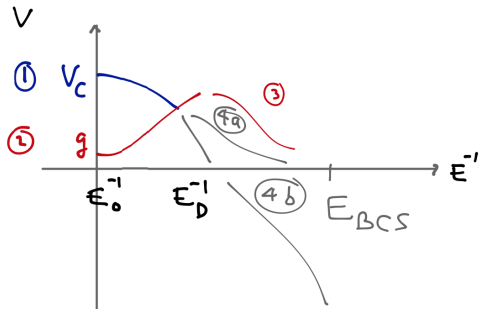
The summary of this discussion is: phonons do not destroy the Fermi surface, but they *do* produce an attractive contribution to the 4-fermion interaction, which is relevant in some range of scales (above the Debye energy). Below the Debye energy, it amounts to an addition to V that goes like $-g^2$: 

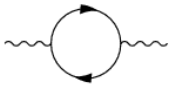
Notice that the scale at which the coupling V becomes strong ($V(E_{\rm BCS}) \equiv 1$ in (2.22)) is

$$E_{\rm BCS} \sim E_D e^{-\frac{1}{NV_D}}.$$

Two comments about this: First, it is non-perturbative in the interaction V_D. Second, it provides some verification of the role of phonons, since $E_D \sim M^{-1/2}$ can be varied by studying the same material with different isotopes and studying how the critical superconducting temperature ($\sim E_{\rm BCS}$) scales with the nuclear mass.

Actually, we can make some headway towards understanding the result of this interaction going strong. Because the diagrams with the special kinematics are marginal and hence unsuppressed, while all other interactions flow to zero at low energy, certain diagrams dominate. In particular, bubble-chains dominate.



Here's the narrative, proceeding as a function of decreasing energy scale, beginning at E_0, the Planck scale of solids: (1) Electrons repel each other by the Coulomb interaction. However, in a metal, this interaction is *screened* by processes like this:  (the intermediate state is an electron-hole pair) and is short-ranged. It is still repulsive, however. As we coarse-grain more and more, we see more and more electron-hole pairs and the force weakens. (2) While this is happening, the electron-phonon interaction is relevant and growing. This adds an attractive bit to V. This lasts until E_D. (3) At E_D the restoring force term in the phonon lagrangian dominates (for the purposes of their interactions with the electrons) and we can integrate them out. (4) What happens next depends on the sign of $V(E_D)$. If it's positive, V flows harmlessly to zero. If

it's negative, it becomes moreso until we exit the perturbative analysis at $E_{\rm BCS}$, and vindicate our choice of Hubbard-Stratonovich channel above.

Further brief comments, for which I refer you to Shankar:

1. Putting back the possible angular dependence of the BCS interaction, the result at one loop is

$$\frac{dV(\theta_1-\theta_3)}{d\ell} = -\frac{1}{8\pi^2}\int_0^{2\pi} đ\theta V(\theta_1-\theta)V(\theta-\theta_3)$$

   or in terms of angular momentum components,

$$\frac{dV_l}{d\ell} = -\frac{V_l^2}{4\pi}.$$

2. This example is interesting and novel in that it is a (family of) fixed point(s) characterized by a dimensionful quantity, namely k_F. This leads to a phenomenon called *hyperscaling violation* where thermodynamic quantities need not have their naive scaling with temperature.

3. The one loop analysis gives the right answer to all loops in the limit that $N \equiv k_F/\Lambda \gg 1$, where Λ is the UV cutoff on the momentum.

4. The forward scattering interaction (for any choice of function $F(\theta_{13})$) is not renormalized at one loop. This means it is exactly marginal at leading order in N.

5. Like in ϕ^4 theory, the sunrise diagram at two loops is the first appearance of wavefunction renormalization. In the context of the Fermi liquid theory, this leads to the renormalization of the effective mass which is called $m^\star$.

Another consequence of the FS kinematics which I should emphasize more: it allows the quasiparticle to be stable. The leading contribution to the decay rate of a one-quasiparticle state with momentum k can be obtained applying the optical theorem to the following process.

The intermediate state is two electrons with momenta $k'+q$ and $k-q$, and one *hole* with momentum k'. The hole propagator has the opposite $\mathbf{i}\eta$ prescription. After doing the frequency integrals by residues, we get

$$\Sigma(k,\epsilon) = \int đq\, đk' \frac{|u_q|^2}{D-\mathbf{i}\eta}$$

$$D \equiv \epsilon_k(1+\mathbf{i}\eta) + \epsilon_{k'}(1-\mathbf{i}\eta) - \epsilon_{k'+q}(1+\mathbf{i}\eta) - \epsilon_{k-q}(1+\mathbf{i}\eta)$$

(Notice that this is the eyeball diagram which gives the lowest-order contribution to the wavefunction renormalization of a field with quartic interactions.) By the optical theorem, its imaginary part is the (leading contribution to the) inverse-lifetime of the quasiparticle state with fixed k:

$$\tau^{-1}(k) = \mathrm{Im}\,\Sigma(k,\epsilon) = \pi \int đq\, đk' \delta(D) |u_q|^2 f(-\epsilon_{k'}) f(\epsilon_{k'+q}) f(\epsilon_{k-q})$$

where

$$f(\epsilon) = \lim_{T\to 0} \frac{1}{e^{\frac{\epsilon-\epsilon_F}{T}}+1} = \theta(\epsilon < \epsilon_F)$$

is the Fermi function. This is just the demand that a particle can only scatter into an empty state and a hole can only scatter into a filled state. These constraints imply that all the energies are near the Fermi energy: both $\epsilon_{k'+q}$ and $\epsilon_{k'}$ lie in a shell of radius ϵ about the FS; the answer is proportional to the density of possible final states, which is thus

$$\tau^{-1} \propto \left(\frac{\epsilon}{\epsilon_F}\right)^2 .$$

So the width of the quasiparticle resonance is

$$\tau^{-1} \propto \epsilon^2 \ll \epsilon$$

much smaller than its frequency – it is a sharp resonance, a well-defined particle.

[End of Lecture 6]

3 Geometric and topological terms in field theory actions

Resolving the identity. The following is an advertisement: When studying a quantum mechanical system, isn't it annoying to have to worry about the order in which you write the symbols? What if they don't commute?! If you have this problem, too, the path integral is for you. In the path integral, the symbols are just integration variables – just ordinary numbers, and you can write them in whatever order you want. You can write them upside down if you want. You can even change variables in the integral (Jacobian not included).

(In what order do the operators end up? As we showed last quarter, in the kinds of path integrals we're thinking about, they end up in *time-order*. If you want a different order, you will want to use the *Schwinger-Keldysh* extension package, sold separately.)

Much of the following is about how to go back and forth from Hilbert space to path integral representations, aka Hamiltonian and Lagrangian descriptions of QFT. You make a path integral representation of some physical quantity by sticking lots of 𝟙s in there, and then resolving each of the identity operators in some basis that you like. Different bases, different integrals. Some are useful, mostly because we have intuition for the behavior of integrals.

3.1 Coherent state path integrals for bosons

[Wen §3.3] Let's develop a path integral for a collection of bosons, using as our basis of the local Hilbert space ordinary SHO (simple harmonic oscillator) coherent states. What I mean by 'bosons' is a many-body system whose Hilbert space can be written as $\mathcal{H} = \otimes_k \mathcal{H}_k$ where k is a label (could be real space, could be momentum space) and

$$\mathcal{H}_k = \text{span}\{|0\rangle_k, \mathbf{a}_k^\dagger |0\rangle_k, \frac{1}{\sqrt{2!}}\left(\mathbf{a}_k^\dagger\right)^2 |0\rangle_k, ...\} = \text{span}\{|n\rangle_{\vec{k}}, n = 0, 1, 2...\}$$

is the SHO Hilbert space. Assume the modes satisfy

$$[\mathbf{a}_{\vec{k}}, \mathbf{a}_{\vec{k}'}^\dagger] = \delta^d(\vec{k} - \vec{k}').$$

A good example hamiltonian to keep in mind is the free one,

$$\mathbf{H}_0 = \sum_{\vec{k}} \left(\epsilon_{\vec{k}} - \mu\right) \mathbf{a}_{\vec{k}}^\dagger \mathbf{a}_{\vec{k}} \ .$$

The object $\epsilon_{\vec{k}} - \mu$ determines the energy of the state with one boson of momentum $\vec{k}$: $\mathbf{a}_{\vec{k}}^\dagger |0\rangle$. The chemical potential μ shifts the energy of any state by an amount

proportional to

$$\left\langle \sum_{\vec{k}} \mathbf{a}^\dagger_{\vec{k}} \mathbf{a}_{\vec{k}} \right\rangle = N$$

the number of bosons. To this we could consider adding an interaction term $\mathbf{H} = \mathbf{H}_0 + \mathbf{V}$, such as

$$V = \sum_{x,y} V_{xy} a^\dagger_x a_x a^\dagger_y a_y.$$

For each normal mode $\mathbf{a}$, coherent states are[9]

$$\mathbf{a} \left|\phi\right\rangle = \phi \left|\phi\right\rangle; \qquad \left|\phi\right\rangle = \mathcal{N} e^{\phi \mathbf{a}^\dagger} \left|0\right\rangle.$$

The eigenbra of $\mathbf{a}^\dagger$ is $\langle\phi|$, with

$$\left\langle\phi\right| \mathbf{a}^\dagger = \left\langle\phi\right| \phi^\star, \quad \left\langle\phi\right| = \left\langle 0\right| e^{+\phi^\star \mathbf{a}} \mathcal{N}.$$

(In this case, this equation is the adjoint of the previous one.) Their overlap is[10]:

$$\langle\phi_1|\phi_2\rangle = e^{\phi_1^\star \phi_2}.$$

If we choose $\mathcal{N} = e^{-|\phi|^2/2}$, they are normalized, but it is more convenient to set $\mathcal{N} = 1$. The overcompleteness relation on $\mathcal{H}_k$ is[11]

$$\mathbb{1}_k = \int \frac{d\phi d\phi^\star}{\pi} e^{-|\phi|^2} \left|\phi\right\rangle \left\langle\phi\right|.$$

It will be convenient to arrange all our operators into sums of *normal-ordered* operators:

$$:\mathbf{a}_k \mathbf{a}^\dagger_l: =: \mathbf{a}^\dagger_l \mathbf{a}_k: = \mathbf{a}^\dagger_l \mathbf{a}_k$$

[9]The right equation is true because

$$\mathbf{a} e^{\phi \mathbf{a}^\dagger} \left|0\right\rangle = \sum_{n=0}^{\infty} \frac{\phi^n}{n!} \underbrace{\mathbf{a} \left(\mathbf{a}^\dagger\right)^n \left|0\right\rangle}_{n(\mathbf{a}^\dagger)^{n-1}|0\rangle} = \sum_{m=n-1} \frac{\phi^{m+1}}{m!} \left(\mathbf{a}^\dagger\right)^m \left|0\right\rangle.$$

[10]You can check this by expanding the coherent states in the number basis and using $\langle 0|a^n a^{\dagger m}|0\rangle = \delta_{mn} n!$.

[11]Again we can go to the number basis and do the integrals:

$$\int \frac{d\phi d\phi^\star}{\pi} e^{-\phi\phi^\star} \phi^n \left(\phi^\star\right)^{n'} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{\mathbf{i}(n-n')\theta} \int_0^\infty du e^{-u} u^{\frac{n+n'}{2}}$$

to get $\mathbb{1} = \sum_n \left|n\right\rangle \left\langle n\right|$.

with all annihilation operators to the right of all creation operators. Coherent state expectation values of such operators can be built from the monomials

$$\langle\phi| \prod_k \left(\mathbf{a}_k^\dagger\right)^{M_k} (\mathbf{a}_k)^{N_k} |\phi\rangle = \prod_k (\phi_k^\star)^{M_k} (\phi_k)^{N_k} .$$

Also useful will be the representation of the trace in this basis:

$$\mathrm{tr}\cdot = \int \frac{d\phi d\phi^\star}{\pi} e^{-|\phi|^2} \langle\phi| \cdot |\phi\rangle .$$

Let the Hamiltonian be $\mathbf{H} = H(\{\mathbf{a}_k^\dagger\}, \{\mathbf{a}_k\}) =: \mathbf{H} :$, normal ordered. First let's study a single mode. To derive a path integral (for *e.g.* the thermal partition function) using this resolution of the identity $\mathbb{1} = \prod_{\vec{k}} \mathbb{1}_{\vec{k}}$, we write

$$Z = \mathrm{tr} e^{-\mathbf{H}/T} = \int \frac{d\phi d\phi^\star}{\pi} e^{-|\phi|^2} \langle\phi| \underbrace{e^{-\mathbf{H}/T}}_{= e^{-\Delta\tau\mathbf{H}} e^{-\Delta\tau\mathbf{H}} \cdots e^{-\Delta\tau\mathbf{H}}} |\phi\rangle$$

and repeatedly insert $\mathbb{1}$ in this form:

$$\begin{aligned} Z &= \mathrm{tr} e^{-\mathbf{H}/T} \\ &= \int_{\phi_{N+1}=\phi_0} \prod_{l=0}^{N} d\phi_l \; e^{-\sum_{l=0}^{N} \left(\phi_{l+1}(\phi_{l+1}-\phi_l) - \Delta\tau H(\phi_{l+1}^\star \phi_l)\right)} \\ &\simeq \int_{\phi(0)=\phi(1/T)} [D\phi] \; e^{-\int_0^{1/T} d\tau (\phi^\star \partial_\tau \phi + H(\phi^\star, \phi))} . \end{aligned} \tag{3.1}$$

Here we used

$$\langle\phi_{\ell+1}| e^{-\Delta\tau\mathbf{H}} |\phi_\ell\rangle \simeq \langle\phi_{\ell+1}| (1 - \Delta\tau\mathbf{H}) |\phi_\ell\rangle = \langle\phi_{\ell+1}| \left(1 - \Delta\tau H(\phi_{\ell+1}^\star, \phi_\ell)\right) |\phi_\ell\rangle \simeq e^{-\Delta\tau H(\phi_{\ell+1}^\star, \phi_\ell)} e^{-\phi_{\ell+1}^\star \phi_\ell}$$

for small enough $\Delta\tau$. Putting back the mode labels, this is

$$Z = \int [Da] e^{\int dt \sum_{\vec{k}} \left(\frac{1}{2}\left(a_{\vec{k}}^\star \dot{a}_{\vec{k}} - a_{\vec{k}} \dot{a}_{\vec{k}}^\star\right) - \left(\epsilon_{\vec{k}} - \mu\right) a_{\vec{k}}^\star a_{\vec{k}}\right)} .$$

In real space $a_{\vec{k}} \equiv \int \mathrm{d}^{D-1} x e^{\mathbf{i}\vec{k}\cdot\vec{x}} \psi(\vec{x})$, Taylor expanding $\epsilon_{\vec{k}} - \mu = -\mu + \frac{\vec{k}^2}{2m} + \mathcal{O}(k^4)$, this is

$$Z = \int [D\psi] e^{\int \mathrm{d}^d \vec{x} \mathrm{d}t \left(\frac{1}{2}(\psi^\star \partial_t \psi - \psi \partial_t \psi^\star) - \frac{1}{2m} \vec{\nabla}\psi^\star \cdot \vec{\nabla}\psi - \mu\psi^\star\psi\right)} .$$

The first term in the exponent is sometimes called a Berry phase term.

Real time. If you are interested in real-time propagation, rather than euclidean time, just replace the euclidean propagator $e^{-\tau\mathbf{H}} \mapsto e^{-\mathbf{i}t\mathbf{H}}$. The result, for example, for the amplitude to propagate from one bose coherent state to another is

$$\langle\phi_f, t_f| e^{-\mathbf{i}t\mathbf{H}} |\phi_0, t_0\rangle = \int_{\phi(t_0)=\phi_0}^{\phi(t_f)=\phi_f} D\phi^\star D\phi \; e^{\frac{\mathbf{i}}{\hbar}\int_{t_0}^{t_f} dt (\mathbf{i}\hbar\phi^\star \partial_t \phi - H(\phi, \phi^\star))} .$$

(Actually there is some funny business with the boundary conditions in this expression. In particular, notice that with a first-order kinetic term, we only get to impose one (complex) initial condition. A more careful treatment can be found here, §1.9.)

Note that a distinguishing feature of the Berry phase term (because it has a single time derivative) is that it produces a complex term in the real-time action. Another distinguishing feature of the Berry phase term is that it is *geometric*: for a history of field configurations, $\Phi(t)$, the Berry phase term

$$\int_{t_0}^{t_f} dt \Phi^\star(t)\dot{\Phi}(t) = \int_{\Phi_0}^{\Phi_f} \Phi^\star d\Phi$$

doesn't depend on how fast we go, only on the path in field space.

Question to ponder: how would you get a second-order-in-time kinetic term?

This is the same non-relativistic field theory you would find (see below) by taking the $E \ll m$ limit of a relativistic scalar field. Notice that the field ψ is actually the coherent state eigenvalue!

If instead we had an interaction term in H, say $\Delta H = \int d^dx \int d^dy \frac{1}{2}\psi^\star(x,t)\psi(x,t)V(x-y)\psi^\star(y,t)\psi(y,t)$, it would lead to a term in the path integral action

$$S_i = -\int \mathrm{d}t \int d^dx \int d^dy \frac{1}{2}\psi^\star(x,t)\psi(x,t)V(x-y)\psi^\star(y,t)\psi(y,t) \ .$$

In the special case $V(x-y) = V(x)\delta^d(x-y)$, this is the local quartic interaction we considered briefly earlier.

Non-relativistic scalar fields. [Zee §III.5, V.1, Kaplan nucl-th/0510023 §1.2.1] In the previous discussion of the EFT for a superconductor, I spoke as if the complex scalar were relativistic.

In superconducting materials, it is generally not. In real superconductors, at least. How should we think about a non-relativistic field? A simple answer comes from realizing that a relativistic field which can make a boson of mass m can certainly make a boson of mass m which is moving slowly, with $v \ll c$. By taking a limit of the relativistic model, then, we can make a description which is useful for describing the interactions of an indefinite number of bosons moving slowly in some Lorentz frame. A situation that calls for such a description is a large collection of ^{4}He atoms.

Reminder: non-relativistic limit of a relativistic scalar field. A non-relativistic particle in a relativistic theory (consider massive ϕ^4 theory) has energy

$$E = \sqrt{p^2+m^2} \overset{\text{if } v \ll c}{=} m + \frac{p^2}{2m} + ...$$

This means that the field that creates and annihilates it looks like

$$\phi(\vec{x},t)=\sum_{\vec{k}}\frac{1}{\sqrt{2E_{\vec{k}}}}\left(a_{\vec{k}}e^{-\mathbf{i}E_{\vec{k}}t-\mathbf{i}\vec{k}\cdot\vec{x}}+h.c.\right)$$

In particular, we have

$$\dot{\phi}^2\simeq m^2\phi^2$$

and the BHS of this equation is large. To remove this large number let's change variables:

$$\phi(x,t)\equiv\frac{1}{\sqrt{2m}}\left(e^{-\mathbf{i}mt}\underbrace{\Phi(x,t)}_{\text{complex},\dot{\Phi}\ll m\Phi}+h.c.\right)\ .$$

Notice that Φ is complex, even if ϕ is real.

Let's think about the action governing this NR sector of the theory. We can drop terms with unequal numbers of Φ and $\Phi^\star$ since such terms would come with a factor of $e^{\mathbf{i}mt}$ which gives zero when integrated over time. Starting from $(\partial\phi)^2-m^2\phi^2-\lambda\phi^4$ we get:

$$L_{\text{real time}}=\Phi^\star\left(\mathbf{i}\partial_t+\frac{\vec{\nabla}^2}{2m}\right)\Phi-g^2\left(\Phi^\star\Phi\right)^2+... \tag{3.2}$$

with $g^2=\frac{\lambda}{4m^2}$.

Notice that Φ is a complex field and its action has a U(1) symmetry, $\Phi\to e^{\mathbf{i}\alpha}\Phi$, even though the full theory did not. The associated conserved charge is the number of particles:

$$j_0=\Phi^\star\Phi,j_i=\frac{\mathbf{i}}{2m}\left(\Phi^\star\partial_i\Phi-\partial_i\Phi^\star\Phi\right),\quad\partial_t j_0-\nabla\cdot\vec{j}=0\ .$$

Notice that the 'mass term' $\Phi^\star\Phi$ is then actually the chemical potential term, which encourages a nonzero density of particles to be present.

This is an example of an *emergent* symmetry: a symmetry of an EFT that is not a symmetry of the microscopic theory. The ... in (3.2) include terms which break this symmetry, but they are irrelevant.

Actually the theory with $\mu=0$ can have another emergent symmetry, which is scale invariance. To keep this non-relativistic kinetic term fixed we must scale time and space differently:

$$x\to\tilde{x}=sx,\ t\to\tilde{t}=s^2t,\ \Phi\to\tilde{\Phi}(\tilde{x},\tilde{t})=\zeta\Phi(sx,s^2t)\ .$$

A fixed point with this scaling rule has dynamical exponent $z=2$. The scaling of the

bare action (with no mode elimination step) is

$$S_E^{(0)} = \int \underbrace{dtd^d\vec{x}}_{=s^{d+z}d\tilde{t}d^d\tilde{x}} \left(\Phi^\star\left(sx, s^2t\right) \underbrace{\left(\partial_t - \frac{\vec{\nabla}^2}{2m}\right)}_{=s^{-2}\left(\tilde{\partial}_t - \frac{\tilde{\vec{\nabla}}^2}{2m}\right)} \Phi(sx, s^2t) - g^2\left(\Phi^\star\Phi(sx, s^2t)\right)^2 + ... \right)$$

$$= \underbrace{s^{d+z-2}\zeta^{-2}}_{\stackrel{!}{=}1 \implies \zeta \stackrel{d=3}{=} s^{3/2}} \int \mathrm{d}\tilde{t}\mathrm{d}^d\tilde{x} \left(\tilde{\Phi}^\star \left(\tilde{\partial}_t - \frac{\tilde{\vec{\nabla}}^2}{2m}\right) \tilde{\Phi} - \zeta^{-2}s^2g^2\left(\tilde{\Phi}^\star\tilde{\Phi}(\tilde{x}, \tilde{t})\right)^2 + ... \right) \quad (3.3)$$

From this we learn that $\tilde{g}^2 = s^2\zeta^{-2}g^2 = s^{4-(d+z)}g^2 \stackrel{d=3,z=2}{\to} 0$ in the IR – the quartic term is irrelevant in $D = d + 1 = 3 + 1$ with nonrelativistic scaling! Where does it become marginal? Recall the delta function potential for a particle in two dimensions.

Number and phase angle. In the NR theory, the canonical momentum for Φ is just $\frac{\partial L}{\partial \dot{\Phi}} \sim \Phi^\star$, with no derivatives. This statement becomes more shocking if we change variables to $\Phi = \sqrt{\rho}e^{\mathbf{i}\theta}$. This is a useful change of variables, if for example we knew ρ didn't want to be zero, as would happen if we add to (3.2) a term of the form $-\mu\Phi^\star\Phi$. So consider the action density

$$L = L_{\text{real time}} = \Phi^\star\left(\mathbf{i}\partial_t + \frac{\vec{\nabla}^2}{2m}\right)\Phi - V(\Phi^\star\Phi), \quad V(\Phi^\star\Phi) \equiv g^2\left(\Phi^\star\Phi\right)^2 - \mu\Phi^\star\Phi.$$

In polar coordinates this is

$$L = \frac{\mathbf{i}}{2}\partial_t\rho - \rho\partial_t\theta - \frac{1}{2m}\left(\rho\left(\nabla\theta\right)^2 + \frac{1}{4\rho}\left(\nabla\rho\right)^2\right) - V(\rho). \quad (3.4)$$

The first term is a total derivative. The second term says that the canonical momentum for the phase variable θ is $\rho = \Phi^\star\Phi = j_0$, the particle number density. Quantumly, then:

$$[\hat{\rho}(\vec{x}, t), \hat{\theta}(\vec{x}', t)] = \mathbf{i}\delta^d(\vec{x} - \vec{x}'). \quad (3.5)$$

Number and phase are canonically conjugate variables. If we fix the phase, the amplitude is maximally uncertain.

If we integrate over space, $N \equiv \int d^dx\rho(\vec{x}, t)$ gives the total number of particles, which is time independent, and satisfies $[N, \theta] = \mathbf{i}$.

What is the term $\mu\Phi^\star\Phi = \mu\rho$? It is a chemical potential for the boson number.

This relation (3.5) explains why there's no Higgs boson in most non-relativistic superconductors and superfluids (in the absence of some extra assumption of particle-hole

symmetry). In the NR theory with first order time derivative, the would-be amplitude mode which oscillates about the minimum of $V(\rho)$ is actually just the conjugate momentum for the goldstone boson!

Superfluids. [Zee §V.1] Let me amplify the previous remark. A superconductor is just a superfluid coupled to a weakly-coupled $\mathsf{U}(1)$ gauge field, so we've already understood something about superfluids.

The effective field theory has the basic lagrangian (3.4), with $\langle\rho\rangle = \bar{\rho} \neq 0$. This nonzero density can be accomplished by adding an appropriate chemical potential to (3.4); up to an uninteresting constant, this is

$$L = \frac{\mathbf{i}}{2}\partial_t\rho - \rho\partial_t\theta - \frac{1}{2m}\left(\rho\left(\nabla\theta\right)^2 + \frac{1}{4\rho}\left(\nabla\rho\right)^2\right) - g^2\left(\rho-\bar{\rho}\right)^2.$$

Expand around such a condensed state in small fluctuations $\sqrt{\rho} = \sqrt{\bar{\rho}}+h$, $h \ll \sqrt{\bar{\rho}}$:

$$L = -2\sqrt{\bar{\rho}}h\partial_t\theta - \frac{\bar{\rho}}{2m}\left(\vec{\nabla}\theta\right)^2 - \frac{1}{2m}\left(\vec{\nabla}h\right)^2 - 4g^2\bar{\rho}h^2 + ...$$

Notice that h, the fluctuation of the amplitude mode, is playing the role of the canonical momentum of the goldstone mode θ. The effects of the fluctuations can be incorporated by doing the gaussian integral over h (What suppresses self-interactions of h?), and the result is

$$\begin{aligned} L &= \sqrt{\bar{\rho}}\partial_t\theta\frac{1}{4g^2\bar{\rho} - \frac{\nabla^2}{2m}}\sqrt{\bar{\rho}}\partial_t\theta - \frac{\bar{\rho}}{2m}\left(\vec{\nabla}\theta\right)^2 \\ &= \frac{1}{4g^2}\left(\partial_t\theta\right)^2 - \frac{\bar{\rho}}{2m}\left(\nabla\theta\right)^2 + ... \end{aligned} \tag{3.6}$$

where in the second line we are expanding in the small wavenumber k of the modes, that is, we are constructing an action for Goldstone modes whose wavenumber is $k \ll \sqrt{9g^2\bar{\rho}m}$ so we can ignore higher gradient terms.

The linearly dispersing mode in this superfluid that we have found is sometimes called the phonon. This is a good name because the wave involves oscillations of the density:

$$\underline{h} = \frac{1}{4g^2\bar{\rho} - \frac{\nabla^2}{2m}}\sqrt{\bar{\rho}}\partial_t\theta$$

is the saddle point solution for h. The phonon has dispersion relation

$$\omega^2 = \frac{2g^2\bar{\rho}}{m}\vec{k}^2.$$

This mode has an emergent Lorentz symmetry with a lightcone with velocity $v_c = g\sqrt{2\bar{\rho}/m}$. The fact that the sound velocity involves g – which determined the steepness

of the walls of the wine-bottle potential – is a consequence of the non-relativistic dispersion of the bosons. In the relativistic theory, we have $L = \partial_\mu \Phi^\star \partial^\mu \Phi - g\left(\Phi^\star \Phi - v^2\right)^2$ and we can take $g \to \infty$ fixing v and still get a linearly dispersing mode by plugging in $\Phi = e^{\mathbf{i}\theta} v$.

The importance of the linearly dispersing phonon mode of the superfluid is that there is *no other* low energy excitation of the fluid. With a classical pile of (*e.g.* non interacting) bosons, a chunk of moving fluid can donate some small momentum $\vec{k}$ to a single boson at energy cost $\frac{(\hbar\vec{k})^2}{2m}$. A quadratic dispersion means more modes at small k than a linear dispersion (the density of states is $N(E) \propto k^{D-1}\frac{dk}{dE}$). With only a linearly dispersing mode at low energies, there is a critical velocity below which a non-relativistic chunk of fluid cannot give up any momentum [Landau]: conserving momentum $M\vec{v} = M\vec{v}' + \hbar\vec{k}$ says the change in energy (which must be negative for this to happen on its own) is

$$\frac{1}{2}M(v')^2 + \hbar\omega(k) - \frac{1}{2}Mv^2 = -\hbar k v + \frac{(\hbar k)^2}{2M} + \hbar\omega(k) = (-v + v_c)\hbar k + \frac{(\hbar k)^2}{2M} \ .$$

For small k, this is only negative when $v > v_c = \partial_k \omega|_{k=0}$. This means that the flow is dissipationless, and is the origin of the name 'superfluid'.

You can ask: an ordinary liquid also has a linearly dispersing sound mode; why doesn't Landau's argument mean that it has superfluid flow? The answer is that it has *other* modes with softer dispersion (so more contribution at low energies), in particular diffusion modes, with $\omega \propto k^2$ (there is an important factor of $\mathbf{i}$ in there).

The Goldstone boson has a compact target space, $\theta(x) \equiv \theta(x) + 2\pi$, since, after all, it is the phase of the boson field. This is significant because it means that as the phase wanders around in space, it can come back to its initial value after going around the circle – such a loop encloses a *vortex*. Somewhere inside, we must have $\Phi = 0$. There is much more to say about this.

[End of Lecture 7]

3.2 Coherent state path integral for fermions

[Shankar, *Principles of QM*, path integrals revisited. In this chapter of his great QM textbook, Shankar sneaks in lots of insights useful for modern condensed matter physics. For much more see Shankar, *Quantum field theory and condensed matter*.]

Consider the algebra of a single fermion mode operator[12]:

$$\{\mathbf{c}, \mathbf{c}\} = 0, \quad \{\mathbf{c}^\dagger, \mathbf{c}^\dagger\} = 0, \quad \{\mathbf{c}, \mathbf{c}^\dagger\} = 1 \ .$$

With a single mode, the most general Hamiltonian is

$$\mathbf{H} = \mathbf{c}^\dagger \mathbf{c} \, (\omega_0 - \mu)$$

(ω_0 and μ are (redundant when there is only one mode) constants, and I've omitted an additive constant). This algebra is represented on a two-state system $|1\rangle = \mathbf{c}^\dagger |0\rangle$. We might be interested in its thermal partition function

$$Z = \text{tr } e^{-\frac{\mathbf{H}}{T}} \ .$$

(In this example, it happens to equal $Z = 1 + e^{-\frac{\omega_0 - \mu}{T}}$, as you can see by computing the trace in the eigenbasis of $\mathbf{n} = \mathbf{c}^\dagger \mathbf{c}$. But never mind that; the one mode is a proxy for many, where it's not quite so easy to sum. The kind of model we really want to think about looks like

$$\mathbf{H} = \sum_k \mathbf{c}_k^\dagger \mathbf{c}_k \, (\omega_k - \mu) + \sum_{x,y} \mathbf{c}_x^\dagger \mathbf{c}_x V_{xy} \mathbf{c}_y^\dagger \mathbf{c}_y .)$$

How do we trotterize this? That is, what is 'the' corresponding classical system? (One answer is to use the (0d) Jordan-Wigner map which relates spins and fermions. Perhaps more about that later. Here's another, different, answer.) We can do the Trotterizing using any resolution of the identity on $\mathcal{H}$, so there can be many very-different-looking answers to this question.

Let's define coherent states for fermionic operators:

$$\mathbf{c} \, |\psi\rangle = \psi \, |\psi\rangle \, . \tag{3.7}$$

Here ψ is a c-number (not an operator), but acting twice with $\mathbf{c}$ we see that we must have $\psi^2 = 0$. ψ is a grassmann number. These satisfy

$$\psi_1 \psi_2 = -\psi_2 \psi_1, \psi \mathbf{c} = -\mathbf{c} \psi \tag{3.8}$$

[12] For many modes,

$$\{\mathbf{c}_i, \mathbf{c}_j\} = 0, \quad \{\mathbf{c}_j^\dagger, \mathbf{c}_j^\dagger\} = 0, \quad \{\mathbf{c}_j, \mathbf{c}_j^\dagger\} = \mathbb{1} \delta_{ij} \ .$$

– they anticommute with each other and with fermionic operators, and commute with ordinary numbers and bosons. They seem weird but they are easy. We'll need to consider multiple grassmann numbers when we have more than one fermion mode, where $\{\mathbf{c}_1, \mathbf{c}_2\} = 0$ will require that they anticommute $\{\psi_1, \psi_2\} = 0$ (as in the definition (3.8)); note that we will be simultaneously diagonalizing operators which *anti*commute.

The solution to equation (3.7) is very simple:

$$|\psi\rangle = |0\rangle - \psi\,|1\rangle = |0\rangle - \psi\mathbf{c}^\dagger\,|0\rangle = e^{-\psi\mathbf{c}^\dagger}\,|0\rangle$$

where as above $|0\rangle$ is the empty state ($\mathbf{c}\,|0\rangle = 0$) and $|1\rangle = \mathbf{c}^\dagger\,|0\rangle$ is the filled state. (Check: $\mathbf{c}\,|\psi\rangle = \mathbf{c}\,|0\rangle - \mathbf{c}\psi\,|1\rangle = +\psi\mathbf{c}\,|1\rangle = \psi\,|0\rangle = \psi\,|\psi\rangle$.)

Similarly, the left-eigenvector of the creation operator is

$$\langle\bar\psi|\,\mathbf{c}^\dagger = \langle\bar\psi|\,\bar\psi, \quad \langle\bar\psi| = \langle 0| - \langle 1|\,\bar\psi = \langle 0| + \bar\psi\,\langle 1|\,.$$

Notice that these states are weird in that they are elements of an enlarged hilbert space with grassmann coefficients (usually we just allow complex numbers). Also, $\bar\psi$ is *not* the complex conjugate of ψ and $\langle\bar\psi|$ is not the adjoint of $|\psi\rangle$. Rather, their overlap is

$$\langle\bar\psi|\psi\rangle = 1 + \bar\psi\psi = e^{\bar\psi\psi}.$$

Grassmann calculus summary. In the last expression we have seen an example of the amazing simplicity of Taylor's theorem for grassmann functions:

$$f(\psi) = f_0 + f_1\psi\ .$$

Integration is just as easy and its the same as taking derivatives:

$$\int \psi d\psi = 1, \quad \int 1 d\psi = 0.$$

With more than one grassmann we have to worry about the order:

$$1 = \int \bar\psi\psi d\psi d\bar\psi = -\int \bar\psi\psi d\bar\psi d\psi.$$

The only integral, really, is the gaussian integral:

$$\int e^{-a\bar\psi\psi} d\bar\psi d\psi = a.$$

Many of these give

$$\int e^{-\bar\psi\cdot A\cdot\psi} d\bar\psi d\psi = \det A.$$

Here $\bar\psi\cdot A\cdot\psi\equiv\left(\bar\psi_1,\cdots,\bar\psi_M\right)\begin{pmatrix}A_{11}&A_{12}&\cdots\\A_{21}&\ddots&\cdots\\\vdots&\vdots&\ddots\end{pmatrix}\begin{pmatrix}\psi_1\\\vdots\\\psi_M\end{pmatrix}$. One way to get this expression is to change variables to diagonalize the matrix A.

$$\left\langle\bar\psi\psi\right\rangle\equiv\frac{\int\bar\psi\psi e^{-a\bar\psi\psi}d\bar\psi d\psi}{\int e^{-a\bar\psi\psi}d\bar\psi d\psi}=-\frac{1}{a}=-\left\langle\psi\bar\psi\right\rangle.$$

If for many grassman variables we use the action $S=\sum_i a_i\bar\psi_i\psi_i$ (diagonalize A above) then

$$\left\langle\bar\psi_i\psi_j\right\rangle=\frac{\delta_{ij}}{a_i}\equiv\langle\bar ij\rangle \tag{3.9}$$

and Wick's theorem here is

$$\left\langle\bar\psi_i\bar\psi_j\psi_k\psi_l\right\rangle=\langle\bar il\rangle\langle\bar jk\rangle-\langle\bar ik\rangle\langle\bar jl\rangle.$$

Back to quantum mechanics: The resolution of $\mathbb{1}$ in this basis is

$$\mathbb{1}=\int d\bar\psi d\psi\; e^{-\bar\psi\psi}\left|\psi\right\rangle\left\langle\bar\psi\right| \tag{3.10}$$

And if $\mathbf{A}$ is a bosonic operator (made of an even number of grassmann operators),

$$\mathrm{tr}\mathbf{A}=\int d\bar\psi d\psi\; e^{-\bar\psi\psi}\left\langle-\bar\psi\right|\mathbf{A}\left|\psi\right\rangle\ .$$

(Note the minus sign; it will lead to a deep statement.) So the partition function is:

$$Z=\int d\bar\psi_0 d\psi_0\; e^{-\bar\psi_0\psi_0}\left\langle-\bar\psi_0\right|\underbrace{e^{-\frac{\mathbf{H}}{T}}}_{=\underbrace{(1-\Delta\tau\mathbf{H})\cdots(1-\Delta\tau\mathbf{H})}_{M\ \text{times}}}\left|\psi_0\right\rangle$$

Now insert (3.10) in between each pair of Trotter factors to get

$$Z=\int\prod_{l=0}^{M-1}d\bar\psi_l d\psi_l e^{-\bar\psi_l\psi_l}\left\langle\bar\psi_{l+1}\right|(1-\Delta\tau\mathbf{H})\left|\psi_l\right\rangle\ .$$

Because of the $-\bar\psi$ in (3.10), to get this nice expression we had to define an extra letter

$$\bar\psi_M=-\bar\psi_0,\quad\psi_M=-\psi_0 \tag{3.11}$$

so we could replace $\left\langle-\bar\psi_0\right|=\left\langle\bar\psi_M\right|$.

Now we use the coherent state property to turn the matrix elements into grassmann-valued functions:

$$\left\langle \bar{\psi}_{l+1} \right| \left(1 - \Delta\tau H(\mathbf{c}^\dagger, \mathbf{c})\right) \left|\psi_l\right\rangle = \left\langle \bar{\psi}_{l+1} \right| \left(1 - \Delta\tau H(\bar{\psi}_{l+1}, \psi_l)\right) \left|\psi_l\right\rangle \overset{\Delta\tau\to 0}{=} e^{\bar{\psi}_{l+1}\psi_l} e^{-\Delta\tau H(\bar{\psi}_{l+1},\psi_l)}.$$

It was important that in $\mathbf{H}$ all $\mathbf{c}$s were to the right of all $\mathbf{c}^\dagger$s, *i.e.* that $\mathbf{H}$ was normal ordered.

So we have

$$\begin{aligned} Z &= \int \prod_{l=0}^{M-1} d\bar{\psi}_l d\psi_l e^{-\bar{\psi}_l\psi_l} e^{\bar{\psi}_{l+1}\psi_l} e^{-\Delta\tau H(\bar{\psi}_{l+1},\psi_l)} \\ &= \int \prod_{l=0}^{M-1} d\bar{\psi}_l d\psi_l \exp\left(\Delta\tau \left(\underbrace{\frac{\bar{\psi}_{l+1} - \bar{\psi}_l}{\Delta\tau}}_{=\partial_\tau\bar{\psi}} \psi_l - H(\bar{\psi}_{l+1}, \psi_l)\right)\right) \\ &\simeq \int [D\bar{\psi}D\psi] \exp\left(\int_0^{1/T} d\tau\ \bar{\psi}(\tau)\left(-\partial_\tau - \omega_0 + \mu\right)\psi(\tau)\right) = \int [D\bar{\psi}D\psi] e^{-S[\bar{\psi},\psi]}. \end{aligned} \tag{3.12}$$

Points to note:

- In the penultimate step we defined, as usual, continuum fields

$$\psi(\tau_l = \Delta\tau l) \equiv \psi_l, \qquad \bar{\psi}(\tau_l = \Delta\tau l) \equiv \bar{\psi}_l.$$

- We elided the difference $H(\bar{\psi}_{l+1}, \psi_l) = H(\bar{\psi}_l, \psi_l) + \mathcal{O}(\Delta\tau)$ in the last expression. This difference is usually negligible and sometimes helpful (an example where it's helpful is the discussion of the number density below).

- The APBCs (3.11) on $\psi(\tau + \frac{1}{T}) = -\psi(\tau)$ mean that in its fourier representation[13]

$$\psi(\tau) = T\sum_n \psi(\omega) e^{-\mathbf{i}\omega_n\tau}, \qquad \bar{\psi}(\tau) = T\sum_n \bar{\psi}(\omega) e^{\mathbf{i}\omega_n\tau} \tag{3.13}$$

  the *Matsubara frequencies*

$$\omega_n = (2n+1)\pi T, \quad n \in \mathbb{Z}$$

  are half-integer multiples of πT. This has the important physical consequence that at finite temperature, there is no zero-mode of a fermion field – all the modes have a restoring force. This means that the quantum Fermi gas at any $T \neq 0$ is adiabatically connected to the classical gas at $T = \infty$, unlike the case of bosons.

[13] $\bar{\psi}$ is still not the complex conjugate of ψ but the relative sign is convenient.

- The measure $[D\bar{\psi}D\psi]$ is defined by this equation, just as in the bosonic path integral.
- The derivative of a grassmann function is also defined by this equation; note that $\psi_{l+1} - \psi_l$ is not 'small' in any sense.
- In the last step we integrated by parts, *i.e.* relabeled terms in the sum, so

$$\sum_l \left(\bar{\psi}_{l+1} - \bar{\psi}_l\right)\psi_l = \sum_l \bar{\psi}_{l+1}\psi_l - \sum_l \bar{\psi}_l\psi_l = \sum_{l'=l-1} \bar{\psi}_{l'}\psi_{l-1} - \sum_l \bar{\psi}_l\psi_l = -\sum_l \bar{\psi}_l\left(\psi_l - \psi_{l-1}\right).$$

  Note that no grassmanns were moved through each other in this process.

The punchline of this discussion for now is that the euclidean action is

$$\boxed{S[\bar{\psi},\psi] = \int d\tau \left(\bar{\psi}\partial_\tau\psi + H(\bar{\psi},\psi)\right)}\,.$$

The first-order kinetic term we've found $\bar{\psi}\partial_\tau\psi$ is sometimes called a 'Berry phase term'. Note the funny-looking sign.

Continuum limit warning (about the red $\simeq$ in (3.12)). The Berry phase term is *actually*

$$\sum_{l=0}^{N-1} \bar{\psi}_{l+1}\left(\psi_{l+1} - \psi_l\right) = T\sum_{\omega_n} \bar{\psi}(\omega_n)\left(1 - e^{\mathbf{i}\omega_n\tau}\right)\psi(\omega_n)$$

and in (3.12) we have kept only the leading nonzero term:

$$\left(1 - e^{\mathbf{i}\omega_n\tau}\right) \to \mathbf{i}\omega_n\tau.$$

Clearly this replacement is just fine if

$$\omega_n\tau \ll 1$$

for all ω_n which matter. Which ω_n contribute? I claim that if we use a reasonable $\mathbf{H} = \mathbf{H}_{\text{quadratic}} + \mathbf{H}_{\text{int}}$, reasonable quantities like Z, $\left\langle \mathcal{O}^\dagger\mathcal{O}\right\rangle$, are dominated by $\omega_n \ll \tau^{-1}$.

There's more we can learn from what we've done here that I don't want to pass up. Let's use this formalism to compute the fermion density at $T = 0$:

$$\langle \mathbf{N}\rangle = \frac{1}{Z}\mathrm{tr}e^{-\mathbf{H}/T}\mathbf{c}^\dagger\mathbf{c}.$$

This is an example where the annoying $\Delta\tau$s in the path integral not only matter, but are extremely friendly to us.

Frequency space, $T \to 0$.

Let's change variables to frequency-space fields, which diagonalize S. The Jacobian is 1 (since fourier transform is unitary):

$$D\bar{\psi}(\tau)D\psi(\tau) = \prod_n d\bar{\psi}(\omega_n)d\psi(\omega_n) \overset{T\to 0}{\to} D\bar{\psi}(\omega)D\psi(\omega).$$

The partition function is

$$Z = \int D\bar{\psi}(\omega)D\psi(\omega)\ \exp\left(T\sum_{\omega_n}\bar{\psi}(\omega_n)\left(\mathbf{i}\omega_n - \omega_0 + \mu\right)\psi(\omega_n)\right).$$

Notice that in the zero-temperature limit

$$T\sum_{\omega_n} \mapsto \int \frac{d\omega}{2\pi} \equiv \int \text{đ}\omega.$$

(This is the same fact as $V\sum_k \mapsto \int \text{đ}^d k$ in the thermodynamic limit.) So the zero-temperature partition function is

$$Z \overset{T\to 0}{=} \int D\bar{\psi}(\omega)D\psi(\omega)\ \exp\left(\int_{-\infty}^{\infty}\text{đ}\omega\bar{\psi}(\omega)\left(\mathbf{i}\omega - \omega_0 + \mu\right)\psi(\omega)\right).$$

Using the gaussian-integral formula (3.9) you can see that the *propagator* for ψ is

$$\left\langle\bar{\psi}(\omega_1)\psi(\omega_2)\right\rangle = \underbrace{\frac{\delta_{\omega_1,\omega_2}}{T}}_{\overset{T\to 0}{\to}\delta(\omega_1-\omega_2)}\ \frac{2\pi}{\mathbf{i}\omega_1 - \omega_0 + \mu}. \tag{3.14}$$

In particular $\left\langle\bar{\psi}(\omega)\psi(\omega)\right\rangle = \frac{2\pi/T}{\mathbf{i}\omega-\omega_0+\mu}$. $\delta(\omega = 0) = 1/T$ is the 'volume' of the time direction.

Back to the number density. Using the same strategy as above, we have

$$\langle\mathbf{N}\rangle = \frac{1}{Z}\int \prod_{l=0}^{M-1+1}\left(d\bar{\psi}_l d\psi_l e^{-\bar{\psi}_l\psi_l}\right)\prod_{l=1}^{M-1}\langle\bar{\psi}_{l+1}|(1-\Delta\tau\mathbf{H}(\mathbf{c}^\dagger\mathbf{c}))|\psi_l\rangle\ \underbrace{\left\langle\bar{\psi}_{N+1}\right|\mathbf{c}^\dagger\mathbf{c}\left|\psi_N\right\rangle}_{=\bar{\psi}_{N+1}\psi_N=\bar{\psi}(\tau_N+\Delta\tau)\psi(\tau_N)},$$

where τ_N is any of the time steps. This formula has a built-in point-splitting of the operators!

$$\langle\mathbf{N}\rangle = \frac{1}{Z}\int D\bar{\psi}D\psi\ e^{-S[\bar{\psi},\psi]}\bar{\psi}(\tau_N+\Delta\tau)\psi(\tau_N)$$

$$= \int_{-\infty}^{\infty} \text{đ}\omega \frac{e^{\mathbf{i}\omega\Delta\tau}}{\mathbf{i}\omega - \omega_0 + \mu} = \theta(\mu - \omega_0). \tag{3.15}$$

Which is the right answer: the mode is occupied in the groundstate only if $\omega_0 < \mu$. In the last step we used the fact that $\Delta\tau > 0$ to close the contour in the UHP; so we only pick up the pole if it is in the UHP. Notice that this quantity is very *UV sensitive*: if we put a frequency cutoff on the integral, $\int^{\Lambda} \frac{d\omega}{\omega} \sim \log\Lambda$, the integral diverges logarithmically. For most calculations the $\Delta\tau$ can be ignored, but here it told us the right way to treat the divergence. [14]

[14] The calculation between the first and second lines of (3.15) is familiar to us – it is a single Wick contraction, and can be described as a feynman diagram with one line between the two insertions. More prosaically, it is

$$\left\langle \bar{\psi}(\tau_N + \Delta\tau)\psi(\tau_N) \right\rangle \overset{(3.13)}{=} T^2 \sum_{nm} e^{\mathbf{i}(\omega_n - \omega_m)\tau + \mathbf{i}\omega_n\Delta\tau} \left\langle \bar{\psi}(\omega_n)\psi(\omega_m) \right\rangle \overset{(3.14)}{=} T \sum_m \frac{e^{\mathbf{i}\omega_n\Delta\tau}}{\mathbf{i}\omega_n - \omega_0 + \mu} \overset{T\to 0}{\to} \int \text{đ}\omega \frac{e^{\mathbf{i}\omega\Delta\tau}}{\mathbf{i}\omega - \omega_0 + \mu}.$$

3.3 Path integrals for spin systems

In this subsection we develop path integral descriptions of spin systems.

Quantum spin systems. To be clear, let me say a few introductory words about quantum spin systems, the flagship family of examples of well-regulated QFTs. These include useful models for magnetic insulators, and for possible hardware platforms for quantum computing.

Such a thing is a collection of two-state systems (aka qbits) $\mathcal{H}_j = \text{span}\{|\uparrow_j\rangle, |\downarrow_j\rangle\}$ distributed over space and coupled somehow:

$$\mathcal{H} = \bigotimes_j \mathcal{H}_j \ , \quad \dim(\mathcal{H}) = 2^N$$

where N is the number of sites.

One qbit: To begin, consider just one two-state system. There are four independent hermitian operators acting on this Hilbert space. Besides the identity, there are the three Paulis, which I will denote by $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ instead of $\boldsymbol{\sigma}^x, \boldsymbol{\sigma}^y, \boldsymbol{\sigma}^z$:

$$\mathbf{X} \equiv \boldsymbol{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{Y} \equiv \boldsymbol{\sigma}^y = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad \mathbf{Z} \equiv \boldsymbol{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This notation (which comes to us from the quantum information community) makes the important information larger and is therefore better, especially for those of us with limited eyesight.

They satisfy

$$\mathbf{XY} = \mathbf{iZ}, \quad \mathbf{XZ} = -\mathbf{ZX}, \quad \mathbf{X}^2 = \mathbb{1},$$

and all cyclic permutations $\mathbf{X} \to \mathbf{Y} \to \mathbf{Z} \to \mathbf{X}$ of these statements.

Multiple qbits: If we have more than one site, the paulis on different sites *commute*:

$$[\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_l] = 0, \quad j \neq l \quad i.e. \quad \mathbf{X}_j \mathbf{Z}_l = (-1)^{\delta_{jl}} \mathbf{Z}_l \mathbf{X}_j,$$

where $\boldsymbol{\sigma}_j$ is any of the three paulis acting on $\mathcal{H}_j$.

Quantum-classical correspondence. We've seen by now that *any* resolution of the identity on the local Hilbert space will give us some form of 'sum over histories'. In a spin system, perhaps the most obvious 'path integral' is the one associated with the $\mathbf{Z}$-basis resolution, $\mathbb{1} = |+\rangle\langle+| + |-\rangle\langle-|$. In this case, the *labels* on the states are classical spins ± 1 (or equivalently, classical bits). I put 'path integral' in quotes because it is instead a 'path sum', since the integration variables are discrete. Using this basis would allow us to further harness our knowledge of stat mech for QFT purposes. An important conclusion from that analysis is the (inverse) relationship between the correlation length and the energy gap above the groundstate.

To see the idea, consider the quantum system consisting of a single spin with $\mathbf{H} = E_0 - \frac{\Delta}{2}\mathbf{X} + \bar{h}\mathbf{Z}$. Set $\bar{h} = 0$ for a moment. Then Δ is the energy gap between the groundstate and the first excited state (hence the name). The thermal partition function is

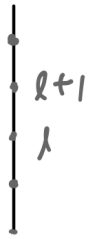
$$Z_Q(T) = \mathrm{tr} e^{-\mathbf{H}/T} = \sum_{s=\pm} \langle s| e^{-\beta \mathbf{H}} |s\rangle \,, \tag{3.16}$$

where we've evaluated the trace in the $\mathbf{Z}$ basis, $\mathbf{Z}|s\rangle = s|s\rangle$. I emphasize that T here is the temperature to which we are subjecting our quantum spin; $\beta = \frac{1}{T}$ is the length of the euclidean time circle. Break up the euclidean time circle into M_τ intervals of size $\Delta\tau = \beta/M_\tau$. Insert many resolutions of unity (this is called 'Trotter decomposition')

$$Z_Q = \sum_{s_1 \ldots s_{M_\tau}} \langle s_{M_\tau}| e^{-\Delta\tau \mathbf{H}} |s_{M_\tau - 1}\rangle \langle s_{M_\tau - 1}| e^{-\Delta\tau \mathbf{H}} |s_{M_\tau - 2}\rangle \cdots \langle s_1| e^{-\Delta\tau \mathbf{H}} |s_{M_\tau}\rangle \ .$$

The RHS is the partition function of a classical Ising chain,

$$Z_1 = \sum_{\{s_l = \pm 1\}} e^{-S}, \qquad S = -K \sum_{l=1}^{M_\tau} s_l s_{l+1} - h \sum_{l=1}^{M_\tau} s_l \tag{3.17}$$



These ss are now just M_τ numbers, each ± 1 – there are 2^{M_τ} terms in this sum. The parameter $K > 0$ is the 'inverse temperature' in the Boltzmann distribution; I put these words in quotes because I want you to think of it as merely a parameter in the classical hamiltonian.

In the Z_1 obtained from Trotterizing the single qubit, we have $h = 0$ and K determined by the relation

$$e^{-2K} = \tanh\left(\frac{\beta\Delta}{2M_\tau}\right) \ . \tag{3.18}$$

Notice that if our interest is in the quantum model with couplings E_0, Δ, we can use any M_τ we want – there are many classical models we could use[15]. For given M_τ, the couplings we should choose are related by (3.18).

A quantum system with just a single spin (for any $\mathbf{H}$ not proportional to $\mathbb{1}$) clearly has a unique groundstate; this statement means the absence of a phase transition in the 1d Ising chain.

Dictionary. More generally, this set of steps establishes a mapping between classical systems in $d+1$ dimensions and quantum systems in d space dimensions. Here's the dictionary:

[15] If we include the $\mathbf{Z}$ term, we'll want to take $\Delta\tau$ small enough so that we can write

$$e^{-\Delta\tau \mathbf{H}} = e^{\Delta\tau \frac{\Delta}{2}\mathbf{X}} e^{-\Delta\tau\left(E_0 - \bar{h}\mathbf{Z}\right)} + \mathcal{O}(\Delta\tau^2)$$

statistical mechanics in $d+1$ dimensions	quantum system in d space dimensions
transfer matrix	euclidean-time propagator, $e^{-\Delta\tau \mathbf{H}}$
statistical 'temperature'	(lattice-scale) coupling K
free energy in infinite volume	groundstate energy: $e^{-F} = Z = \mathrm{tr} e^{-\beta \mathbf{H}} \overset{\beta\to\infty}{\to} e^{-\beta E_0}$
periodicity of euclidean time L_τ	temperature: $\beta = \frac{1}{T} = \Delta\tau M_\tau$
statistical averages	groundstate expectation values of time-ordered operators

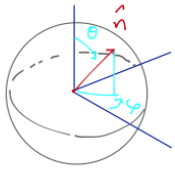
Note that this correspondence between classical and quantum systems is not an isomorphism. For one thing, we've seen that *many* classical systems are related to the same quantum system, which does not care about the lattice spacing in time. There is a set of physical quantities which agree between these different classical systems, called *universal*, which is the information in the quantum system.

There is a lot more to say about this relationship between QM in d dimensions and stat mech in $d+1$ dimensions. Should we say more in this class? Maybe we will.

[End of Lecture 8]

3.3.1 Geometric quantization and coherent state quantization of spin systems

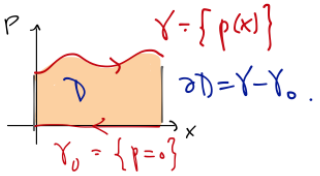
[Zinn-Justin, Appendix A3; XGW §2.3] We're going to spend some time talking about QFT in $D = 0{+}1$, then we'll work our way up to $D = 1{+}1$, and beyond. Consider the nice, round two-sphere. It has an area element which can be written



$$\omega = s\mathrm{d}\cos\theta \wedge \mathrm{d}\varphi \qquad \text{and satisfies} \qquad \int_{S^2} \omega = 4\pi s.$$

s is a number. Suppose we think of this sphere as the *phase space* of some dynamical system. We can use ω as the symplectic form. What is the associated quantum mechanics system?

Let me remind you what I mean by 'the symplectic form'. Recall the phase space formulation of classical dynamics. The action associated to a trajectory is



$$\mathcal{A}[x(t), p(t)] = \int_{t_1}^{t_2} \mathrm{d}t\,(p\dot{x} - H(x,p)) = \int_\gamma p(x)\mathrm{d}x - \int H\mathrm{d}t$$

where γ is the trajectory through the phase space. The first term is the area 'under the graph' in the classical phase space – the area between (p, x) and $(p = 0, x)$. We can rewrite it as

$$\int p(t)\dot{x}(t)\mathrm{d}t = \int_{\partial D} p\mathrm{d}x = \int_D \mathrm{d}p \wedge \mathrm{d}x$$

using Stokes' theorem; here ∂D is the closed curve made by the classical trajectory and some reference trajectory ($p = 0$) and it bounds some region D. Here $\omega = \mathrm{d}p \wedge \mathrm{d}x$ is the symplectic form. More generally, we can consider an $2n$-dimensional phase space with coordinates $u_\alpha, \alpha = 1..2n$ and symplectic form

$$\omega = \omega_{\alpha\beta}\mathrm{d}u^\alpha \wedge \mathrm{d}u^\beta$$

and action

$$\mathcal{A}[u] = \int_D \omega - \int_{\partial D} \mathrm{d}t H(u,t).$$

The symplectic form says who is canonically conjugate to whom. It's important that $\mathrm{d}\omega = 0$ so that the equations of motion resulting from $\mathcal{A}$ depend only on the trajectory $\gamma = \partial D$ and not on the choice of D. The equations of motion from varying u are

$$\omega_{\alpha\beta}\dot{u}^\beta = \frac{\partial H}{\partial u^\alpha}.$$

Locally, we can find coordinates p, x so that $\omega = \mathrm{d}(p\mathrm{d}x)$. Globally on the phase space this is not guaranteed – the symplectic form needs to be closed, but need not be exact.

So the example above of the two-sphere is one where the symplectic form is closed (there are no three-forms on the two sphere, so $\mathrm{d}\omega = 0$ automatically), but is not exact. One way to see that it isn't exact is that if we integrate it over the whole two-sphere, we get the area:

$$\int_{S^2} \omega = 4\pi s \ .$$

On the other hand, the integral of an exact form over a closed manifold (meaning a manifold without boundary, like our sphere) is zero:

$$\int_C \mathrm{d}\alpha = \int_{\partial C} \alpha = 0.$$

So there can't be a *globally defined* one-form α such that $\mathrm{d}\alpha = \omega$. Locally, we can find one; for example:

$$\alpha = s\cos\theta \mathrm{d}\varphi \ ,$$

but this is singular at the poles, where φ is not a good coordinate.

So: what I mean by "what is the associated quantum system..." is the following: let's construct a system whose path integral is

$$Z = \int [\mathrm{d}\theta \mathrm{d}\varphi] e^{\frac{\mathrm{i}}{\hbar}\mathcal{A}[\theta,\varphi]} \tag{3.19}$$

with the action above, and where $[\mathrm{d}x]$ denotes the path integral measure:

$$[\mathrm{d}x] \equiv \aleph \prod_{i=1}^{N} \mathrm{d}x(t_i)$$

where $\aleph$ involves lots of awful constants that drop out of ratios. It is important that the measure does not depend on our choice of coordinates on the sphere.

- Hint 1: the model has an action of $\mathsf{O}(3)$, by rotations of the sphere.
- Hint 2: We actually didn't specify the model yet, since we didn't choose the Hamiltonian. For definiteness, let's pick the hamiltonian to be

$$H = -s\vec{h}\cdot\vec{n}$$

where $\vec{n} \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. WLOG, we can take the polar axis to be along the 'magnetic field': $\vec{h} = \hat{z}h$. The equations of motion are then

$$0 = \frac{\delta\mathcal{A}}{\delta\theta(t)} = -s\sin\theta\,(\dot{\varphi} - h)\,, \quad 0 = \frac{\delta\mathcal{A}}{\delta\varphi(t)} = -\partial_t\,(s\cos\theta)$$

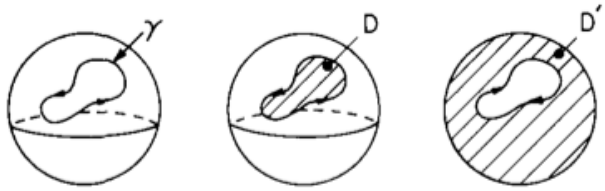
which by rotation invariance can be written better as

$$\partial_t\vec{n} = \vec{h}\times\vec{n}. \tag{3.20}$$

This is a big hint about the answer to the question.

- Hint 3: Semiclassical expectations. Semiclassically, each patch of phase space of area $2\pi\hbar$ contributes one quantum state. Therefore we expect that if our whole phase space has area $4\pi s$, we should get approximately $\frac{4\pi s}{2\pi\hbar} = \frac{2s}{\hbar}$ states, at least at large $s/\hbar$. (Notice that s appears out front of the action.) This will turn out to be very close – the right answer is $2s+1$ (when the spin is measured in units with $\hbar = 1$)!

In QM we care that the action produces a well-defined phase – the action must be defined modulo additions of 2π times an integer. We should get the same answer whether we fill in one side D of the trajectory γ or the other D'. The difference between them is



[from Witten]

$$s\left(\int_D - \int_{D'}\right)\text{area} = s\int_{S^2}\text{area}\ .$$

So in this difference s multiplies $\int_{S^2}\text{area} = 4\pi$ (actually, this can be multiplied by an integer which is the number of times the area is covered). Our path integral will be well-defined (*i.e.* independent of our arbitrary choice of 'inside' and 'outside') only if $4\pi s \in 2\pi\mathbb{Z}$, that is if $\boxed{2s \in \mathbb{Z} \text{ is an integer}}$.

The conclusion of this discussion is that the coefficient of the area term must be an integer. We will interpret this integer below.

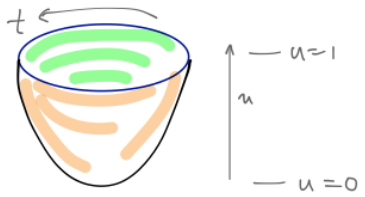
WZW term. We have a nice geometric interpretation of the 'area' term in our action $\mathcal{A}$ – it's the solid angle swept out by the particle's trajectory. But how do we write it in a manifestly SU(2) invariant way? We'd like to be able to write it, not in terms of the annoying coordinates θ, ϕ, but directly in terms of

$$n^a \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^a.$$

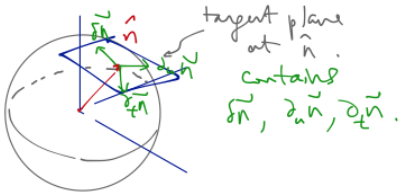
One way to do this is to add an extra dimension (!):

$$\frac{1}{4\pi}\int \mathrm{d}t\cos\theta\partial_t\phi = \frac{1}{8\pi}\int_0^1 \mathrm{d}u\int \mathrm{d}t\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c\epsilon^{abc} \equiv W_0[\vec{n}]$$

where $x^\mu = (t, u)$, and the ϵ tensors are completely antisymmetric in their indices with all nonzero entries 1 and -1.



In order to write this formula we have to extend the $\vec{n}$-field into the extra dimension whose coordinate is u. We do this in such a way that the real spin lives at $u = 1$: $\vec{n}(t, u = 1) = \vec{n}(t)$, and $\vec{n}(t, u = 0) = (0, 0, 1)$ – it goes to the north pole at the other end of the extra dimension for all t. If we consider periodic boundary conditions in time $n(\beta) = n(0)$, then this means that the space is really a disk with the origin at $u = 0$, and the boundary at $u = 1$. Call this disk B, its boundary ∂B is the real spacetime ('B' is for 'ball').



This WZW term has the property that its variation with respect to $\vec{n}$ depends only on the values at the boundary (that is: δW_0 is a total derivative). The crucial reason is that allowed variations $\delta\vec{n}$ lie on the 2-sphere, as do derivatives $\partial_\mu\vec{n}$; this means $\epsilon^{abc}\delta n^a \partial_\mu n^b \partial_\nu n^c = 0$, since they all lie in a two-dimensional tangent plane to the 2-sphere at $\vec{n}(t)$. Therefore:

$$\begin{aligned}
\delta W_0 &= \int_0^1 \mathrm{d}u \int \mathrm{d}t \frac{1}{4\pi}\epsilon^{\mu\nu} n^a \partial_\mu \delta n^b \partial_\nu n^c \epsilon^{abc} = \int_B \frac{1}{4\pi} n^a \mathrm{d}\delta n^b \wedge \mathrm{d}n^c \epsilon^{abc} \\
&= \int_0^1 \mathrm{d}u \int \mathrm{d}t\, \partial_\mu \left(\frac{1}{4\pi}\epsilon^{\mu\nu} n^a \delta n^b \partial_\nu n^c \epsilon^{abc} \right) = \int_B \mathrm{d}\left(\frac{1}{4\pi} n^a \delta n^b \mathrm{d}n^c \epsilon^{abc} \right) \\
&\overset{\text{Stokes}}{=} \frac{1}{4\pi} \int \mathrm{d}t \delta\vec{n} \cdot \left(\dot{\vec{n}} \times \vec{n} \right).
\end{aligned} \tag{3.21}$$

(Note that $\epsilon^{abc} n^a m^b \ell^c = \vec{n} \cdot \left(\vec{m} \times \vec{\ell} \right)$. The right expressions in red in each line are a rewriting in terms of differential forms; notice how much prettier they are.) So the equations of motion coming from this term do not depend on how we extend it into the auxiliary dimension.

And in fact they are the same as the ones we found earlier:

$$0 = \frac{\delta}{\delta\vec{n}(t)} \left(4\pi s W_0[n] + s\vec{h} \cdot \vec{n} + \lambda \left(\vec{n}^2 - 1 \right) \right) = s\partial_t \vec{n} \times \vec{n} + s\vec{h} + 2\lambda\vec{n}$$

(λ is a Lagrange multiplier to enforce unit length.) The cross product of this equation with $\vec{n}$ is $\partial_t \vec{n} = \vec{h} \times \vec{n}$.

In QM we also care that the action produces a well-defined phase – the action must be defined modulo additions of 2π times an integer. There may be many ways to

extend $\hat{n}$ into an extra dimension; another obvious way is shown in the figure above. The demand that the action is the same modulo $2\pi\mathbb{Z}$ gives the same quantization law as above for the coefficient of the WZW term. So the WZW term is topological in the sense that because of topology its coefficient must be quantized.

(This set of ideas generalizes to many other examples, with other fields in other dimensions. WZW stands for Wess-Zumino-Witten.)

Coherent quantization of spin systems. [Wen §2.3.1, Fradkin, Sachdev, *QPT*, chapter 13 and §2.2 of cond-mat/0109419] To understand more about the path integral we've just constructed, we now go in the opposite direction. Start with a spin one-half system, with

$$\mathcal{H}_{\frac{1}{2}} \equiv \mathrm{span}\{|\uparrow\rangle, |\downarrow\rangle\}.$$

Define spin coherent states $|\vec{n}\rangle$ by[16]:

$$\vec{\boldsymbol{\sigma}} \cdot \vec{n} \, |\vec{n}\rangle = |\vec{n}\rangle \ .$$

These states form another basis for $\mathcal{H}_{\frac{1}{2}}$; they are related to the basis where $\boldsymbol{\sigma}^z$ is diagonal by:

$$|\vec{n}\rangle = z_1 |\uparrow\rangle + z_2 |\downarrow\rangle, \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{\mathbf{i}\varphi/2} \cos\frac{\theta}{2} e^{\mathbf{i}\psi/2} \\ e^{-\mathbf{i}\varphi/2} \sin\frac{\theta}{2} e^{\mathbf{i}\psi/2} \end{pmatrix} \tag{3.22}$$

as you can see by diagonalizing $\vec{n} \cdot \vec{\boldsymbol{\sigma}}$ in the $\boldsymbol{\sigma}^z$ basis. Notice that

$$\vec{n} = z^\dagger \vec{\boldsymbol{\sigma}} z, \quad |z_1|^2 + |z_2|^2 = 1$$

and the phase of z_α does not affect $\vec{n}$ (this is the Hopf fibration $S^3 \to S^2$). In (3.22) I chose a representative of the phase. The space of independent states is a two-sphere:

$$S^2 = \{(z_1, z_2) || z_1|^2 + |z_2|^2 = 1\} / (z_\alpha \simeq e^{\mathbf{i}\chi} z_\alpha).$$

It is just the ordinary Bloch sphere of pure states of a qbit.

These states are not orthogonal (there are infinitely many of them and the Hilbert space is only 2-dimensional!):

$$\langle \check{n}_1 | \check{n}_2 \rangle = z_1^\dagger z_2$$

as you can see using the σ^z-basis representation (3.22). The (over-)completeness relation in this basis is:

$$\int \frac{\mathrm{d}^2 \vec{n}}{2\pi} |\vec{n}\rangle \langle \vec{n}| = \mathbb{1}_{2\times 2}. \tag{3.23}$$

[16] For more general spin representation with spin $s > \frac{1}{2}$, and spin operator $\vec{\mathbf{S}}$, we would generalize this equation to

$$\vec{\mathbf{S}} \cdot \vec{n} \, |\vec{n}\rangle = s \, |\vec{n}\rangle \, .$$

As always, we can construct a path integral representation of any amplitude by inserting many copies of $\mathbb{1}$ in between successive time steps. For example, we can construct such a representation for the propagator using (3.23) many times:

$$\begin{aligned}\mathbf{i}G(\vec{n}_f,\vec{n}_1,t) &\equiv \langle\vec{n}_f|\, e^{-\mathbf{i}\mathbf{H}t}\,|\vec{n}_1\rangle \\ &= \int \prod_{i=1}^{N\equiv\frac{t}{\mathrm{d}t}} \frac{\mathrm{d}^2\vec{n}(t_i)}{2\pi} \lim_{\mathrm{d}t\to 0} \langle\vec{n}(t)|\vec{n}(t_N)\rangle \ldots \langle\vec{n}(t_2)|\vec{n}(t_1)\rangle\, \langle\vec{n}(t_1)|\vec{n}(0)\rangle\,. \end{aligned} \quad (3.24)$$

(Notice that $\mathbf{H}=0$ here, so $\mathbf{U}\equiv e^{-\mathbf{i}\mathbf{H}t}$ is actually the identity.) The crucial ingredient is

$$\langle\vec{n}(t+\epsilon)|\vec{n}(t)\rangle = z^\dagger(\mathrm{d}t)z(0) = 1 - z^\dagger(\mathrm{d}t)\,(z(\mathrm{d}t)-z(0)) \approx e^{-z^\dagger\partial_t z\mathrm{d}t}.$$

$$\mathbf{i}G(\vec{n}_2,\vec{n}_1,t) = \int \left[\frac{D\vec{n}}{2\pi}\right] e^{\mathbf{i}S_B[\vec{n}(t)]}, \qquad S_B[\vec{n}(t)] = \int_0^t \mathrm{d}t\mathbf{i}z^\dagger\dot{z}\,. \quad (3.25)$$

Even though the Hamiltonian of the spins was zero – whatever their state, they have no potential energy and no kinetic energy – the action in the path integral is not zero. This phase $e^{\mathbf{i}S_B}$ is a quantum phenomenon (again) called a Berry phase.

Starting from the action S_B and doing the Legendre transform to find the Hamiltonian you will get zero. The first-derivative action says that $z^\dagger$ is the canonical momentum conjugate to z: the space with coordinates $(z,z^\dagger)$ becomes the phase space (just like position and momentum)! But this phase space is curved. In fact it is the two-sphere

$$S^2 = \{(z_1,z_2)||z_1|^2+|z_2|^2=1\}/(z_\alpha \simeq e^{\mathbf{i}\psi}z_\alpha).$$

In terms of the coordinates θ,φ above, we have

$$S_B[z] = S_B[\theta,\varphi] = \int dt\frac{1}{2}\left(\cos\theta\dot{\phi}+\dot{\psi}\right)|_{\psi=0} = 4\pi s W_0[\hat{n}]|_{s=\frac{1}{2}}. \quad (3.26)$$

At the last step we chose a gauge $\psi=0$. BIG CONCLUSION: This is the 'area' term that we studied above, with $s=\frac{1}{2}$! So the expression in terms of z in (3.25) gives another way to write the area term which is manifestly SU(2) invariant; this time the price is introducing these auxiliary z variables.

Making different choices of for the phase ψ at different times can shift the constant in front of the second term in (3.26); as we observed earlier, this term is a total derivative. Different choices of ψ change the overall phase of the wavefunction, which doesn't change physics (recall that this is why the space of normalized states of a qbit is a two-sphere and not a three-sphere). Notice that $\mathcal{A}_t = z^\dagger\partial_t z$ is like the time component of a gauge field. Adding a total derivative to the action (by changing $\psi(t)$) imparts a gauge transformation.

The Berry phase $S_B[n]$ is *geometric*, in the sense that it depends on the trajectory of the spin through time, but not on its parametrization, or speed or duration. It is called the Berry phase of the spin history because it is the phase acquired by a spin which follows the instantaneous groundstate (*i.e.* adiabatic evolution) $|\Psi_0(t)\rangle$ of $H(\check{n}(t),t) \equiv -h(t)\check{n}(t)\cdot\mathbf{S}$, with $h>0$. This is Berry's adiabatic phase, $S_B = -\lim_{\partial_t h\to 0}\int dt \mathrm{Im}\,\langle\Psi_0(t)|\,\partial_t\,|\Psi_0(t)\rangle$.

Since S_B is geometric, like integrals of differential forms, let's take advantage of this to make it pretty and relate it to familiar objects. Introduce a vector potential (the Berry connection) on the sphere $A^a, a=x,y,z$ so that

$$S_B = \oint d\tau \dot{n}_a A^a = \oint_\gamma A \overset{\text{Stokes}}{=} \int_D F$$

where $\gamma = \partial D$ is the trajectory. ($F = \mathrm{d}A$ is the Berry curvature.) What is the correct form? We must have $(\nabla\times A)\cdot\check{n} = \epsilon^{abc}\partial_{n^a}A^b n^c = 1$ (for spin half). This is a monopole field. Two choices which work are

$$A^{(1)} = -\cos\theta d\varphi, \quad \text{and} \quad A^{(2)} = (1-\cos\theta)d\varphi.$$

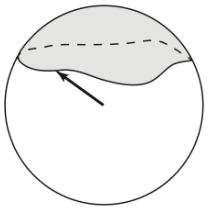
These two expressions differ by the gauge transformation $\mathrm{d}\varphi$, which is locally a total derivative. The first is singular at the N and S poles, $\check{n}=\pm\check{z}$. The second is singular only at the S pole. Considered as part of a 3d field configuration, this codimension two singularity is the 'Dirac string'. The demand of invisibility of the Dirac string quantizes the Berry flux. The gauge transformations which move around the singularities of A are accomplished by adding total derivatives to the action, *i.e.*by choosing $\psi(t)$. For example, by choosing $\psi(t)=\pm\varphi(t)$ we find the gauge which is nonsingular away from the north and south poles.

If we redo the above coherent-state quantization for a spin-s system we'll get the expression with general s (see below). Notice that this only makes sense when $2s\in\mathbb{Z}$.

We can add a nonzero Hamiltonian for our spin; for example, we can put it in an external Zeeman field $\vec{h}$, which adds $\mathbf{H} = -\vec{h}\cdot\vec{\mathbf{S}}$. This will pass innocently through the construction of the path integral, adding a term to the action $S = S_B + S_h$,

$$S_h = \int \mathrm{d}t \left(s\vec{h}\cdot\vec{n}\right)$$

where s is the spin.



We are back at the system (3.19). We see that the system we get by 'geometric quantization' of the sphere is a quantum spin. The quantized coefficient of the area is $2s$: it determines the dimension of the spin space to be $2s+1$. Here the

quantization of the WZW term is just quantization of angular momentum. (In higher-dimensional field theories, it is something else.)

Deep statement: the purpose in life of the WZW term is to enforce the commutation relation of the SU(2) generators, $[\mathbf{S}^i, \mathbf{S}^j] = \mathbf{i}\epsilon^{ijk}\mathbf{S}^k$. It says that the different components of the spin don't commute, and it says precisely what they don't commute to.

Incidentally, another way to realize this system whose action is proportional to the area of the sphere is to take a particle on the sphere, put a magnetic monopole in the center, and take the limit that the mass of the particle goes to zero. In that context, the quantization of $2s$ is Dirac quantization of magnetic charge. And the degeneracy of $2s+1$ states is the degeneracy of states in the lowest Landau level for a charged particle in a magnetic field; the $m \to 0$ limit gets rid of the higher Landau levels (which are separated from the lowest by the cylotron frequency, $\frac{eB}{mc}$).

In the crucial step, we assumed the path $z(t)$ was smooth enough in time that we could do calculus, $z(t+\epsilon) - z(t) = \epsilon\dot{z}(t) + \mathcal{O}(\epsilon^2)$. Is this true of the important contributions to the path integral? Sometimes not, and we'll come back to this later.

Digression on $s > \frac{1}{2}$. [Auerbach, *Interacting Electrons and Quantum Magnetism*] I want to say something about larger-spin representations of $\mathsf{SU}(2)$, partly to verify the claim above that it results in a factor of $2s$ in front of the Berry phase term. Also, large s allows us to approximate the integral by stationary phase.

In general, a useful way to think about the coherent state $|\check{n}\rangle$ is to start with the maximal-spin eigenstate $|s,s\rangle$ of $\mathbf{S}^z$ (the analog of spin up for general s), and *rotate it* by the rotation that takes $\mathbf{S}^z$ to $\mathbf{S}\cdot\check{n}$:

$$|\check{n}\rangle = \mathcal{R}(\chi,\theta,\varphi)\,|s,s\rangle\,.$$

The form of $\mathcal{R}$ involves Euler angles; let's find a better route than remembering about Euler angles.

Schwinger bosons. The following is a helpful device for spin matrix elements. Consider two copies of the harmonic oscillator algebra, with modes a, b satisfing $[a, a^\dagger] = 1 = [b, b^\dagger], [a, b] = [a, b^\dagger] = 0$. Then the objects

$$\mathbf{S}^+ = a^\dagger b,\ \mathbf{S}^- = b^\dagger a,\ \mathbf{S}^z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right)$$

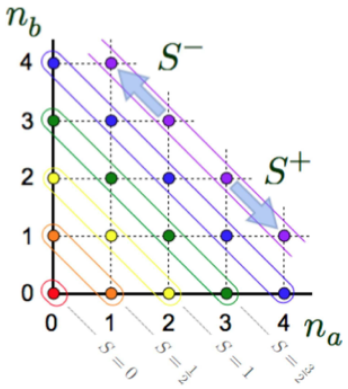
satisfy the $\mathsf{SU}(2)$ algebra. The no-boson state $|0\rangle$ is a singlet of this $\mathsf{SU}(2)$, and the one-boson states $\begin{pmatrix} a^\dagger|0\rangle \\ b^\dagger|0\rangle \end{pmatrix}$ form a spin-half doublet.

More generally, the states

$$\mathcal{H}_s \equiv \text{span}\{|n_a, n_b\rangle \, | a^\dagger a + b^\dagger b \equiv n_a + n_b = 2s\}$$

form a spin-s representation. Algebraic evidence for this is the fact that $\vec{S}^2 P_s = s(s+1)P_s$, where P_s is the projector onto $\mathcal{H}_s$. The spin-s eigenstates of $\mathbf{S}^z$ are

$$|s, m\rangle = \frac{(a^\dagger)^{s+m}}{\sqrt{(s+m)!}} \frac{(b^\dagger)^{s-m}}{\sqrt{(s-m)!}} |0\rangle \, .$$



[nice figure from Arovas and Auerbach, 0809.4836.]

The fact that $\begin{pmatrix} a^\dagger |0\rangle \\ b^\dagger |0\rangle \end{pmatrix}$ form a doublet means that $\begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}$ must be a doublet. But we know how a doublet transforms under a rotation, and this means we know how to write the coherent state:

$$|\check{n}\rangle = \mathcal{R}\,|s, s\rangle = \mathcal{R}\frac{(a^\dagger)^{2s}}{\sqrt{(2s)!}}|0\rangle = \mathcal{R}\frac{(a^\dagger)^{2s}}{\sqrt{(2s)!}}\mathcal{R}^{-1}\mathcal{R}\,|0\rangle = \frac{(a'^\dagger)^{2s}}{\sqrt{(2s)!}}|0\rangle = \frac{(z_1 a^\dagger + z_2 b^\dagger)^{2s}}{\sqrt{(2s)!}}|0\rangle \, .$$

Here $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{\mathbf{i}\varphi/2}\cos\frac{\theta}{2}e^{\mathbf{i}\psi/2} \\ e^{-\mathbf{i}\varphi/2}\sin\frac{\theta}{2}e^{\mathbf{i}\psi/2} \end{pmatrix}$ as above[17].

But now we can compute the crucial ingredient in the coherent state path integral, the overlap of successive coherent states:

$$\langle\check{n}|\check{n}'\rangle = \frac{e^{-\mathbf{i}s(\psi-\psi')}}{(2s)!}\underbrace{\langle 0|\,(z_1^\star a + z_2^\star b)^{2s}(z_1' a^\dagger + z_2' b^\dagger)^{2s}\,|0\rangle}_{\overset{\text{Wick}}{=}(2s)!\left([z_1^\star a + z_2^\star b, z_1' a^\dagger + z_2' b^\dagger]\right)^{2s}} = e^{-\mathbf{i}s(\psi-\psi')}(z_1^\star z_1' + z_2^\star z_2')^{2s} = \left(e^{-\mathbf{i}(\psi-\psi')/2} z^\dagger \cdot z'\right)^{2s} .$$

Here's the point: this is the same as the spin-half answer, raised to the $2s$ power. This means that the Berry phase just gets multiplied by $2s$, $S_B^{(s)}[n] = 2s S_B^{\left(\frac{1}{2}\right)}[n] = 4\pi s W_0[n]$, as we claimed. [End of Lecture 9]

Semi-classical spectrum. Above we found a path integral representation for the Green's function of a spin as a function of time, $G(n_t, n_0; t)$. The information this contains about the spectrum of the hamiltonian can be extracted by Fourier transforming

$$G(n_t, n_0; E) \equiv -\mathbf{i}\int_0^\infty dt G(n_t, n_0; t) e^{\mathbf{i}(E+\mathbf{i}\epsilon)t}$$

and taking the trace

$$\Gamma(E) \equiv \int \frac{d^2 n_0}{2\pi} G(n_0, n_0; E) = \text{Tr}\,\frac{1}{E - \mathbf{H} + \mathbf{i}\epsilon} .$$

[17] Sometimes (such as in lecture) you may see the notation $z_1 \equiv u, z_2 \equiv v$.

This function has poles at the eigenvalues of $\mathbf{H}$. Its imaginary part is the spectral density, $\rho(E) = \frac{1}{\pi}\mathrm{Im}\Gamma(E) = \sum_\alpha \delta(E - E_\alpha)$.

The path integral representation is

$$\Gamma(E) = -\mathbf{i}\int dt \oint D\check{n}\; e^{\mathbf{i}((E+\mathbf{i}\epsilon)t + sS[n])}.$$

The $\oint$ indicates periodic boundary conditions, $\check{n}(0) = \check{n}(t)$, and $S[n] = S_B[n] - \int^t dt' H_{\rm cl}[n]/s$. Here $H_{\rm cl}[n] \equiv \langle \check{n}|\,\mathbf{H}\,|\check{n}\rangle$.

At large s, field configurations which vary too much in time are cancelled out by the rapidly oscillating phase, that is: we can try to do these integrals by stationary phase. The stationarity condition for the n integral is the equations of motion $0 = \dot{n}\times n - \partial_n H_{\rm cl}$. If $\mathbf{H} = \vec{h}\cdot\mathbf{S}$, this gives the Landau-Lifshitz equation (3.20) for precession. We keep only solutions periodic with $t = nT$ an integer multiple of the period T. The stationarity condition for the t integral is

$$0 = E + \partial_t S[n] = E - H_{\rm cl}[n].$$

In the second equality we used the fact that the Berry phase is geometric, it depends only on the trajectory, not on t (how long it takes to get there). So the semiclassical trajectories are periodic solutions to the EOM with energy $E = H_{\rm cl}[n^E]$. The exponent evaluated on such a trajectory is then just the Berry term. Denoting by n_1^E such trajectories which traverse once ('prime' orbits),

$$\Gamma(E) \sim \sum_{n_1^E}\sum_{m=0}^{\infty} e^{\mathbf{i}ms S_B[n]} = \sum_{n_1^E} \frac{e^{\mathbf{i}sS_B[n]}}{1 - e^{\mathbf{i}sS_B[n]}}.$$

This is an instance of the Gutzwiller trace formula. The locations of poles of this function approximate the eigenvalues of $\mathbf{H}$. They occur at $E = E^m_{sc}$ such that $S_B[\vec{n}^{E_m}] = \frac{2\pi m}{s}$. The actual eigenvalues are $E^m = E^m_{sc} + \mathcal{O}(1/s)$.

If the path integral in question were a 1d particle in a potential, with $S_B = \int p dx$, and $H_{\rm cl} = p^2 + V(x)$, the semiclassical condition would reduce to

$$2\pi m = \oint_{x^{E_m}} p(x)dx = \int_{\text{turning points}} \sqrt{E_m - V(x)}$$

the Bohr-Sommerfeld condition.

3.4 Topological terms from integrating out fermions

[Abanov ch 7] Here is a quick application of fermionic path integrals. Consider a 0+1 dimensional model of *spinful* fermions $\mathbf{c}_\alpha, \alpha = \uparrow, \downarrow$ coupled to a single spin s, $\vec{\mathbf{S}}$. Let's couple them in an $\mathsf{SU}(2)$-invariant way:

$$H_K = M\left(\mathbf{c}^\dagger \vec{\sigma} \mathbf{c}\right) \cdot \vec{\mathbf{S}}$$

by coupling the spin of the fermion $\mathbf{c}^\dagger_\alpha \vec{\sigma}_{\alpha\beta} \mathbf{c}_\beta$ to the spin. 'K' is for 'Kondo'. Notice that M is an energy scale. (Ex: find the spectrum of H_K.)

Now apply both of the previous coherent state path integrals that we've learned to write the (say euclidean) partition sum as

$$Z = \int [D\psi D\bar{\psi} D\vec{n}] e^{-S_0[n] - \int_0^T \mathrm{d}t \bar{\psi}(\partial_t - M\vec{n}\cdot\vec{\sigma})\psi}$$

where $\psi = (\psi_\uparrow, \psi_\downarrow)$ is a two-component Grassmann spinor, and $\vec{\sigma}$ are Pauli matrices acting on its spinor indices. $\vec{n}^2 = 1$. Let $S_0[n] = \int K\dot{n}^2 + (2s)2\pi W_0[n]$, where I've added a second-order kinetic term for fun.

First of all, consider a fixed, say static, configuration of $\check{n}$. What does this do to the propagation of the fermion? I claim that it gaps out the fermion excitations, in the sense that

$$\left\langle \mathbf{c}^\dagger_\alpha(t) \mathbf{c}_\beta(0) \right\rangle \equiv \left\langle \bar{\psi}_\alpha(t) \psi_\beta(0) \right\rangle$$

will be short-ranged in time. Let's see this using the path integral.

We can do the (gaussian) integral over the fermion:

$$Z = \int [D\vec{n}] e^{-S_{\text{eff}}[\vec{n}]}$$

with

$$S_{\text{eff}}[\vec{n}] = S_0[\vec{n}] - \log\det\left(\partial_t - M\vec{n}\cdot\vec{\sigma}\right) \equiv S_0 - \log\det D \equiv S_0 + S_1.$$

The variation of the new term in the effective action under a variation of $\vec{n}$ is:

$$\delta S_1 = -\mathrm{tr}\left(\delta D D^{-1}\right) = -\mathrm{tr}\left(\delta D D^\dagger \left(D D^\dagger\right)^{-1}\right)$$

where $D^\dagger \equiv -\partial_t - M\vec{n}\cdot\vec{\sigma}$. This is

$$\delta S_{\text{eff}} = M\mathrm{tr}\left(\delta\vec{n}\cdot\vec{\sigma}\left(\partial_t + M\vec{n}\cdot\vec{\sigma}\right)\left(\underbrace{-\partial_t^2 + M^2 - M\dot{\vec{n}}\cdot\vec{\sigma}}_{=DD^\dagger}\right)^{-1}\right). \tag{3.27}$$

We can expand the denominator in $\dot{\vec{n}}/M$ (and use $n^2 = 1$) to get

$$\delta S_1 = \int \mathrm{d}t \left(-\frac{M}{|M|}\frac{1}{2}\delta\vec{n} \cdot \left(\vec{n} \times \dot{\vec{n}}\right) + \frac{1}{4M}\delta\dot{\vec{n}}\dot{\vec{n}} + ... \right)$$

where ... is higher order in the expansion and we ignore it. But we know this is the variation of

$$S_1 = -2\pi \frac{M}{|M|} W_0 + \int_0^T \mathrm{d}t \left(\frac{1}{8M}\dot{\vec{n}}^2 \right) + \mathcal{O}\left(\frac{\dot{n}}{M}\right)^2$$

where W_0 is the WZW term. Integrating out the fermions has shifted the coefficient of the WZW term from $s \to s \mp \frac{1}{2}$ depending on the sign of M. This is satisfying: we are adding angular momenta, $s \otimes \frac{1}{2} = \left(s - \frac{1}{2}\right) \oplus \left(s + \frac{1}{2}\right)$. If $M > 0$, it is an antiferromagnetic interaction whose groundstates will be the ones with smaller eigenvalue of $\vec{S}^2$. If $M < 0$, it is ferromagnetic, and the low-energy manifold grows. This agrees precisely with the coefficient of the WZW term in our effective action, which is $4\pi\left(s - \frac{1}{2}\mathrm{sign}(M)\right)$.

Here is a more direct (?) calculation of the fermion determinant S_1 (also from Abanov).

$$S_1 = -\ln\det D = -\mathrm{Tr}\ \ln D \stackrel{?}{=} -\mathrm{Tr}\ \ln \tilde{D}$$

where $\tilde{D} \equiv U^\dagger D U = \partial_t - \mathbf{i}a - M\sigma^3$ where we've defined the unitary transformation U so that

$$\sigma^3 \stackrel{!}{=} U^\dagger \vec{n} \cdot \vec{\sigma} U, \qquad \text{and} \qquad a \equiv U^\dagger \mathbf{i} \partial_t U.$$

In terms of the free propagator $G_0^{-1} \equiv \partial_t - M\sigma^3$, we can write

$$\tilde{D} = G_0^{-1}(1 - G_0 \mathbf{i} a).$$

Then we can expand in powers of a

$$S_1 = -\mathrm{Tr}\ \ln \tilde{D} = \mathrm{Tr}\ \left(\ln G_0 + G_0 \mathbf{i} a + \frac{1}{2}\left(G_0 \mathbf{i} a\right)^2 + \cdots \right) \equiv S_{(0)} + S_{(1)} + \cdots .$$

The first term is some constant which we ignore. The term linear in a is

$$S_{(1)} = \mathrm{tr} G_0 \mathbf{i} a = \mathrm{tr}_\sigma \underbrace{\int \text{đ}\omega \frac{e^{\mathbf{i}\omega dt}}{-\mathbf{i}\omega - M\sigma^3}}_{=\theta(M\sigma^3)} \mathbf{i} a_{\omega=0} = -\mathrm{sign}(M)\mathbf{i} \int dt a^3(t).$$

Here $a^3 \equiv \mathrm{tr}_\sigma a\sigma^3 = \frac{1}{2}\cos\theta\dot{\varphi}$, from which we conclude

$$S_{(1)} = -\mathbf{i} 2\pi \mathrm{sign}(M) W_0[n].$$

Similarly, the next term is

$$S_{(2)} = \frac{1}{2}\mathrm{tr}(G_0 \mathbf{i}a)^2 = \frac{1}{2}\int đ\omega_1 \int đ\omega_2 \mathrm{tr}_\sigma \left(\frac{1}{-\mathbf{i}\omega_1 - M\sigma^3} \mathbf{i}a_{-\omega_2} \frac{1}{-\mathbf{i}(\omega_1+\omega_2) - M\sigma^3} \mathbf{i}a_{\omega_2} \right) \tag{3.28}$$

$$= \frac{1}{8M}\int đ\omega \mathrm{tr}_\sigma \left(a_{-\omega}a_\omega - \sigma^3 a_{-\omega}\sigma^3 a_\omega \right) \tag{3.29}$$

$$= \frac{1}{2M}\int dt \left(a_1^2 + a_2^2\right) = \frac{1}{8M}\int dt \left(\partial_t \vec{n}\right)^2 . \tag{3.30}$$

The second term in S_1 is a shift of K. Higher-order terms are suppressed by more powers of $\frac{\dot{n}}{M}$, so for $\dot{n} \ll M$, this is a local action. That means that the coupling to n must have gapped out the fermions. That the term proportional to M is a funny mass term for the fermions is clear from the expression for $DD^\dagger$ in (3.27): when n is static, $DD^\dagger = -\partial_t^2 + M^2$, so that the fermion propagator is

$$\left\langle \bar{\psi}_\alpha(t)\psi_\beta(0)\right\rangle = \left(\frac{1}{D}\right)_t = \left(\frac{D}{DD^\dagger}\right)_t = \int đ\omega \frac{e^{\mathbf{i}\omega t}\left(\omega + \mathbf{i}M\vec{n}\cdot\sigma_{\alpha\beta}\right)}{\omega^2 + M^2} \sim e^{-Mt}$$

which is short-ranged in time. So indeed the fermions are fast modes in the presence of the coupling to the n-field.

Such topological terms are one way in which some (topological) information from short distances can persist in the low energy effective action. Being quantized, they can't change under the continuous RG evolution. Here the WZW term manages to be independent of M, the mass scale of the fermions. Here the information is that the system is made of fermions (or at least a half-integer spin representation of $\mathsf{SU}(2)$).

The above calculation generalizes well to higher dimensions. The general idea is that integrating out fermions with Yukawa terms involving bosons ϕ produces WZW terms for ϕ. For many examples of its application, see this paper. (The context for this paper will become clearer in §7.3).

3.5 Pions

[Schwartz §28.1] Below the scale of electroweak symmetry breaking, we can forget the W and Z bosons. Besides the 4-Fermi interactions, the remaining drama is QCD and electromagnetism:

$$\mathcal{L}_{QCD_2} = -\frac{1}{4}F_{\mu\nu}^2 + \mathbf{i}\sum_{\alpha=L,R}\sum_f \bar{q}_{\alpha f}\not{D}q_{\alpha f} - \bar{q}Mq.$$

Here f is a sum over quark flavors, which includes the electroweak doublets, u and d. Let's focus on just these two lightest flavors, u and d. We can diagonalize the mass matrix by a field redefinition (this is what makes the CKM matrix meaningful): $M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$. If it were the case that $m_u = m_d$, we would have *isospin* symmetry

$$\begin{pmatrix} u \\ d \end{pmatrix} \to U \begin{pmatrix} u \\ d \end{pmatrix}, \quad U \in \mathsf{SU}(N_f = 2).$$

If, further, there were no masses $m = 0$, then L and R decouple and we also have chiral symmetry, $q \to e^{\mathbf{i}\gamma_5\alpha}q$, *i.e.*

$$q_L \to Vq_L, q_R \to V^{-1}q_R, \; V \in \mathsf{SU}(N_f = 2).$$

Why do I restrict to $\mathsf{SU}(2)$ and not $\mathsf{U}(2)$? The central bit of the axial symmetry $\mathsf{U}(1)_A$ is anomalous – its divergence is proportional to the *gluon* theta term operator $F \wedge F$, which has all kinds of nonzero matrix elements. It's not a symmetry (see Peskin page 673 for more detail). The missing non-Goldstone boson is called the η'. The central bit of the vectorlike transformation $q \to e^{\mathbf{i}\alpha}q$ is *baryon number*, B. (Actually this is anomalous under the full electroweak symmetry, but $B - L$ is not).

The groundstate of QCD is mysterious, because of infrared slavery. Here's one piece of input from experiment and numerical simulation. Apparently it is the case that in the groundstate

$$\langle \bar{q}_f q_f \rangle = V^3 \tag{3.31}$$

independent of flavor f. This condensate spontaneously breaks

$$\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R \to \mathsf{SU}(2)_{\text{isospin}}, \tag{3.32}$$

the diagonal combination. $\begin{pmatrix} u \\ d \end{pmatrix}$ is a doublet. Since $p = u_\alpha u_\beta d_\gamma \epsilon_{\alpha\beta\gamma}, n = u_\alpha d_\beta d_\gamma \epsilon_{\alpha\beta\gamma}$, this means that $\begin{pmatrix} p \\ n \end{pmatrix}$ is also a doublet. This symmetry is (explicitly) weakly broken by

the difference of the masses $m_d = 4.7\text{MeV} \neq m_u = 2.15\text{MeV}$ and by the electromagnetic interactions, since $q_d = -1/3 \neq q_u = 2/3$.

This symmetry-breaking structure enormously constrains the dynamics of the color singlets which are the low-energy excitations above the QCD vacuum (hadrons). Let us use the EFT strategy. We know that the degrees of freedom must include (pseudo-)Goldstone bosons for the symmetry breaking (3.32) ('pseudo' because of the weak explicit breaking). [End of Lecture 10]

Effective field theory. Since QCD is strongly coupled in this regime, let's use the knowing-the-answer trick: the low energy theory must include some fields which represent the breaking of the symmetry (3.32). One way to do this is to introduce a field Σ which transforms like

$$\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R : \Sigma \to g_L \Sigma g_R^\dagger, \quad \Sigma^\dagger \to g_R \Sigma^\dagger g_L^\dagger$$

(this will be called a *linear* sigma model, because Σ transforms linearly) – we have in mind $\bar{q}_\alpha q_\beta \sim \Sigma_{\alpha\beta}$, like the Hubbard-Stratonovich variable. We can make singlets (hence an action) out of $\Sigma_{\alpha\beta}\Sigma^\dagger_{\beta\alpha} = \text{tr}\Sigma\Sigma^\dagger \equiv |\Sigma|^2$:

$$\mathcal{L} = |\partial_\mu \Sigma|^2 + m^2 \text{tr}\Sigma\Sigma^\dagger - \frac{\lambda}{4}\left(\text{tr}\Sigma\Sigma^\dagger\right)^2 - g\text{tr}\Sigma\Sigma^\dagger\Sigma\Sigma^\dagger + \cdots \tag{3.33}$$

which is designed to have a minimum at $\langle\Sigma\rangle = \frac{V}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, with (when $g \to 0$) $V = 2m/\sqrt{\lambda}$ (here V is from (3.31)), which preserves $\mathsf{SU}(2)_{\text{isospin}}$ (under which $\Sigma \to g\Sigma g^\dagger$). We can parametrize the fluctuations about this configuration as

$$\Sigma(x) = \frac{V + \sigma(x)}{\sqrt{2}} e^{\frac{2\mathbf{i}\pi^a(x)\tau^a}{F_\pi}}$$

where $F_\pi = V = \frac{2m}{\sqrt{\lambda}}$ is be chosen to give $\pi^a(x)$ canonical kinetic terms. The π^a parametrize the directions of field space in which the potential is flat (like the field θ that goes around the minimum of a wine-bottle potential). Under $g_{L/R} = e^{\mathbf{i}\theta_{L/R}\tau^a}$, the pion field transforms as

$$\pi^a \to \pi^a + \underbrace{\frac{F_\pi}{2}\left(\theta_L^a - \theta_R^a\right)}_{\text{nonlinear realization of } \mathsf{SU}(2)_{\text{axial}}} - \underbrace{\frac{1}{2} f^{abc}\left(\theta_L^a + \theta_R^a\right)\pi^c}_{\text{linear realiz'n (adj rep) of } \mathsf{SU}(2)_{\text{isospin}}} .$$

The fields $\pi^\pm, \pi^0$ create pions, they transform in the adjoint representation of the diagonal $\mathsf{SU}(2)_{\text{isospin}}$, and they *shift* under the broken symmetry. This shift symmetry forbids mass terms π^2. The radial excitation σ, on the other hand, is a fiction which we've introduced in (3.33), and which has no excuse to stick around at low energies

(and does not). We can put it out of its misery by taking $m \to \infty, \lambda \to \infty$ fixing F_π. In the limit, the useful field to use is

$$U(x) \equiv \frac{\sqrt{2}}{V}\Sigma(x)|_{\sigma=0} = e^{\frac{2\mathrm{i}\pi^a \tau^a}{F_\pi}}$$

which is unitary $UU^\dagger = U^\dagger U = \mathbb{1}$. This last identity means that all terms in an action for U require derivatives, so (again) no mass for π. The most general Lagrangian for U can be written as an expansion in derivatives, and is called the *chiral Lagrangian*:

$$\mathcal{L}_\chi = \frac{F_\pi^2}{4}\mathrm{tr}D_\mu U D^\mu U^\dagger + L_1 \mathrm{tr}\left(D_\mu U D^\mu U^\dagger\right)^2 + L_2 \mathrm{tr}D_\mu U D_\nu U^\dagger \mathrm{tr}D^\nu U^\dagger D_\mu U + L_3 \mathrm{tr}D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger + \cdots \tag{3.34}$$

In terms of π, the leading term expands into

$$L_\chi = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{F_\pi^2}\left(-\frac{1}{3}\pi^0\pi^0 D_\mu \pi^+ D^\mu \pi^- + \cdots\right) + \frac{1}{F_\pi^4}\left(\frac{1}{18}\left(\pi^-\pi^+\right)^2 D_\mu \pi^0 D^\mu \pi^0 + \cdots\right)$$

This fixes the relative coefficients of many irrelevant interactions, all with two derivatives, suppressed by powers of F_π. The expansion of the L_i terms have four derivatives, and are therefore suppressed by further powers of E/F_π, the promised small parameter of this EFT.

Pion masses. The pions aren't actually massless: $m_{\pi^\pm} \sim 140$MeV. In terms of quarks, one source for such a thing is the quark mass term $\mathcal{L}_{QCD} \ni \bar{q}Mq$. This explicitly breaks the isospin symmetry if the eigenvalues of M aren't equal. But an *invariance* of $\mathcal{L}_{QCD}$ is

$$q_{L/R} \to g_{L/R} q_{L/R}, \;\; M \to g_L M g_R^\dagger. \tag{3.35}$$

Think of M as a background field (such a thing is sometimes called a *spurion*). If M were an actual dynamical field, then (3.35) would be a symmetry. In the effective action which summarizes all the drama of strong-coupling QCD in terms of pions, the field M must still be there, and if we transform it as in (3.35), it should still be an invariance. Maybe we're going to do the path integral over M later. (This 'spurion' trick has applications all over physics.)

So the chiral lagrangian $\mathcal{L}_\chi$ should depend on M and (3.35) should be an invariance. This determines

$$\Delta\mathcal{L}_\chi = \frac{V^3}{2}\mathrm{tr}\left(MU + M^\dagger U^\dagger\right) + \cdots = V^3(m_u + m_d) - \frac{V^3}{2F_\pi^2}(m_u + m_d)\sum_a \pi_a^2 + \mathcal{O}(\pi^2).$$

The coefficient V^3 is chosen so that the first term matches $\langle \bar{q}Mq\rangle = V^3(m_u + m_d)$. The second term then gives

$$m_\pi^2 \simeq \frac{V^3}{F_\pi^2}(m_u + m_d)$$

which is called the Gell-Mann Oakes Renner relation.

Electroweak interactions. You may have noticed that I used covariant-looking Ds in (3.34). That's because the $\mathsf{SU}(2)_L$ symmetry we've been speaking about is actually gauged by W^a_μ. (The electroweak gauge boson kinetic terms are in the $\cdots$ of (3.34).) Recall that

$$\mathcal{L}_{\text{Weak}} \ni gW^a_\mu \left(\underbrace{J^a_\mu - J^{5a}_\mu}_{\text{'V' - 'A'}} \right) = gW^a_\mu \left(V_{ij}\bar{Q}_i\gamma^\mu \frac{1-\gamma^5}{2}\tau^a Q_j + \bar{L}_i\gamma^\mu\tau^a\frac{1-\gamma^5}{2}L_i \right)$$

where $Q_1 = \begin{pmatrix} u \\ d \end{pmatrix}, L_1 = \begin{pmatrix} e \\ \nu_e \end{pmatrix}$ are doublets of $\mathsf{SU}(2)_L$.

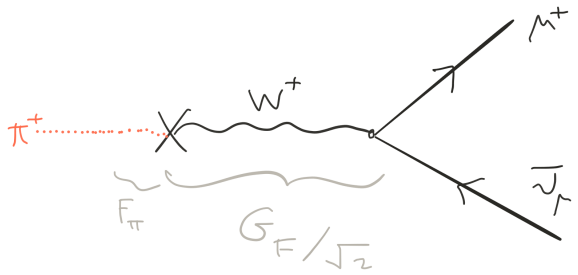
Now, in equations, the statement "a pion is a Goldstone boson for the axial $\mathsf{SU}(2)$" is:

$$\langle 0| J^{5a}_\mu(x) \left|\pi^b(p)\right\rangle = \mathbf{i}p_\mu F_\pi e^{-\mathbf{i}p\cdot x}\delta^{ab}$$

where the state $\left|\pi^b(p)\right\rangle$ is a one-pion state of momentum p. If the vacuum were invariant under the symmetry transformation generated by J_μ, the BHS would vanish. The momentum dependence implements the fact that a global rotation ($p_\mu = 0$) does not change the energy. Contracting the BHS with p^μ and using current conservation (ignoring the explicit breaking just mentioned) would give $0 = p^2F_\pi^2 = m_\pi^2 F_\pi^2$, a massless dispersion for the pions.

Combining the previous two paragraphs, we see that the following process can happen

$$\pi \overset{\text{Goldstone}}{\to} J^5_\mu \overset{\text{electroweak interaction}}{\to} \text{leptons}$$



(3.36)

and in fact is responsible for the dominant decay channel of charged pions. (Time goes from left to right in these diagrams, sorry.)

$$\mathcal{M}(\pi^+ \to \mu^+\nu_\mu) = \frac{G_F}{\sqrt{2}}F_\pi p^\mu \bar{v}_{\nu_\mu}\gamma^\mu(1-\gamma^5)u_\mu$$

where the Fermi constant $G_F \sim 10^{-5}GeV^{-2}$ (known from *e.g.* $\mu^- \to e^-\bar{\nu}_e\nu_\mu$) is a good way to parametrize the Weak interaction amplitude. Squaring this and integrating

over two-body phase space gives the decay rate

$$\Gamma(\pi^+ \to \mu^+ \nu_\mu) = \frac{G_F^2 F_\pi^2}{4\pi} m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 .$$

(You can see from the answer why the decay to muons is more important than the decay to electrons, since $m_\mu/m_e \sim 200$. This is called *helicity suppression* – the decay of the helicity-zero π^+ into back-to-back spin-half particles by the weak interaction (which only produces L particles and R antiparticles) can't happen if helicity is conserved – the mass term is required to flip the e_L into an e_R.) This contributes most of $\tau_{\pi^+} = \Gamma^{-1} = 2.6 \cdot 10^{-8} s$.

Knowing further the mass of the muon $m_\mu = 106\text{MeV}$ then determines $F_\pi = 92\text{MeV}$ which fixes the leading terms in the chiral Lagrangian. This is why F_π is called the *pion decay constant*. This gives a huge set of predictions for *e.g.* pion scattering $\pi^0\pi^0 \to \pi^+\pi^-$ cross sections.

Note that the neutral pion can decay by an anomaly into two photons:

$$q_\mu \langle p_1, p_2| J_\mu^{5,a=3}(q) |0\rangle = -c \frac{e^2}{4\pi^2} \epsilon^{\nu\lambda\alpha\beta} p_1^\nu \epsilon_1^\lambda p_2^\alpha \epsilon_2^\beta$$

where $\langle p_1, p_2|$ is a state with two photons of polarizations $\epsilon_{1,2}$. I know this because it is a matrix element of the $J_e J_e J_{\mathsf{SU}(2)-\text{axial}}$ anomaly,

$$\partial_\mu J^{\mu 5a} = -\frac{e^2}{16\pi^2} \epsilon^{\nu\lambda\alpha\beta} F_{\nu\lambda} F_{\alpha\beta} \text{tr}\left(\tau^a Q^2\right)$$

where $Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix}$ is the quark charge matrix. Comments: (1) this symmetry acts by $u \to e^{\mathbf{i}\theta\gamma^5} u, d \to e^{-\mathbf{i}\theta\gamma^5}$, and is *not* the same as the anomalous $\mathsf{U}(1)_A$ (which does $q_i \to e^{\mathbf{i}\theta\gamma^5} q_i$ for every flavor), and it's also not the same as isospin $u \to e^{\mathbf{i}\theta} u, d \to e^{-\mathbf{i}\theta}$, which is not chiral, and not spontaneously broken. Confusing! (2) The rate of π^0 decay (known since the 1940s) gives a measurement of the number of colors of QCD! (3) This effect can be encoded in the Lagrangian for the pions by a term

$$L \ni N_c \frac{e^2}{16\pi^2} \pi^0 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma},$$

where $N_c = 3$ is the number of colors. The effective field theory consistently realizes the anomalies of the microscopic theory. This is an example of *'t Hooft anomaly matching*, a principle which can be used, for example, to prove that QCD *must* spontaneously break the $\mathsf{SU}(3)_L \times \mathsf{SU}(3)_R$ chiral symmetry (see Schwartz §30.6).

Wait – what $\mathsf{SU}(3)$?

SU(3) and baryons. The strange quark mass is also pretty small $m_s \sim 95\text{MeV}$, and $\langle \bar{s}s \rangle \sim V^3$. This means the approximate invariance and symmetry breaking pattern is actually $\mathsf{SU}(3)_L \times \mathsf{SU}(3)_R \to \mathsf{SU}(3)_{\text{diag}}$, meaning that there are $16 - 8 = 8$ pseudo NGBs. Besides $\pi^{\pm,0}$, the others are the kaons $K^{\pm,0}$ and η. It's still only the $\mathsf{SU}(2)_L$ that's gauged.

We can also include baryons $B = \epsilon_{\alpha\beta\gamma} q_\alpha q_\beta q_\gamma$. Since $q = (u, d, s) \in 3$ of the flavor $\mathsf{SU}(3)$, the baryons are in the representation

$$\begin{aligned} 3 \otimes 3 \otimes 3 &= (6 \oplus \bar{3}) \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1 \\ \square \otimes \square \otimes \square &= (\square\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}) \otimes \square = \square\square\square \oplus \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \end{aligned} \tag{3.37}$$

The proton and neutron are in one of the octets. This point of view brought some order (and some predictions) to the otherwise-bewildering zoo of hadrons.

Returning to the two-flavor $\mathsf{SU}(2)$ approximation, we can include the nucleons $N_{L/R} = \begin{pmatrix} p \\ n \end{pmatrix}_{L/R}$ and couple them to pions by the symmetric coupling

$$\mathcal{L} \ni \lambda_{NN\pi} \bar{N}_L \Sigma N_R.$$

The expectation value for Σ gives a nucleon mass: $m_N = \lambda_{NN\pi} F_\pi$, where $\lambda_{NN\pi}$ can be measured by scattering. This is a cheap version of the *Goldberger-Treiman relation*; for a better one see Peskin pp. 670-672.

WZW terms in the chiral Lagrangian. Finally, I would be remiss not to mention that the chiral Lagrangian must be supplemented by WZW terms to have the correct realization of symmetries (in order to encode all the effects of anomalies, and in order to violate $\pi \to -\pi$ which is not a symmetry of QCD). This is an important additional ingredient in the EFT recipe book: although we wrote all the local Lagrangian terms which were manifestly consistent with the symmetries, this actually did *not* account for all the symmetric terms that we can add to the action!

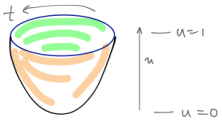
The chiral Lagrangian governs a *non-linear sigma model* (NLσM)– a QFT whose fields are maps from spacetime into some *target space*. In this case the target space is the coset space G/H, where G is the full symmetry group ($\mathsf{SU}(N_f)_L \times \mathsf{SU}(N_f)_R$) and H is the *unbroken* subgroup $\mathsf{SU}(N_f)_{\text{diagonal}}$. We can parametrize this space by $U = e^{\mathbf{i}\pi^a T^a \frac{2}{F_\pi}}$ where the T^a includes only generators of the broken part of the group, so the π^a are coordinates on G/H.

A WZW term is a term which we can sometimes add to a NLσM action; it is defined by the fact that it is symmetric under some group G, but *isn't* the integral of a symmetric local Lagrangian density in D dimensions. Making it manifestly symmetric requires the introduction of a fictitious extra dimension. This has the dramatic and surprising consequence that its coefficient is quantized.

To get the idea, consider again a model in $D = 0 + 1$ where the field variable $\check{n}$ takes values on the unit sphere S^2, $1 = \sum_{a=1,2,3} \check{n}_a^2$. This is a special case of a coset space $\mathsf{G}/\mathsf{H} = \mathsf{SU}(2)/\mathsf{U}(1)$.

In order to write the WZW term in a manifestly symmetric way (under the $\mathsf{SO}(3)$ of rotations of the sphere, we have to extend the field into a (possibly fictitious) extra dimension whose coordinate is u.
We do this in such a way that the real system lives at $u = 1$:

$$\check{n}(t, u = 1) \equiv \check{n}(t), \quad \check{n}(t, u = 0) \equiv (0, 0, 1)$$



it goes to the north pole at the other end of the extra dimension for all t. Consider periodic boundary conditions in time $\check{n}(2\pi) = \check{n}(0)$. Then this means that the full space is really a disk with the origin at $u = 0$, and the boundary at $u = 1$. Call this disk B, its boundary $\partial B = \mathcal{M}$ is the real spacetime (here a circle).

We can write the WZW term in terms of the S^2-valued field $\check{n}^{1,2,3}$ as

$$\mathcal{W}_1[\check{n}] = \frac{2\pi}{\Omega_2} \int_{B_2} \check{n}^a \mathrm{d}\check{n}^b \wedge \mathrm{d}\check{n}^c \epsilon_{abc} = \frac{1}{4\pi} \int_{\mathcal{M}} \mathrm{d}t \, (1 - \cos\theta) \, \partial_t \phi.$$

The integrand here is the volume element of the image of a chunk of spacetime in the target S^2. If we integrate over the union of two balls with cancelling boundaries $B_2 \cup \bar{B}_2$, we get an integer multiple of 2π (the integer is the winding number of the map).

The coefficient k of $\mathcal{W}_1$ in the action $\Delta S[\check{n}] = k\mathcal{W}_1[\check{n}]$ must be an integer since B_1 and $\bar{B}_1$ give equally good definitions of $\mathcal{W}_2$, which differ by $2\pi k$. So this ambiguity will not affect the path integral if $k \in \mathbb{Z}$.

The generalization to a group-valued variable U in any dimension is of the form

$$\mathcal{W}_D = c \int_{B_{D+1}} \mathrm{tr} \underbrace{U^{-1} dU \wedge U^{-1} dU \wedge \cdots \wedge U^{-1} dU}_{D+1 \text{ of these}}.$$

Such terms are interesting when $\pi_{D+1}(\mathcal{M})$ is nontrivial, where $\mathcal{M}$ is the space where the fields live (the target space), that is, there are maps from S^{D+1} to $\mathcal{M}$ which cannot be smoothly deformed to the trivial map where every point in the base space goes to

the same point in the target. The variation of $\mathcal{W}_D$ with respect to U is (for even D)[18]:

$$\delta\mathcal{W}_D = (D+1)c\int_{B_{D+1}} \mathrm{tr}\left\{\left(U^{-1}dU\right)^D \underbrace{\delta\left(U^{-1}dU\right)}_{=U^{-1}d(\delta UU^{-1})U}\right\} \tag{3.39}$$

$$= (D+1)c\int_{B_{D+1}} \mathrm{tr}\left\{\left(dUU^{-1}\right)^D d(\delta UU^{-1})\right\} \tag{3.40}$$

$$= (D+1)c\int_{B_{D+1}} d\mathrm{tr}\left\{\left(U^{-1}dU\right)^D U^{-1}\delta U\right\} \tag{3.41}$$

$$\overset{\text{Stokes}}{=} (D+1)c\int_{\mathcal{M}} \mathrm{tr}\left\{\left(U^{-1}dU\right)^D U^{-1}\delta U\right\}$$

which only depends on the field configuration on $\mathcal{M}$, not on the extension to B_{D+1}. Again there can be topologically distinct ways to make the extension; demanding that they always give the same answer determines c in terms of volumes of spheres (so that $c\int_{S^{D+1}}\mathrm{tr}(U^{-1}dU)^{D+1}\in\mathbb{Z}$ is the winding number), and the coefficient must be an integer. (In $D=4$, we have $c=\frac{\mathbf{i}}{240\pi^2}$.)

This WZW term is *less* topological than the theta term we discussed above, in the sense that it affects the equations of motion for $\check{n}(t)$. The variation of $\mathcal{W}$ *is* local in D dimensions. The following table gives a comparison between theta terms and WZW terms for a field theory in D spacetime dimensions, on a spacetime M_D:

[End of Lecture 11]

[18] Why do I restrict to even D?

$$\mathrm{tr}\left(U^{-1}dU\right)^{D+1} = \epsilon^{\mu_1\cdots\mu_{D+1}}\mathrm{tr}\left(U^{-1}\partial_{\mu_1}U\cdots U^{-1}\partial_{\mu_{D+1}}U\right)$$

but $\epsilon^{\mu_1\cdots\mu_{D+1}} = -(-1)^{D+1}\epsilon^{\mu_{D+1}\mu_1\cdots\mu_D}$ so $\mathcal{W}_D = (-1)^D\mathcal{W}_D$ vanishes in odd dimensions. The step from (3.40) to (3.41) also relies on this fact. Using $1=U^{-1}U$ and hence $0=\delta(U^{-1}U)=d(U^{-1}U)$, so that

$$dU^{-1} = -U^{-1}dUU^{-1}, \tag{3.38}$$

the term by which (3.40) and (3.41) differ is

$$\mathrm{tr}\left\{\left(d\left(U^{-1}dU\right)^D\right)\delta UU^{-1}\right\}$$

$$\overset{\text{product rule}}{=} \mathrm{tr}\left\{\left(dU^{-1}\wedge dU\wedge\left(U^{-1}dU\right)^{D-1} - \left(U^{-1}dU\wedge dU^{-1}\wedge dU\wedge\left(U^{-1}dU\right)^{D-2}+\cdots\right)\delta UU^{-1}\right\}$$

$$\overset{(3.38)}{=} -\mathrm{tr}\left\{\left(U^{-1}dU\wedge U^{-1}dU\wedge\left(U^{-1}dU\right)^{D-1} - U^{-1}dUU^{-1}\wedge dUU^{-1}\wedge dU\wedge\left(U^{-1}dU\right)^{D-2}+\cdots\right)\delta UU^{-1}\right\}$$

$$= \mathrm{tr}\left\{\underbrace{(1-1+1-1...)}_{D-1\text{ of these}}\left(U^{-1}dU\right)^{D-1}\delta UU^{-1}\right\} \overset{D-1\text{ even}}{=} 0.$$

See Weinberg, vol 2, §23.4 for more.

theta term	WZW term
$\mathcal{H} = \int_{M_D} h$	$\mathcal{W}_D = \int_{B_{D+1}} w, \quad \partial B_{D+1} = M_D$
$h = \mathrm{d}q$	$w = \mathrm{d}v$
Doesn't affect EOM	Affects EOM
Invisible in perturbation theory	Appears in perturbation theory, *e.g.* in beta functions
$\mathcal{H} \in \mathbb{Z}$ for M_D closed	Coefficient of $\mathcal{W} \in 2\pi\mathbb{Z}$ in order for path integral to be well-defined.

Pion physics is the context where these terms were first discovered, and where it was realized that their coefficients are quantized. In particular the coefficient of the WZW term $W_4[U]$ here is N_c, the number of colors, as Witten shows by explicitly coupling to electromagnetism, and finding the term that encodes $\pi^0 \to \gamma\gamma$. Apparently Witten realized that such a term was required because without it the chiral Lagrangian had an extra symmetry under $\pi \to -\pi$ which is absent in QCD; the WZW term produces a 5-pion amplitude which violates this symmetry.

One dramatic consequence here is that the chiral Lagrangian (with some higher-derivative terms) has a topological soliton solution (the skyrmion) which is a *fermion* if the number of colors of QCD is odd. The field configuration $U(x,t)$ is constant in time and approaches the vacuum at infinity, so we can regard it as a map U : (space $\cup\infty \sim S^d$) $\to G/H$, where G is the full symmetry group and H is the unbroken subgroup, so G/H is the space of Goldstones (in the chiral Lagrangian, $G/H = \mathsf{SU}(3)\times \mathsf{SU}(3)/\mathsf{SU}(3)_{\text{preserved}} \simeq \mathsf{SU}(3)_{\text{broken}}$). The configuration is topological in the sense that as a map from $S^3 \to G/H$, it cannot be smoothly deformed to the trivial map – it represents a nontrivial element of $\pi_3(G/H)$. Its nontriviality is witnessed by a winding number, which can be written as the integral of a local density. In fact, the baryon number of this configuration comes from the anomalous (WZW) contribution to the baryon number current $B_\mu = \frac{\epsilon_{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{tr}U^{-1}\partial_\nu U U^{-1}\partial_\alpha U U^{-1}\partial_\beta U$ whose conserved charge $\int_{\text{space}} B_0$ is exactly the winding number of the map from space (plus the point at infinity) to the space of goldstones. And finally this object a fermion because the WZW term evaluates to π on a spacetime trajectory where the soliton makes a 2π rotation. So this object is a fermionic particle which carries baryon number. It also carries isospin. It's a nucleon! Above we added nucleon fields to the chiral Lagrangian, but we actually didn't need to – they were already there as solitonic excitations. Note that the *size* of the soliton (the region of space over which the fields vary) is determined by the

higher-derivative terms in the chiral lagrangian, so we shouldn't take too seriously the substructure of the proton predicted by this picture. But it doesn't do too badly.

As a preview, I should also mention that WZW terms are important in the study of interacting spin systems, for example in our understanding of the dependence on the s of Heisenberg spin-s chains (§4.2), and in phase transitions beyond the Landau-Ginzburg (symmetry-breaking) paradigm (*i.e.* deconfined quantum criticality, §7.3).

Constraints on solitons in scalar field theories. The fact that the chiral Lagrangian has nontrivial, stable, static solitonic particle solutions merits some further comment. The irrelevant terms actually play an important role. Without them, we can show that no such stable solutions exist.

Derrick's argument: Consider a field theory of scalars with 0-derivative and 2-derivative terms. For purposes of finding static solutions, extremizing the action is the same as extremizing the energy:

$$E[\phi] = \int d^d\phi \left(g(\phi) \left(\vec{\nabla}\phi\right)^2 + V(\phi)\right)^2 \equiv I_1 + I_2.$$

There could be multiple scalars, so for example, the argument applies to the leading term in the chiral lagrangian $L = \mathrm{tr}\,(U^{-1}\partial U)^2$. We'll assume $I_1 > 0$, since otherwise there is an obvious gradient instability of the theory.

Suppose we have a solution $\underline{\phi}$ which extremizes E. To describe a particle excitation of the vacuum, it must approach the vacuum value far away, $\underline{\phi}(x) \overset{x\to\infty}{\to} \phi_0$.

Now consider a dilated configuration $\phi_\lambda(x) \equiv \underline{\phi}(\lambda x)$. Plugging in and changing integration variables gives

$$E[\phi_\lambda] = \frac{I_1}{\lambda^{2-d}} + \frac{I_2}{\lambda^d}.$$

Demanding that $\underline{\phi}$ is a stationary point implies

$$0 = \partial_\lambda E[\phi_\lambda]|_{\lambda=1} = (2-d)I_1 - dI_2 \quad \Longrightarrow \quad I_2 = \frac{2-d}{d} I_1$$

and then

$$\partial_\lambda^2 E[\phi_\lambda]|_{\lambda=1} = -(2-d)(1-d)I_1 + d(d+1)I_2 = -2(d-2)I_1 < 0.$$

So the solution is unstable to dilations.

If we add a term with more derivatives, like $I_3 = \frac{1}{M^4}\int(\vec{\nabla}\phi)^6$, it will contribute positively to $\partial_\lambda^2 E[\phi_\lambda]|_{\lambda=1}$ and the argument is no longer valid. The length scale $1/M$ in front of this higher-derivative term then determines the size of the soliton.

4 Field theory of spin systems

Where do spin systems come from? Just as the classical ising model arises in many ways (lattice gas, double well potential....), quantum spin systems arise in many ways. for example:

1. Spinful fermions at half-filling, with charge dofs quenched.
2. Similarly, spinless fermions hopping around on a lattice also realize a spin system: there are two states per site. How to relate fermion operators and spin operators? In 1+1 dimensions, we can answer this in complete explicitness, below.
3. If we view down-spin as vacuum and up-spin as the presence of a boson, we can view a hard-core boson system as a spin-1/2 system. Here it is natural to preserve a $\mathsf{U}(1) \subset \mathsf{SU}(2)$ symmetry which rotates the phase of $\mathbf{X} + \mathbf{i}\mathbf{Y}$; the conserved quantity is $\mathbf{Z} = (2\mathbf{n}_B - 1)$ where $\mathbf{n}_B$ is the boson number. First we'll study a situation with just $\mathbb{Z}_2$ (Ising) symmetry.

4.1 Transverse-Field Ising Model

The Ising model has many guises. There is this from statistical mechanics:

$$Z = \sum_{\{s_j\}} e^{-K \sum_{\langle jl \rangle} s_j s_l}.$$

There is this quantum spin system:

$$\mathbf{H}_{TFIM} = -J \sum_j \left(g_x \mathbf{X}_j + g_z \mathbf{Z}_j \mathbf{Z}_{j+1} \right) \ .$$

And there is this 2d conformal field theory:

$$S[\chi] = \int \mathrm{d}^2 z \left(\chi \overline{\partial}_z \chi \right) \tag{4.1}$$

which I first encountered on the worldsheet of a superstring. An important part of our job is to understand the connections between these things. One thing they have in common is a $\mathbb{Z}_2$ symmetry, $s_j \to -s_j$ or $\mathbf{Z}_j \to -\mathbf{Z}_j$ or $\chi \to -\chi$.

Whether or not you liked the derivation above of its relation to the euclidean statistical mechanics Ising model, we are going to study the quantum system whose hamiltonian is

$$\mathbf{H}_{\text{TFIM}} = -J \left(g \sum_j \mathbf{X}_j + \sum_{\langle jl \rangle} \mathbf{Z}_j \mathbf{Z}_l \right). \tag{4.2}$$

Some of the things we say next will be true in one or more spatial dimensions.

Notice that J has units of energy; we could choose units where it's 1. In 1d (or on bipartite lattices), the sign of J does not matter for determining what state of matter we realize: if $J < 0$, we can relabel our operators: $\tilde{Z}_j = (-1)^j \mathbf{Z}_j$ and turn an antiferromagnetic interaction into a ferromagnetic one. So let's assume $g, J > 0$.

This model is interesting because of the competition between the two terms: the $\mathbf{X}_j$ term wants *each* spin (independently of any others) to be in the state $|{\rightarrow}\rangle_j$ which satisfies

$$\mathbf{X}_j |{\rightarrow}\rangle_j = |{\rightarrow}\rangle_j \,. \quad |{\rightarrow}\rangle_j = \frac{1}{\sqrt{2}} \Big(|{\uparrow}\rangle_j + |{\downarrow}\rangle_j \Big) \,.$$

In conflict with this are the desires of $-\mathbf{Z}_j \mathbf{Z}_{j+1}$, which is made happy (*i.e.* smaller) by the more cooperative states $|{\uparrow_j}{\uparrow_{j+1}}\rangle$, or $|{\downarrow_j}{\downarrow_{j+1}}\rangle$. In fact, it would be just as happy about any linear combination of these $a\,|{\uparrow_j}{\uparrow_{j+1}}\rangle + b\,|{\downarrow_j}{\downarrow_{j+1}}\rangle$ and we'll come back to this point.

Another model which looks like it might have some form of competition is

$$\mathbf{H}_{\text{boring}} = \cos\theta \sum_j \mathbf{Z}_j + \sin\theta \sum_j \mathbf{X}_j \ , \quad \theta \in [0, \frac{\pi}{2}]$$

Why is this one boring? Notice that we can continuously interpolate between the states enjoyed by these two terms: the groundstate of $\mathbf{H}_1 = \cos\theta \mathbf{Z} + \sin\theta \mathbf{X}$ is

$$|\theta\rangle = \cos\frac{\theta}{2} |{\uparrow}\rangle + \sin\frac{\theta}{2} |{\downarrow}\rangle$$

– as we vary θ from 0 to $\pi/2$ we just smoothly rotate from $|{\uparrow_z}\rangle$ to $|{\uparrow_x}\rangle$. And it is always a product state.

How do we know the same thing can't happen in the transverse-field Ising chain? Symmetry. We've already seen that the Ising model has a $\mathsf{G} = \mathbb{Z}_2$ symmetry which acts by $\mathbf{Z}_j \to \mathbf{S}\mathbf{Z}_j\mathbf{S}^\dagger = -\mathbf{Z}_j, \mathbf{X}_j \to \mathbf{S}\mathbf{X}_j\mathbf{S}^\dagger = +\mathbf{X}_j$, where the unitary $\mathbf{S}$ commutes with $\mathbf{H}_{\text{TFIM}}$: $\mathbf{S}\mathbf{H}_{\text{TFIM}}\mathbf{S}^\dagger = \mathbf{H}_{TFIM}$. Here $\mathbf{S} = \prod_i \mathbf{X}_i$. The difference with $\mathbf{H}_{\text{boring}}$ is that $\mathbf{H}_{\text{TFIM}}$ has two phases in which G is realized differently on the groundstate.

$\boxed{g = \infty}$: First, let's take g so big that we may ignore the ZZ ferromagnetic term, so

$$\mathbf{H}_{g\to\infty} = -\sum_j \mathbf{X}_j \ .$$

(The basic idea of this discussion will apply in any dimension, on any lattice.) Since all terms commute, the groundstate is the simultaneous groundstate of each term:

$$\mathbf{X}_j |\text{gs}\rangle = + |\text{gs}\rangle \,, \quad \forall j, \qquad \Longrightarrow \ |\text{gs}\rangle = \otimes_j |{\rightarrow}\rangle_j \,.$$

Notice that this state *preserves the symmetry* in the sense that $\mathbf{S}\,|\mathrm{gs}\rangle = |\mathrm{gs}\rangle$. Such a symmetry-preserving groundstate is called a *paramagnet*.

$\boxed{g = 0}$: Begin with $g = 0$.

$$\mathbf{H}_0 = -J \sum_j \mathbf{Z}_j \mathbf{Z}_{j+1}$$

has groundstates

$$|+\rangle \equiv |\uparrow\uparrow \cdots \uparrow\rangle\,, \quad |-\rangle \equiv |\downarrow\downarrow \cdots \downarrow\rangle\,,$$

or any linear combination. Note that the states $|\pm\rangle$ are not symmetric: $\mathbf{S}\,|\pm\rangle = |\mp\rangle$, and so we are tempted to declare that the symmetry is broken by the groundstate.

You will notice, however, that the states

$$\left| \text{  }_{\pm} \right\rangle \equiv \frac{1}{\sqrt{2}} \left(|+\rangle \pm |-\rangle\right)$$

are symmetric – they are $\mathbf{S}$ eigenstates, so $\mathbf{S}$ maps them to themselves up to a phase. It gets worse: In fact, in finite volume (finite number of sites of our chain), with $g \neq 0$, $|+\rangle$ and $|-\rangle$ are *not* eigenstates, and $\left| \text{  }_{+} \right\rangle$ is the groundstate. **BUT:**

1. The two states $|+\rangle$ and $|-\rangle$ only mix at order N in perturbation theory in g, since we have to flip all N spins using the perturbing hamiltonian $\Delta\mathbf{H} = -gJ \sum_j \mathbf{X}_j$ to get from one to the other. The tunneling amplitude is therefore

$$T \sim g^N \langle -|\, \mathbf{X}_1 \mathbf{X}_2 \cdots \mathbf{X}_N\, |+\rangle \overset{N\to\infty}{\to} 0.$$

2. There's a reason for the symbol I used to denote the symmetric states: at large N, these 'cat states' are superpositions of macroscopically distinct quantum states. Such things don't happen, because of decoherence: if even a single dust particle in the room measures the spin of a single one of the spins, it measures the value of the whole chain. In general, this happens very rapidly.

3. Imagine we add a small symmetry-breaking perturbation: $\Delta\mathbf{H} = -\sum_j h\mathbf{Z}_j$; this splits the degeneracy between $|+\rangle$ and $|-\rangle$. If $h > 0$, $|+\rangle$ is for sure the groundstate. Consider preparing the system with a tiny $h > 0$ and then setting $h = 0$ after it settles down. If we do this to a finite system, $N < \infty$, it will be in an excited state of the $h = 0$ Hamiltonian, since $|+\rangle$ will not be stationary (it will have a nonzero amplitude to tunnel into $|-\rangle$). But if we take the thermodynamic

limit *before* taking $h \to 0$, it will stay in the state we put it in with the 'training field' h. So beware that there is a singularity of our expressions (with physical significance) that means that the limits do not commute:

$$\lim_{N\to\infty}\lim_{h\to 0} Z \neq \lim_{h\to 0}\lim_{N\to\infty} Z.$$

The physical one is to take the thermodynamic limit first.

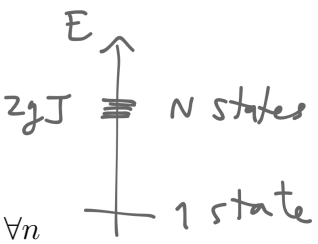
The conclusion of this brief discussion is that spontaneous symmetry breaking actually happens in the $N \to \infty$ limit. At finite N, $|+\rangle$ and $|-\rangle$ are approximate eigenstates which become a better approximation as $N \to \infty$.

This state of a $\mathbb{Z}_2$-symmetric system which spontaneously breaks the $\mathbb{Z}_2$ symmetry is called a ferromagnet.

So the crucial idea I want to convey here is that there must be a sharp phase transition at some finite g: the situation cannot *continuously* vary from one unique, symmetric groundstate $\mathbf{S}\left|\mathrm{gs}_{g\ll 1}\right\rangle = \left|\mathrm{gs}_{g\ll 1}\right\rangle$ to two symmetry-breaking groundstates: $\mathbf{S}\left|\mathrm{gs}_{\pm}\right\rangle = \left|\mathrm{gs}_{\mp}\right\rangle$.

Quasiparticles. Next, let's ask what are the low-lying excitations, and see what happens when we try to vary the coupling away from the extreme points.

$\boxed{g \gg 1}$ An excited state of the paramagnet, deep in the phase, is achieved by flipping one spin. With $\mathbf{H} = \mathbf{H}_\infty = -gJ\sum_j \mathbf{X}_j$, this costs energy $2gJ$ above the groundstate. There are N such states, labelled by which spin we flipped:



$$|n\rangle \equiv \left|\to\cdots\to\underbrace{\leftarrow}_{nth\ \text{site}}\to\cdots\right\rangle, \qquad (\mathbf{H}_\infty - E_0)\,|n\rangle = 2gJ\,|n\rangle\,, \ \forall n$$

When g is not infinite, we can learn a lot from (1st order) degenerate perturbation theory in the ferromagnetic term. The key information is the matrix elements of the perturbing hamiltonian between the degenerate manifold of states. Using the fact that $\mathbf{Z}_j\,|{\to}\rangle = |{\leftarrow}\rangle$, so,

$$\mathbf{Z}_j\mathbf{Z}_{j+1}\,|{\to_j}{\leftarrow_{j+1}}\rangle = |{\leftarrow_j}{\to_{j+1}}\rangle$$
$$\langle n \pm 1|\sum_j \mathbf{Z}_j\mathbf{Z}_{j+1}\,|n\rangle = 1,$$

the ferromagnetic term hops the spin flip by one site. Within the degenerate subspace, it acts as

$$\mathbf{H}_{\text{eff}}\,|n\rangle = -J\left(|n+1\rangle + |n-1\rangle\right) + (E_0 + 2gJ)\,|n\rangle\,.$$

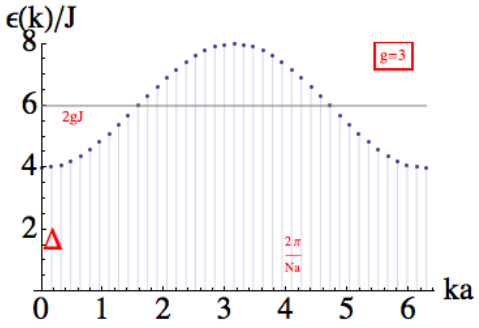
It is a kinetic, or 'hopping' term for the spin flip.

Let's see what this does to the spectrum. Assume periodic boundary conditions and N sites total. Again this is a translation invariant problem (in fact the same one, basically), which we solve by Fourer transform:

$$|n\rangle \equiv \frac{1}{\sqrt{N}} \sum_j e^{-\mathbf{i}kx_j} |k\rangle \,, \quad \begin{cases} x_j \equiv ja, \\ k = \frac{2\pi m}{Na}, \quad m = 1..N \end{cases}$$

On the momentum states, we have

$$(H - E_0)\,|k\rangle = (-2J\cos ka + 2gJ)\,|k\rangle \,.$$



The dispersion of these spinon particles is

$$\epsilon(k) = 2J(g - \cos ka) \overset{k\to 0}{\sim} \Delta + J(ka)^2 \qquad (4.3)$$

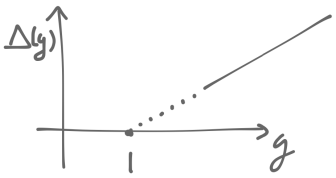
with $\Delta = 2J(g-1)$ – there is an energy gap (notice that Δ does not depend on system size). So these are *massive* particles, with dispersion $\epsilon = \Delta + \frac{k^2}{2M} + ...$ where Δ is the energy to create one at rest (notice that the rest energy is not related to its inertial mass $M^{-1} = 2Ja^2$).

A particle at j is created by the creation operator $\mathbf{Z}_n$:

$$|n\rangle = \mathbf{Z}_n\,|\mathrm{gs}_\infty\rangle \,.$$

And it is annihilated by the annihilation operator $\mathbf{Z}_n$ – you can't have two spin flips at the same location! These particles are their own antiparticles.

The *number* of such particles is counted by the operator $\sum_j (-\mathbf{X}_j)$. The number of particles is only conserved modulo two, however.



What happens as g gets smaller? The gap to creating a spin flip at large g looks like $2J(g-1)$. If we take this formula seriously, we predict that at $g = 1$ it costs zero energy to create spin flips: they should condense in the vacuum. Condensing spin flips means that the spins point in all directions, and the state is paramagnetic. (We shouldn't take it seriously because it's just first order in perturbation theory, but it turns out to be exactly right.)

It's possible to develop some more evidence for this picture and understanding of the physics of the paramagnetic phase in the Ising chain by doing more perturbation theory, and including states with two spin flips. Notice that for a state with two spin-flip particles, the total momentum k no longer uniquely determines the energy, since the two spin-flips can have a relative momentum; this means that there is a

two-particle continuum of states, once we have enough energy to make two spin flips. For more on this, see *e.g.* Sachdev (2d ed) §5.2.2. In particular the two spin-flips can form boundstates, which means the lowest-energy two-particle state is actually slightly below 2Δ.

$\boxed{g \ll 1}$ Now let's consider excitations of the ferromagnet, about the state $|+\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$. It is an eigenstate of $\mathbf{H}_0 = -J\sum_j \mathbf{Z}_j\mathbf{Z}_{j+1}$ and its (groundstate) energy is $E_0 = -JN$. We can make an excitation by flipping one spin:

$$|\cdots\uparrow\ \uparrow\ \uparrow\cdot\downarrow\cdot\uparrow\ \uparrow\ \uparrow\cdots\rangle$$

This makes two bonds unhappy, and costs $2J + 2J = 4J$. But once we make it there are many such states: the hamiltonian is the same amount of unhappy if we also flip the next one.

$$|\cdots\uparrow\ \uparrow\ \uparrow\cdot\downarrow\ \downarrow\cdot\uparrow\ \uparrow\cdots\rangle$$

The actual elementary excitation is a *domain wall* (or *kink*), which only costs $2J$. The domain wall should be regarded as living between the sites. It is not entirely a local object, since with periodic boundary conditions, we must make two, which can then move independently. To create two of them far apart, we must change the state of many spins.

At $g = 0$ the domain walls are *localized* in the sense that a domain wall at a fixed position is an energy eigenstate (just like the spinons at $g = \infty$), with the same energy for any position. But now the paramagnetic term $-\sum_j g\mathbf{X}_j$ is a kinetic term for the domain walls:

$$\mathbf{X}_{j+1}\underbrace{|\cdots\uparrow\uparrow\uparrow_j\cdot\downarrow_{j+1}\downarrow\downarrow\cdots\rangle}_{\bar{j}} = \underbrace{|\cdots\uparrow\uparrow\uparrow_j\uparrow_{j+1}\cdot\downarrow_{j+2}\downarrow\cdots\rangle}_{=|\bar{j}+1\rangle}.$$

Just like in our $g \gg 1$ discussion, acting on a state with a single domain wall[19]

$$(H_{\text{eff}} - E_0)\,|\bar{j}\rangle = -gJ\,(|\bar{j}+1\rangle + |\bar{j}-1\rangle) + 2J\,|\bar{j}\rangle$$

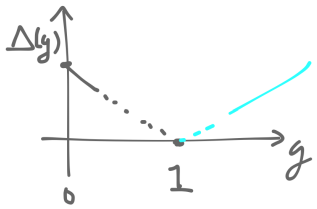
where the diagonal term is the energy cost of one domain wall at rest. Again this is diagonalized in k-space with energy

$$\epsilon_{\text{one dwall}}(k) = 2J(1 - g\cos ka)$$

[19] A state with a single domain wall can't happen with periodic boundary conditions, but with open boundary conditions, it can.

Again, this calculation is almost ridiculously successful at predicting the location of the phase transition:

$$\Delta_{DW} = 2J(1-g) \overset{g\to 1}{\to} 0.$$



Although our discussion of the paramagnetic state $g \gg 1$ can be applied in any $d \geq 1$, the physics of domain walls is very dimension-dependent.

[End of Lecture 12]

Pause to reassess goals. You may be wondering: why in this class about QFT are we lavishing so much attention on this spin chain? It may seem like this discussion involves lots of lattice-scale details which are divorced from our fancy Lorentz-invariant continuum QFT considerations. First: Recall that my definition of QFT is the study of extensive quantum systems, *i.e.* (quantum) degrees of freedom distributed over space. A spin chain exactly fits this definition. Second: It's true that this kind of system (and the TFIM model in particular) arises in many discussions of condensed matter. But the real reason is: In this example, we can understand everything. In particular, we can understand everything about how QFT emerges from a completely well-defined quantum mechanical system, in which there are obviously no divergences or infinities or anomalies. And yet, the system exhibits a quantum phase transition, described by a relativistic conformal field theory (CFT)! (In particular, Lorentz invariance emerges.) And moreover, we will discover that a description in terms of gauge theory is useful. So I think the TFIM is particularly useful as an answer to the question 'where do QFTs come from?'

Interpretation of the stability of the SSB state in terms of domain walls. If at finite N, with periodic boundary conditions, we prepare the system in the state $|+\rangle$, tunneling to $|-\rangle$ requires creation of a pair of domain walls $\Delta E = 4J$, which then *move* all the way around the circle, giving the tunneling rate

$$\prod_{j=1}^{N} \left(\frac{\langle \bar{j}+1 | \, \mathbf{H}_{\text{eff}} \, | \bar{j} \rangle}{\Delta E} \right) \sim \frac{(gJ)^N}{J^N} \sim g^N \sim e^{-N \log \frac{1}{g}}.$$

(For $g < 1$, $\log \frac{1}{g} > 0$.) The tunneling rate goes like e^{-N} – it is exponentially small in the system size.

Duality between spin flips and domain walls. The discussion we've just made of the small-g physics has a lot in common with the large-g physics. More quantitatively, the dispersion relation $\epsilon_{\text{one dwall}}(k)$ for a single domain wall looks nearly the same as that of one spin flip (4.3). In fact they are mapped to each other by the replacement

$$g \to \frac{1}{g}, \quad J \to Jg. \tag{4.4}$$

Notice that this takes small g (weak coupling of domain walls, strong coupling of spin flips) to large g (strong coupling of domain walls, weak coupling of spin flips).

In fact, there is a change of variables that (nearly) interchanges the two sides of the phase diagram. Suppose the system is on an interval – open boundaries – the chain just stops at $j = 1$ and $j = N$. (We do this to avoid the constraint of an even number of domain walls.) We can specify a basis state in the Z-basis by the direction (up or down along Z) of the first spin and the locations of domain walls.

Consider the operator, diagonal in this basis, which measures whether there is a domain wall between j and $j + 1$:

$$\boldsymbol{\tau}^x_{\bar{j}} \equiv \mathbf{Z}_{\bar{j}-\frac{1}{2}}\mathbf{Z}_{\bar{j}+\frac{1}{2}} = \begin{cases} +1, & \text{if } z_{\bar{j}-\frac{1}{2}} = z_{\bar{j}+\frac{1}{2}} \\ -1, & \text{if } z_{\bar{j}-\frac{1}{2}} = -z_{\bar{j}+\frac{1}{2}} \end{cases} = (-1)^{\text{disagreement}}.$$

Notice that $\boldsymbol{\tau}^2_{\bar{j}} = 1, \boldsymbol{\tau}^\dagger_{\bar{j}} = \boldsymbol{\tau}_{\bar{j}}$. Similarly, consider the operator that *creates* a domain wall at $\bar{j}$:

$$\boldsymbol{\tau}^z_{\bar{j}} \equiv \mathbf{X}_{\bar{j}+\frac{1}{2}}\mathbf{X}_{\bar{j}+\frac{3}{2}}\cdots = \prod_{j>\bar{j}} \mathbf{X}_j.$$

This operator flips all the spins to the right of the link in question (and fixes our reference first spin). It, too, is hermitian and squares to one. Finally, notice that

$$\boldsymbol{\tau}^z_{\bar{j}}\boldsymbol{\tau}^x_{\bar{j}'} = (-1)^{\delta_{\bar{j}\bar{j}'}}\boldsymbol{\tau}^x_{\bar{j}'}\boldsymbol{\tau}^z_{\bar{j}}$$

just like $\mathbf{Z}$ and $\mathbf{X}$ (since when $\bar{j} = \bar{j}'$, they contain a single $\mathbf{Z}$ and $\mathbf{X}$ at the same site). The domain walls can be represented in terms of two-state systems living on the links.

Notice that the inverse of the map from $\mathbf{X}, \mathbf{Z}$ to $\boldsymbol{\tau}^x, \boldsymbol{\tau}^z$ is

$$\mathbf{X}_j = \boldsymbol{\tau}^z_{j-\frac{1}{2}}\boldsymbol{\tau}^z_{j+\frac{1}{2}} \ .$$

(The right hand side is an inefficient way to flip a single spin at j: namely, flip all the spins right of $j - 1$, and then flip back all the spins to the right of j.)

So the 1d TFIM hamiltonian in bulk is

$$\begin{aligned} \mathbf{H}_{\text{TFIM}} &= -J\sum_j \left(g\mathbf{X}_j + \mathbf{Z}_j\mathbf{Z}_{j+1}\right) \\ &= -J\sum_{\bar{j}} \left(g\boldsymbol{\tau}^z_{\bar{j}}\boldsymbol{\tau}^z_{\bar{j}+1} + \boldsymbol{\tau}^x_{\bar{j}}\right). \end{aligned} \tag{4.5}$$

This is the TFIM hamiltonian again with $\mathbf{Z} \to \boldsymbol{\tau}^z$ and $\mathbf{X} \to \boldsymbol{\tau}^x$ and the couplings mapped by (4.4).

This is in fact the same map as Kramers-Wannier duality (or rather it is mapped to it by the quantum-to-classical map). As K&W argued, if there is a single phase transition it must occur at the self-dual point $g = 1$.

Notice that the paramagnetic (disordered) groundstate of the original system is a *condensate* of domain walls, in the following sense. The operator that creates a domain wall has an expectation value:

$$\left\langle \boldsymbol{\tau}^z_{\bar{j}} \right\rangle = \left\langle \mathrm{gs}_{g=\infty} \right| \boldsymbol{\tau}^z_{\bar{j}} \left| \mathrm{gs}_{g=\infty} \right\rangle = \left\langle \mathrm{gs}_{g=\infty} \right| \prod_{j>\bar{j}} \mathbf{X}_j \left| \mathrm{gs}_{g=\infty} \right\rangle = 1 \quad \forall \bar{j} \ .$$

(For $g \in (1, \infty)$, this expectation value is less than one but nonzero, just like how $|\langle \mathbf{Z} \rangle|$ decreases from 1 as g grows from zero.) Although there is a condensate, there is no order, in the sense that an expectation value of $\mathbf{X}$ does not break any symmetry of $\mathbf{H}_{\mathrm{TFIM}}$. (There is another state where $\langle \boldsymbol{\tau}^x \rangle = -1$, but (at large g) it's a high-energy state.)

An important point (and the reason 'duality' is a dangerous word): the two sides of the phase diagram are *not the same*. On one side there are two groundstates related by the broken symmetry, on the other side there is a unique symmetric groundstate. That's how we knew there had to be a phase transition! I will say more about this mismatch.

Open boundaries. Let us make sure we can reproduce the correct number of groundstates in the two phases. To get this right, we have to be precise about the endpoint conditions. Let's study the case where we have N sites in a row; the first and last sites have only one neighbor. The Hamiltonian is

$$\mathbf{H}_{\mathrm{TFIM}} = -J \left(\sum_{j=1}^{N-1} (g\mathbf{X}_j + \mathbf{Z}_j \mathbf{Z}_{j+1}) + g\mathbf{X}_N \right) \ .$$

The duality map is

$$\mathbf{Z}_j \mathbf{Z}_{j+1} = \boldsymbol{\tau}^x_{j+\frac{1}{2}}, j = 1, 2 \ldots N-1, \quad \mathbf{X}_j = \boldsymbol{\tau}^z_{j-\frac{1}{2}} \boldsymbol{\tau}^z_{j+\frac{1}{2}}, \quad j = 1 \ldots N \quad .$$

In terms of the domain-wall variables, the hamiltonian is

$$\mathbf{H}_{\mathrm{TFIM}} = -Jg \left(\sum_{j=1}^{N-1} \left(\boldsymbol{\tau}^z_{j-\frac{1}{2}} \boldsymbol{\tau}^z_{j+\frac{1}{2}} + \frac{1}{g} \boldsymbol{\tau}^x_{j+\frac{1}{2}} \right) + \boldsymbol{\tau}^z_{N-\frac{1}{2}} \boldsymbol{\tau}^z_{N+\frac{1}{2}} \right) \ .$$

But now there are two special cases:
• $\boldsymbol{\tau}^z_{N+\frac{1}{2}} = 1$: this operator flips all the spins with $j > N$; but there are no spins with $j > N$. So it is the identity operator.

• $\boldsymbol{\tau}^x_{\frac{1}{2}}$: this operator measures whether or not there is a domain wall between $j = 1$ and $j = 0$, $\boldsymbol{\tau}^x_{\frac{1}{2}} = \mathbf{Z}_0\mathbf{Z}_1$. But there is no spin at $j = 0$. One way to think about this is to put a "ghost spin" at $j = 0$ which is always in the state $\mathbf{Z}_0 = 1$. So $\boldsymbol{\tau}^x_{\frac{1}{2}} = \mathbf{Z}_1$: it measures the value of our reference spin.

At $g = 0$: $\mathbf{H}_{g=0} = -J\sum_{j=2}^{N} \boldsymbol{\tau}^x_{j-\frac{1}{2}}$ and the groundstate is $\boldsymbol{\tau}^x_{j-\frac{1}{2}} = 1$ for $j = 2...N$. But $\boldsymbol{\tau}^x_{\frac{1}{2}}$ does not appear, so there are two degenerate groundstates, eigenstates of $\boldsymbol{\tau}^x_{\frac{1}{2}}$ with eigenvalue $\pm$, which are just $|\pm\rangle$, the states with no domain walls: all the other spins agree with the first one in a state where $\boldsymbol{\tau}^x_{j>1} = 1$.

At $g = \infty$, $\mathbf{H}_{g=\infty} = -Jg\left(\sum_{j=1}^{N-1} \boldsymbol{\tau}^z_{j-\frac{1}{2}}\boldsymbol{\tau}^z_{j+\frac{1}{2}} + \boldsymbol{\tau}^z_{N-\frac{1}{2}}\right)$. The first term requires agreement between $\boldsymbol{\tau}^z_{j-\frac{1}{2}} = \boldsymbol{\tau}^z_{j+\frac{1}{2}}$ for $j = 1...N-1$, but the second term requires $\boldsymbol{\tau}^z_{N-\frac{1}{2}} = 1$. So all the others must be $+1$, too. This is the unique groundstate of the paramagnet.

Comment: Recall that the classical stat mech model we get by trotterizing the TFIM is the ordinary 2d Ising model, with anisotropic couplings,

$$H_c = -\sum_{\vec{n}} \left(Ks(\vec{n}+\check{\tau})s(\vec{n}) + K_x s(\vec{n}+\check{x})s(\vec{n})\right).$$

In these variables, the critical curve between the two phases is at

$$\sinh(2K_x)\sinh(2K) = 1\ ;$$

this relationship follows by *Kramers-Wannier duality*. In the continuous-time limit, $K_x \sim \Delta\tau \ll 1, K \gg 1$, so the critical condition becomes

$$1 = \sinh(2K_x)\sinh(2K) \simeq 2K_x\frac{e^{2K}}{2} = J\Delta\tau\frac{1}{Jg\Delta\tau} = \frac{1}{g}.$$

This determines the location of the quantum phase transition, in agreement with all the other stuff we've said about it.

4.1.1 Solution of Ising chain in terms of Majorana fermions

The TFIM in $d = 1$ is actually exactly solvable, for all g, including the quantum critical point. Let us see how this works and how the critical field theory emerges near $g = 1$.

Jordan-Wigner in 0+1 dimensions. As a warmup, consider a single fermionic operator $\mathbf{c}$ satisfying

$$\mathbf{c}^2 = 0, \quad \{\mathbf{c}, \mathbf{c}^\dagger\} = 1\ .$$

This constructs the Hilbert space of a qbit as follows:

$$\mathbf{c}\,|{\rightarrow}\rangle = 0, \quad \mathbf{c}^\dagger\,|{\rightarrow}\rangle = |{\leftarrow}\rangle\,, \quad \mathbf{c}^\dagger\,|{\leftarrow}\rangle = 0, \quad \mathbf{c}\,|{\leftarrow}\rangle = |{\rightarrow}\rangle\ . \tag{4.6}$$

The two states of the qbit just describe the presence or absence of the fermion in this state. We can rewrite the Pauli operators as

$$\mathbf{Z} = \mathbf{c} + \mathbf{c}^\dagger,$$

which flips the spin,

$$\mathbf{Y} = \frac{1}{\mathbf{i}}\left(\mathbf{c} - \mathbf{c}^\dagger\right),$$

$$\mathbf{X} = \mathbb{1} - 2\mathbf{c}^\dagger\mathbf{c}.$$

Here $\mathbf{c}^\dagger\mathbf{c}$ counts the number of fermions. Also note that the raising and lowering operators are $\boldsymbol{\sigma}^+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = c^\dagger, \boldsymbol{\sigma}^- = c$ and the number operator $\mathbf{c}^\dagger\mathbf{c}$ (whose eigenvalues are 0 and 1) is $\mathbf{c}^\dagger\mathbf{c} = \frac{1}{2}\left(\mathbb{1} - \mathbf{X}\right)$.

The story is less simple with more than one fermion mode operator, because they must satisfy $\{\mathbf{c}_\alpha, \mathbf{c}_\beta\} = 0$, rather than just commuting, as the would if we just used the above map for each mode.

[0904.2771, Fradkin] Let's look at the TFIM again:

$$\mathbf{H}_{\text{TFIM}} = -J\sum_j \left(g\mathbf{X}_j + \mathbf{Z}_j\mathbf{Z}_{j+1}\right)$$

has a phase transition between large-g and small-g phases.

(Disordered) large g: excitations are created by $\mathbf{Z}_j$ – they are spin flips. The groundstate is a condensate of domain walls: $\langle\boldsymbol{\tau}^z\rangle \neq 0$.

(Ordered) small g: excitations are created by the 'disorder' operator $\boldsymbol{\tau}^z_{\tilde{j}} = \prod_{j>\tilde{j}}\mathbf{X}_j$ – they are domain walls. The groundstate is a condensate of spins $\langle\mathbf{Z}_j\rangle \neq 0$, *i.e.* a ferromagnet.

So we understand what are the 'correct variables' (in the sense that they create the elementary excitations above the groundstate) at large and small g. I claim that the Correct Variables *everywhere* in the phase diagram are obtained by "attaching a spin to a domain wall". These words mean the following: let

$$\begin{aligned} \boldsymbol{\chi}_j &\equiv \mathbf{Z}_j\boldsymbol{\tau}^z_{j+\frac{1}{2}} = \mathbf{Z}_j\prod_{j'>j}\mathbf{X}_{j'} \\ \tilde{\boldsymbol{\chi}}_j &\equiv \mathbf{Y}_j\boldsymbol{\tau}^z_{j+\frac{1}{2}} = -\mathbf{i}\mathbf{Z}_j\prod_{j'\geq j}\mathbf{X}_{j'} \end{aligned} \tag{4.7}$$

The first great virtue of this definition is that these operators agree with the creators of the elementary excitations in both regimes we've studied: When $g \ll 1$, $\langle \mathbf{Z}_j \rangle \simeq 1$ and more strongly, $\mathbf{Z}_j = \langle \mathbf{Z}_j \rangle + \text{small}$, so $\boldsymbol{\chi}_j \simeq \langle \mathbf{Z}_j \rangle \boldsymbol{\tau}^z_{j+\frac{1}{2}} \simeq \boldsymbol{\tau}^z_{j+\frac{1}{2}}$, the domain wall creation operator. Similarly, when $g \gg 1$, $\boldsymbol{\tau}^z_j \simeq 1 + \text{small}$, so $\boldsymbol{\chi}_j \simeq \mathbf{Z}_j \left\langle \boldsymbol{\tau}^z_{j+\frac{1}{2}} \right\rangle \simeq \mathbf{Z}_j$, which is the spin flipper on the paramagnetic vacuum.

Now let us consider the algebra of these $\boldsymbol{\chi}$s:

- They are *real*: $\boldsymbol{\chi}_j^\dagger = \boldsymbol{\chi}_j, \tilde{\boldsymbol{\chi}}_j^\dagger = \tilde{\boldsymbol{\chi}}_j$.

and

- They are *fermions*:

$$\text{if } i \neq j, \ \boldsymbol{\chi}_j \boldsymbol{\chi}_i + \boldsymbol{\chi}_i \boldsymbol{\chi}_j \equiv \{\boldsymbol{\chi}_j, \boldsymbol{\chi}_i\} = 0, \quad \{\tilde{\boldsymbol{\chi}}_j, \tilde{\boldsymbol{\chi}}_i\} = 0, \quad \{\boldsymbol{\chi}_j, \tilde{\boldsymbol{\chi}}_i\} = 0. \tag{4.8}$$

This is because the spin flip $\mathbf{Z}_j$ in $\boldsymbol{\chi}_j$ changes sign when it moves through the domain wall created by $\boldsymbol{\chi}_i$. When they are at the same site:

$$\boldsymbol{\chi}_j^2 = 1 = \tilde{\boldsymbol{\chi}}_j^2. \quad \text{In summary:} \quad \boxed{\{\boldsymbol{\chi}_i, \boldsymbol{\chi}_j\} = 2\delta_{ij}, \ \{\tilde{\boldsymbol{\chi}}_i, \tilde{\boldsymbol{\chi}}_j\} = 2\delta_{ij},}.$$

Notice that (4.8) means that $\boldsymbol{\chi}_i$ cares about $\boldsymbol{\chi}_j$ even if $|i-j| \gg 1$. Fermions are weird and non-local!

Notice that these fermions don't have a spin index. I should mention that this change of variables is called *Jordan-Wigner*.

Real fermion operators like this are called *Majorana* fermion operators. We can make more familiar-looking objects by making complex combinations:

$$\mathbf{c}_j \equiv \frac{1}{2}\left(\boldsymbol{\chi}_j - \mathbf{i}\tilde{\boldsymbol{\chi}}_j\right) \qquad \Longrightarrow \quad \mathbf{c}_j^\dagger = \frac{1}{2}\left(\boldsymbol{\chi}_j + \mathbf{i}\tilde{\boldsymbol{\chi}}_j\right)$$

These satisfy the more familiar anticommutation relations:

$$\{\mathbf{c}_i, \mathbf{c}_j^\dagger\} = \delta_{ij}, \quad \{\mathbf{c}_i, \mathbf{c}_j\} = 0, \quad \{\mathbf{c}_i^\dagger, \mathbf{c}_j^\dagger\} = 0,$$

and in particular, $\left(\mathbf{c}_i^\dagger\right)^2 = 0$, like a good fermion creation operator should.

We can write $\mathbf{H}_{\text{TFIM}}$ in terms of the fermion operators. We need to know how to write $\mathbf{X}_j$ and $\mathbf{Z}_j\mathbf{Z}_{j+1}$.

The operator which *counts* spin flips in the paramagnetic phase is

$$\mathbf{X}_j = -\mathbf{i}\tilde{\boldsymbol{\chi}}_j \boldsymbol{\chi}_j = -2\mathbf{c}_j^\dagger \mathbf{c}_j + 1 = (-1)^{\mathbf{c}_j^\dagger \mathbf{c}_j}.$$

(To get this we can use (4.7) and $\mathbf{YZ} = \mathbf{iX}$ and $\left(\boldsymbol{\tau}^z\right)^2 = 1$. Even better: notice that $\tilde{\boldsymbol{\chi}}_j = +\mathbf{i}X_j\boldsymbol{\chi}_j$.) Here $\mathbf{c}_j^\dagger \mathbf{c}_j = \mathbf{n}_j$ measures the number of fermions at the site j and is either 0 or 1, since they are fermions. At each site

$$|{\rightarrow_j}\rangle = |n_j = 0\rangle\,, \qquad |{\leftarrow_j}\rangle = |n_j = 1\rangle$$

like in the one-mode case, (4.6). The number of spin flips is the number of fermions.

The operator which counts domain walls is

$$\mathbf{Z}_j\mathbf{Z}_{j+1} = \mathbf{i}\tilde{\boldsymbol{\chi}}_{j+1}\boldsymbol{\chi}_j.$$

(Check: $\mathbf{i}\tilde{\boldsymbol{\chi}}_{j+1}\boldsymbol{\chi}_j = \mathbf{i}\mathbf{Y}_{j+1}\prod_{k\geq j+2}\mathbf{X}_k\mathbf{Z}_j\prod_{l\geq j+1}\mathbf{X}_l = \left(\underbrace{\mathbf{i}\mathbf{Y}_{j+1}\mathbf{X}_{j+1}}_{=\mathbf{Z}_{j+1}}\right)\mathbf{Z}_j$.)

$$\mathbf{H}_{\text{TFIM}} = -J\sum_j\left(\mathbf{i}\tilde{\boldsymbol{\chi}}_{j+1}\boldsymbol{\chi}_j + g\mathbf{i}\boldsymbol{\chi}_j\tilde{\boldsymbol{\chi}}_j\right)$$

is quadratic in these variables, for any g! Free at last! (It is quadratic in the $\mathbf{c}$s, too, since they are linear in the $\boldsymbol{\chi}$s.)

Comments:

- Notice that the relation
$$\mathbf{X}_j = 1 - 2\mathbf{c}_j^\dagger\mathbf{c}_j$$
is exactly implementing the simple idea that *spinless* fermions on a lattice produce two-states per site which we can regard as spin up or spin down (in this case it's up or down along x): The states $\mathbf{X} = \pm 1$ correspond to $\mathbf{c}^\dagger\mathbf{c} = 0$ and 1 respectively.

- Notice that the description in terms of majoranas is preferred over the complex fermions because the phase rotation symmetry generated by the fermion number $\mathbf{c}^\dagger\mathbf{c}$ is not a symmetry of $\mathbf{H}_{\text{TFIM}}$ – in terms of the $\mathbf{c}$s, it contains terms of the form $\mathbf{c}_j\mathbf{c}_{j+1}$ which change the total number of $\mathbf{c}$ fermions (by ± 2). It is the hamiltonian for a *superconductor*, in which the continuous fermion number symmetry is broken down to a $\mathbb{Z}_2$ subgroup. Fermion number is still conserved mod two, and this is the $\mathbb{Z}_2$ symmetry of the Ising model, which acts by $\mathbf{Z} \to -\mathbf{Z}$.

- A useful thing to remember about majorana operators $\{\chi, \gamma\} = 0$ is that $(\mathbf{i}\chi\gamma)^\dagger = -\mathbf{i}\gamma\chi = +\mathbf{i}\chi\gamma$ is hermitian.

- Another useful fact:
$$\mathbf{c}(-1)^{\mathbf{c}^\dagger\mathbf{c}} = -\mathbf{c} \tag{4.9}$$
which is true because the BHS only nonzero if the number is nonzero *before* the annihilation operator acts, in which case we get $(-1)^1$. Similarly (the conjugate equation), $(-1)^{\mathbf{c}^\dagger\mathbf{c}}\mathbf{c}^\dagger = -\mathbf{c}^\dagger$, and $(-1)^{\mathbf{c}^\dagger\mathbf{c}}\mathbf{c} = \mathbf{c}$ and so on.

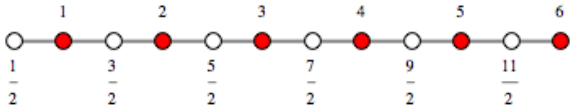
- This procedure of "attaching spin to a domain wall" led to fermions. This maybe isn't so surprising in one dimension. But there are analogs of this procedure

in higher dimensions. In 2+1 dimensions, an analog is to attach charge to a vortex (or to attach magnetic flux to charge). This leads to transmutation of statistics from bosons to fermions and more generally to anyons and the fractional quantized Hall effect. In 3+1 dimensions, an analog is attaching charge to a magnetic monopole to produce a 'dyon'; in this case, the angular momentum carried by the EM fields is half-integer.

Dual fermions

Let

$$\boldsymbol{\gamma}_{j+\frac{1}{2}} = -\tilde{\boldsymbol{\chi}}_{j+1}, \quad \tilde{\boldsymbol{\gamma}}_{j+\frac{1}{2}} = \boldsymbol{\chi}_j.$$



We are dividing up the pair of majorana modes that we had previously associated with the site j between the two neighboring links.

In terms of these $\mathbf{H}$ is

$$\mathbf{H}_{\text{TFIM}} = -J \sum_{\bar{j}=j+\frac{1}{2}} \left(\mathbf{i}\tilde{\boldsymbol{\gamma}}_{\bar{j}}\boldsymbol{\gamma}_{\bar{j}} + g\mathbf{i}\tilde{\boldsymbol{\gamma}}_{\bar{j}+1}\boldsymbol{\gamma}_{\bar{j}} \right).$$

We have interchanged the form of the two terms. The two expressions are related by $\boldsymbol{\chi} \leftrightarrow \boldsymbol{\gamma}, J \leftrightarrow Jg, g \leftrightarrow 1/g$.

Now let's think about the two phases in terms of the fermion fields.

$\boxed{g \gg 1}$ The disordered state, with a unique groundstate is governed by

$$\mathbf{H}_{g\to\infty} = -Jg \sum_j \mathbf{i}\boldsymbol{\chi}_j\tilde{\boldsymbol{\chi}}_j = -Jg \sum_j (-1)^{\mathbf{c}_j^\dagger \mathbf{c}_j}.$$

The groundstate of this is just the state $|0\rangle$ with no fermions, $\mathbf{c}_j|0\rangle = 0$ for all j, on which $(-1)^{\mathbf{c}_j^\dagger \mathbf{c}_j} = 1$.

$\boxed{g \ll 1}$

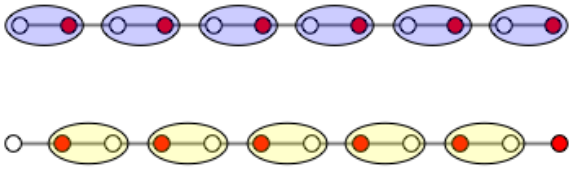
$$\mathbf{H}_{g\to 0} = -J \sum_j \mathbf{i}\tilde{\boldsymbol{\chi}}_{j+1}\boldsymbol{\chi}_j.$$

This is a hopping term for the χ fermions, which will be diagonalized by some band-structure, which will then be partially filled by fermions. It's easier just to use the dual fermion variables, in terms of which

$$\mathbf{H}_{g\to 0} = -J \sum_{\bar{j}=j+\frac{1}{2}} \mathbf{i}\tilde{\boldsymbol{\gamma}}_{\bar{j}}\boldsymbol{\gamma}_{\bar{j}} = -J \sum_{\bar{j}=j+\frac{1}{2}} (-1)^{\check{\mathbf{c}}_{\bar{j}}^\dagger \check{\mathbf{c}}_{\bar{j}}}$$

where $\check{\mathbf{c}}_{\bar{j}} \equiv \frac{1}{2}\left(\boldsymbol{\gamma}_{\bar{j}} - \mathbf{i}\tilde{\boldsymbol{\gamma}}_{\bar{j}}\right)$ are the dual complex fermions. The groundstate of this is just the state with no dual fermions: $\check{\mathbf{c}}_{\bar{j}}\left|\check{0}\right\rangle = 0, \forall \bar{j}$.

Again we should think about the endpoint conditions. Consider what happens for an open chain. It's useful to account for the states as follows: think of dividing up each site into a pair of sites (white and black, as in the figure, one from the lattice, one from the dual lattice) with one majorana mode living at each. To help visualize, let's say $\tilde{\boldsymbol{\chi}}_j = -\boldsymbol{\gamma}_{j-\frac{1}{2}}$ lives at the white site (at $j-\frac{1}{2}$) and $\boldsymbol{\chi}_j = \tilde{\boldsymbol{\gamma}}_{j+\frac{1}{2}}$ lives at the red site to its right (at j). Note that $\boldsymbol{\gamma}_{\frac{1}{2}} = -\tilde{\boldsymbol{\chi}}_1, \tilde{\boldsymbol{\gamma}}_{\frac{1}{2}} = \boldsymbol{\chi}_0$. The g (para) term is a coupling between the white and red sites at fixed j. The ferro term couples a red site at $j-\frac{1}{2}$ to the next white site at j. This means that at small g, in an open chain, one site at each end will be *left out* – will not appear in the hamiltonian at all. This results in *unpaired* majorana zeromodes at the ends of the chain. Let $\mathbf{a}^\dagger \equiv \frac{1}{2}\left(\mathbf{i}\tilde{\boldsymbol{\chi}}_1 + \boldsymbol{\chi}_N\right)$. The algebra $\{\mathbf{a}, \mathbf{a}^\dagger\} = 1$ must be represented on the groundstates. This requires a pair of states



$$|0\rangle = |+\rangle\,, \;\; \mathbf{a}^\dagger\,|0\rangle = |-\rangle\,.$$

The SSB degeneracy of the ferromagnet is mapped by this fermionization map to a *topological* degeneracy in terms of the fermions.

Splitting of the energy these two states is small because the modes are separated in space and the bulk is gapped. The modes are separated by a distance much bigger than the correlation length, $\xi \sim \frac{1}{\Delta}$. The splitting comes from a term

$$\Delta\mathbf{H} = \epsilon\mathbf{a}^\dagger\mathbf{a} = \epsilon\mathbf{i}\tilde{\boldsymbol{\chi}}_1\boldsymbol{\chi}_N$$

and we (again) estimate that $\epsilon \sim e^{-N/\xi}$.

4.1.2 Continuum limit

[Sachdev, p. 139 of 2nd ed, p. 48 of 1st ed] We found above that a quantum lattice model (with continuous time and discrete space) can be Trotterized into a lattice statistical mechanics model (with discrete (euclidean) time and discrete space). We also found a reverse map, via the continuum scaling limit. In this subsection we discuss the analogous scaling limit that takes a quantum lattice model to a continuum quantum field theory (with continuous time *and* continuous space).

Label site j of the lattice by its position $x_j = ja$, where a is the lattice spacing. $L = Na$ is the length of the chain. Wavevectors lie in the interval $k \in (-\frac{\pi}{a}, \frac{\pi}{a}]$.

In terms of complex fermions

$$\mathbf{X}_j = 1 - 2\mathbf{c}_j^\dagger \mathbf{c}_j, \quad \mathbf{Z}_j = -\prod_{i>j}(1-2\mathbf{c}_i^\dagger \mathbf{c}_i)\left(\mathbf{c}_j + \mathbf{c}_j^\dagger\right) = -\prod_{i>j}(-1)^{\mathbf{c}_i^\dagger \mathbf{c}_i}\left(\mathbf{c}_j + \mathbf{c}_j^\dagger\right).$$

and their Fourier modes $\mathbf{c}_k \equiv \frac{1}{\sqrt{N}}\sum_j \mathbf{c}_j e^{-\mathbf{i}kx_j}$, the TFIM hamiltonian is

$$\mathbf{H}_{\text{TFIM}} = J\sum_k \left(2(g-\cos ka)\mathbf{c}_k^\dagger \mathbf{c}_k - \mathbf{i}\sin ka\left(\mathbf{c}_{-k}^\dagger \mathbf{c}_k^\dagger + \mathbf{c}_{-k}\mathbf{c}_k\right) - g\right)$$

Notice that the second set of terms violates fermion number conservation in units of two; this is the same statement that the ferromagnetic term creates spin flips in pairs. The constant at the end is irrelevant unless we are after the Onsager free energy.

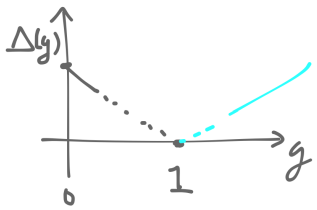
This Hamiltonian is quadratic in $\mathbf{c}_k$s, but not quite diagonal. The final solution for the spectrum involves one more operation the fancy name for which is 'Bogoliubov transformation', which is the introduction of new (complex) mode operators which mix particles and holes:

$$\boldsymbol{\gamma}_k = u_k \mathbf{c}_k - \mathbf{i}v_k \mathbf{c}_{-k}^\dagger$$

Demanding that the new variables satisfy canonical commutators $\{\boldsymbol{\gamma}_k, \boldsymbol{\gamma}_{k'}^\dagger\} = \delta_{k,k'}$ requires $u_k = \cos(\phi_k/2), v_k = \sin(\phi_k/2)$. We fix the angles ϕ_k by demanding that the hamiltonian in terms of $\boldsymbol{\gamma}_k$ be diagonal – no $\boldsymbol{\gamma}_k\boldsymbol{\gamma}_{-k}$ terms. The resulting condition is $\tan\phi_k = \frac{\epsilon_2(k)}{\epsilon_1(k)}$ with $\epsilon_1(k) = 2J(g-\cos ka), \epsilon_2(k) = -J\sin ka$, and $\mathbf{H} = \sum_k \epsilon_k\left(\boldsymbol{\gamma}_k^\dagger \boldsymbol{\gamma}_k - \frac{1}{2}\right)$, with $\epsilon_k = \sqrt{\epsilon_1^2 + \epsilon_2^2}$.

The end result is that the exact single-particle (single $\boldsymbol{\gamma}$) dispersion is

$$\epsilon_k = 2J\sqrt{1+g^2-2g\cos ka}\ .$$



The argument of the sqrt is positive for $g \geq 0$. This is minimized at $k=0$, which tells us the exact gap at all g:

$$\epsilon_k \geq \epsilon_0 = 2J|1-g| = \Delta(g)$$

which, ridiculously, is just what we got from 1st order perturbation theory on each side of the transition.[20]

[20] Notice by the way that this 'single-particle excitation' in terms of the spins is a state with a single domain wall, and therefore only makes sense in the thermodynamic limit; in finite volume, the spin hilbert space only contains states with an even number of fermions. See §4.1.3 below for more on this point.

Comment on relation to Onsager. Notice that by the quantum-to-classical mapping, this solution has all the universal information of Onsager's solution. In particular, his exact free energy can be obtained by computing the groundstate energy of the fermions. (See the homework.)

[End of Lecture 13]

The critical theory is scale invariant. At $g \to 1$, the fermions at $k = 0$ become gapless:

$$\epsilon_k \overset{g=g_c=1}{=} c|k|,$$

where the speed of propagation is $c = 2Ja$. There is an emergent Lorentz symmetry, at least in the spectrum, at the critical point. And the fact that the long-wavelength ($k \sim 0$) modes have the lowest energy allows a continuum description that forgets the lattice details.

Near $g \to g_c = 1$ (*i.e.* $|g - g_c| \ll g_c$),

$$\epsilon_k \overset{g\sim g_c=1}{=} c\sqrt{k^2 + \underbrace{\left(\frac{g-g_c}{a}\right)^2}_{m^2\to 0}}$$

There is a diverging length scale, which is $\xi = \frac{1}{m} = \frac{a}{|g-g_c|}$.

Some useful language:

- The *correlation length critical exponent* is ν in $\xi \sim |g - g_c|^{-\nu}$, the rate at which the correlation length diverges near the critical point. (Here $\nu = 1$.)
- The *dynamical critical exponent* is the power law in the dispersion *at* the critical point:

  $$\epsilon(k) \propto k^z. \tag{4.10}$$

  Notice that if we rescale space and time like

  $$x \to \lambda x, \quad t \to \lambda^z t \tag{4.11}$$

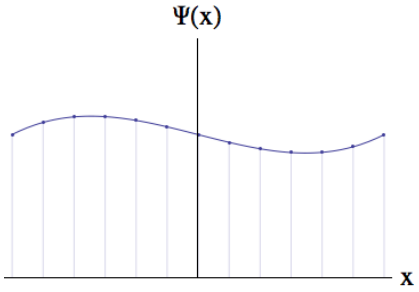
  it will preserve the dispersion (4.10). (Here $z = 1$, as in a relativistic theory where time and space can be boosted into each other.)

We can expect the continuum description (at least the long-wavelength information) to be invariant under the scale transformation (4.11).

When $\xi \gg a$, define the continuum fermion field

$$\mathbf{\Psi}(x_j) \equiv \frac{1}{\sqrt{a}}\mathbf{c}_j$$

where the factor of $\sqrt{a}$ is designed to convert the kronecker deltas in



$$\{\mathbf{c}_j, \mathbf{c}^\dagger_{j'}\} = \delta_{jj'} \quad \text{into} \quad \{\mathbf{\Psi}(x), \mathbf{\Psi}^\dagger(x')\} = \delta(x-x').$$

The fourier transform is

$$\mathbf{c}_k = \int dx \frac{e^{-\mathbf{i}kx}}{\sqrt{L}} \mathbf{\Psi}(x) \ .$$

Let's plug this into $\mathbf{H}_{\text{TFIM}}$, focussing on (the lightest) modes with $ka \ll 1$.

$$J\sum_k (g - \cos ka)\mathbf{c}^\dagger_k \mathbf{c}_k \rightsquigarrow (g-g_c)\int dx\ \mathbf{\Psi}^\dagger(x)\mathbf{\Psi}(x)$$

Here the $\rightsquigarrow$ means we are dropping terms of higher order in the expansion in powers of $a\partial_x$.

$$-\mathbf{i}J\sum_k \sin ka \mathbf{c}^\dagger_{-k}\mathbf{c}^\dagger_k \rightsquigarrow \frac{c}{2}\int dx\ \mathbf{\Psi}(x)^\dagger \partial_x \mathbf{\Psi}(x)^\dagger$$

So

$$\mathbf{H} \rightsquigarrow \frac{c}{2}\int dx\ \left(\mathbf{\Psi}(x)^\dagger\partial_x\mathbf{\Psi}(x)^\dagger - \mathbf{\Psi}(x)\partial_x\mathbf{\Psi}(x)\right) + \Delta\int dx \mathbf{\Psi}^\dagger\mathbf{\Psi}$$

with $\Delta = 2J|g-1|$.

[Fradkin, 2d ed p. 119] To understand the significance of this result it will be best to use the (fermion coherent state) path integral, which we'll do below. But we can learn something just by thinking about the Heisenberg equations of motion.

$$\begin{aligned}\mathbf{H}_{\text{TFIM}} &= -J\sum_j (g\mathbf{X}_j + \mathbf{Z}_j\mathbf{Z}_{j+1}) \\ &= -\mathbf{i}J\sum_j (g\boldsymbol{\chi}(j)\tilde{\boldsymbol{\chi}}(j) - \boldsymbol{\chi}(j)\tilde{\boldsymbol{\chi}}(j+1)) \ . \end{aligned} \tag{4.12}$$

Notice that the $\mathbb{Z}$ symmetry action on the fermions is

$$\mathbf{S} = \prod_j \mathbf{X}_j = \mathbf{i}^N \prod_j \boldsymbol{\chi}(j)\tilde{\boldsymbol{\chi}}(j).$$

The Heisenberg eom are $\mathbf{i}\partial_t\mathcal{O} = [\mathbf{H}, \mathcal{O}]$:

$$\mathbf{i}\partial_t\boldsymbol{\chi}(j) = \mathbf{i}J\left(g\tilde{\boldsymbol{\chi}}(j) - \tilde{\boldsymbol{\chi}}(j+1)\right)$$

$$\mathbf{i}\partial_t\tilde{\boldsymbol{\chi}}(j) = \mathbf{i}J\left(-g\boldsymbol{\chi}(j) + \boldsymbol{\chi}(j-1)\right) \tag{4.13}$$

(Free fields means linear equations of motion.) In the continuum,

$$\boldsymbol{\chi}(j+1) \simeq \boldsymbol{\chi}(x_j) + a\partial_x\boldsymbol{\chi}(x_j) + \mathcal{O}(a^2\partial_x^2)$$

so

$$\begin{aligned}\frac{1}{aJ}\partial_t\boldsymbol{\chi}(x) &\simeq -\left(\frac{1-g}{aJ}\right)\tilde{\boldsymbol{\chi}}(x) - \partial_x\tilde{\boldsymbol{\chi}}(x)\\ \frac{1}{aJ}\partial_t\tilde{\boldsymbol{\chi}}(x) &\simeq \left(\frac{1-g}{aJ}\right)\boldsymbol{\chi}(x) - \partial_x\boldsymbol{\chi}(x)\end{aligned} \tag{4.14}$$

If we let $\boldsymbol{\chi}_\pm = \frac{1}{2}\left(\boldsymbol{\chi} \mp \tilde{\boldsymbol{\chi}}\right)$ and rescale the speed of propagation into the time variable, $t \equiv aJx^0$,

$$\begin{aligned}\partial_0\boldsymbol{\chi}_+ &= \partial_x\boldsymbol{\chi}_+ + m\boldsymbol{\chi}_-\\ \partial_0\boldsymbol{\chi}_- &= -\partial_x\boldsymbol{\chi}_- - m\boldsymbol{\chi}_+\end{aligned} \tag{4.15}$$

with $m = \frac{1-g}{a}$. When $m \to 0$, at the critical point, these are *chiral* fermions:

$$\left(\partial_0 \mp \partial_x\right)\boldsymbol{\chi}_\pm = 0.$$

The $\boldsymbol{\chi}$s are still real, so these are majorana fermions. In a language we will introduce soon, this is an example of a 1+1d CFT with central charge $c = (\frac{1}{2}, \frac{1}{2})$.

Even a bit away from the critical point, we can reconstruct the relativistic symmetry, and organize (4.15) into following Dirac equation

$$0 = \mathbf{i}\gamma^\mu\partial_\mu\boldsymbol{\chi} + \mathbf{i}m\boldsymbol{\chi},$$

Notice that a spin index has emerged to save the spin-statistics theorem. Here we have chosen a basis of gamma matrices where

$$\gamma^0 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

(in the $\pm$ basis) which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ as they should. So the Dirac equation above is

$$0 = \mathbf{i}\begin{pmatrix}m & \partial_0 + \partial_x\\ -\partial_0 + \partial_x & m\end{pmatrix}\begin{pmatrix}\boldsymbol{\chi}_+\\ \boldsymbol{\chi}_-\end{pmatrix}.$$

(The Majorana condition I am imposing is $\bar{\boldsymbol{\chi}} = \boldsymbol{\chi}^T\gamma^0$, aka $\boldsymbol{\chi}^\dagger = \boldsymbol{\chi}^T$, so $\boldsymbol{\chi}_\pm$ are real.)

4.1.3 Duality and Jordan-Wigner with periodic boundary conditions

How do I define the creation operator for a single domain wall with periodic boundary conditions on the spins? I can't. The fact that I can only make an even number means that the *sign* of the operator is not well-defined – in fact you can't measure the single-domain wall operator. And since the fermions $\boldsymbol{\chi}$ are made from this operator, you can't measure a single fermion operator either.

Although there is a sign ambiguity, we can still define the domain wall operators:

$$\boldsymbol{\tau}^x_{\bar{j}} = \mathbf{Z}_{\bar{j}-\frac{1}{2}}\mathbf{Z}_{\bar{j}+\frac{1}{2}}, \quad \boldsymbol{\tau}^z_{\bar{j}} = \prod_{N\geq j>\bar{j}} \mathbf{X}_j$$

– in the second definition, we have made a choice: even though $\mathbf{Z}_{N+1} = \mathbf{Z}_1$, we've decided to pick out the link where $\bar{j} = N + \frac{1}{2}$ to stop the string of $\mathbf{X}$s. As we saw in the case of open boundary conditions, this means that $\boldsymbol{\tau}_{\frac{1}{2}}$ and $\boldsymbol{\tau}_{N+\frac{1}{2}}$ are special:

$$\boldsymbol{\tau}^z_{N+\frac{1}{2}} = 1$$

since the product is over the empty set.

$$\boldsymbol{\tau}^z_{\frac{1}{2}} = \prod_{j=1}^{N} \mathbf{X}_j = \mathbf{S}$$

is the symmetry generator. This is significant because this operator commutes with $\mathbf{H}_{TFIM}$, it means that eigenstates of $\boldsymbol{\tau}^z_{\frac{1}{2}}$ are energy eigenstates. And notice that it determines the boundary conditions on the $\boldsymbol{\tau}$s in following sense: combining the previous two equations,

$$1 = \boldsymbol{\tau}^z_{N+\frac{1}{2}} = \begin{cases} \boldsymbol{\tau}^z_{\frac{1}{2}} & \text{if } \prod_j \mathbf{X}_j = \mathbf{S} = 1 \\ -\boldsymbol{\tau}^z_{\frac{1}{2}} & \text{if } \prod_j \mathbf{X}_j = \mathbf{S} = -1 \end{cases}$$

This operator $\boldsymbol{\tau}^z_{\frac{1}{2}}$ which says whether the $\boldsymbol{\tau}$s are periodic or antiperiodic is a dynamical variable.

4.1.4 Scaling near the fixed-point theory of the Ising phase transition.

Armed with the fermion coherent state path integral, let's go back to the continuum hamiltonian describing the TFIM near $g = 1$:

$$\mathbf{H} \rightsquigarrow \frac{c}{2}\int dx \ \left(\boldsymbol{\Psi}(x)^\dagger \partial_x \boldsymbol{\Psi}(x)^\dagger - \boldsymbol{\Psi}(x)\partial_x \boldsymbol{\Psi}(x)\right) + \Delta \int dx \boldsymbol{\Psi}^\dagger \boldsymbol{\Psi} \equiv \int dx \ \mathfrak{h}$$

with $\Delta = 2J|g-1|$.

The euclidean-time action is $S[\Psi, \Psi^\dagger] = \int d\tau \int dx \mathcal{L}$, with

$$\begin{aligned}\mathcal{L} &= \bar{\Psi}\partial_\tau \Psi + \mathfrak{h} \\ &= \bar{\Psi}\partial_\tau \Psi + \frac{c}{2}\left(\bar{\Psi}\partial_x \bar{\Psi} - \Psi \partial_x \Psi\right) + \Delta \bar{\Psi}\Psi. \end{aligned} \tag{4.16}$$

Here I have simply replaced the field operators $\mathbf{\Psi}(x), \mathbf{\Psi}^\dagger(x)$ with their (right and left, respectively) grassmann eigenvalues $\Psi(x), \bar{\Psi}$. This lagrangian is more appealing in terms of the self-conjugate variables $\psi_\pm$:

$$\Psi = \psi_+ + \psi_- + \mathbf{i}\left(\psi_+ - \psi_-\right), \quad \bar{\Psi} = \psi_+ + \psi_- - \mathbf{i}\left(\psi_+ - \psi_-\right)$$

$$\mathcal{L} = \sum_\pm \psi_\pm \left(\partial_\tau \pm \mathbf{i}\partial_x\right)\psi_\pm + \Delta \psi_+ \psi_-.$$

If we let $z \equiv x + \mathbf{i}\tau$, and turn off the mass perturbation, this is the third avatar of the Ising model I mentioned at the beginning in (4.1).

Let's analyze its behavior under scale transformations. In order to make S invariant, we must scale time and space the same way ($z = 1$), and we must scale

$$x \to \lambda x, t \to \lambda t, \Psi \to \lambda^{-\frac{1}{2}}\Psi.$$

So just by scaling, we can see that its correlators behave as

$$\left\langle \mathbf{\Psi}(x)^\dagger \mathbf{\Psi}(0)\right\rangle \sim \frac{1}{x}.$$

(Since it's a gaussian theory, you can also calculate the coefficient.) More generally, the scaling dimension δ of an operator $\mathcal{O}(x)$ in $\mathcal{O} \to \lambda^{-\delta}\mathcal{O}$ in a scale-invariant QFT determines its vacuum autocorrelation functions to be

$$\left\langle \mathcal{O}(x)^\dagger \mathcal{O}(0)\right\rangle \sim \frac{1}{x^{2\delta}}.$$

(Later we will interpret δ as an eigenvalue of the dilatation operator.)

The mass perturbation Δ will violate the scale invariance, since

$$\int dx d\tau \bar{\Psi}\Psi \to \lambda^{+1} \int dx d\tau \bar{\Psi}\Psi \ .$$

As we make lengths and times bigger, it gets bigger – it's a *relevant* perturbation, which determines the behavior at long distances. The scaling dimension of this relevant perturbation determines the correlation length critical exponent: the correlation length is a length scale $\xi = a\lambda$ at which the relevant coupling $\Delta = \lambda^\nu \Delta_0 \sim 1$ has evolved

to an order-one value (Δ_0 is its 'bare' value, at the lattice scale a, and ν is its scaling dimension). Eliminating $\lambda = \xi/a$ from these equations, we get

$$1 = (\xi/a)^\nu \Delta_0 \implies \xi \sim \frac{1}{\Delta_0^\nu},$$

so ν, the scaling dimension of the relevant operator, is indeed the correlation length critical exponent defined above.

Let's ask whether there are any other *relevant* operators, in the sense that they scale to larger values at larger λ (longer wavelengths). If there are more it means that our fixed point is *multicritical* – we would have to tune more than one parameter to reach it. We're going to demand that the Ising symmetry is preserved; this is the fermion number symmetry, $\Psi(x) \to -\Psi(x)$. So we should only consider operators made of an even number of fermions. And all the other operators we can make out of an even number of Ψ are *irrelevant*, in the sense that they scale to smaller values at larger λ. The marginal-looking operator $\bar\Psi\Psi\bar\Psi\Psi$ vanishes by Fermi statistics. Other operators have more derivatives have no chance:

$$\int dx d\tau \bar\Psi \partial_x^2 \Psi \sim \lambda^{-1}$$

$$\int dx d\tau \bar\Psi \partial_x \Psi \bar\Psi \partial_x \Psi \sim \lambda^{-2}$$

This means we have to tune only one parameter ($g - g_c \sim \Delta$) to get to the critical theory. (Since there are no classically-marginal operators, there is no need to do any perturbating to determine the effects of turning them on a little bit – small quantum corrections cannot overcome the order-1 amounts by which these operators are irrelevant.)

There *are* operators of dimension less than two, namely the operator $\mathbf{Z}$ itself; this is an operator which creates a *branch cut* in the fermions, which after all are made from the domain wall operators. To understand its dimension, it is useful to use a bosonized description, which we will do later (I hope). Its scaling dimension turns out to be 1/8. This is important because it determines the rate at which the magnetization vanishes as $g \to g_c$ from below. It's important that the $\mathbb{Z}_2$ symmetry forbids us from adding this (relevant) operator to the action. So in the presence of the symmetry, the Ising fixed point has only one relevant perturbation (the fermion mass).

Notice how different this critical theory is from the guess would have made from mean field theory: we would have guessed that the critical fluctuations should be those of the mean field θ, which when I set $g = g_c$ becomes a massless free boson, with many relevant operators which preserve the $\theta \to -\theta$ symmetry, since demanding scale

invariance of

$$S_0[\theta] = \int dx d\tau \ \left((\partial_\tau \theta)^2 + (\partial_x \theta)^2 \right)$$

implies that θ is dimensionlesss. So all even powers $\theta^2, \theta^4 ...$ (in fact, all even functions, like $\cos\theta$) have a chance to be relevant. We'll see that it's *not* actually quite so far off. (And in fact is the correct variables in $d > 1$, and the correct fixed point for $d \geq 3$ (the upper critical dimension).)

The nontrivial scaling of $\mathbf{Z}$ is an example of an *anomalous dimension*: the engineering dimension is different from the correct scaling behavior. Dimensional analysis is inviolable. What has happened is that powers of the short-distance scales are involved; that is: it is $J^{1/8}\mathbf{Z}$ (or $a^{-1/8}\mathbf{Z}$) that has finite correlation functions in the scaling limit. In relativistic QFT this phenomenon is called 'wavefunction renormalization' – we must multiplicatively renormalize our operators to get finite (not zero, not infinity) correlators in the long-wavelength limit.

4.1.5 Beyond the quantum Ising chain

The Jordan-Wigner transformation is useful in spin chains with many other choices of Hamiltonian and symmetry group, as you'll see on the problem set, and below.

4.1.6 Gauge theory formulation of the 1d duality

There is a systematic way to understand what's going on with the boundary conditions under the duality transformation, which generalizes to higher dimensions (but it is kind of overkill in this case). It is a reformulation of the TFIM as a lattice gauge theory. This is worth doing because in duality transformations in higher dimensions, the gauge theory formulation cannot be avoided. So this is a context where we can encounter the concepts of gauge theory without lots of other complications.

We begin by enlarging our system, by adding, in addition to the real qbits on the sites of the chain another set of fake qbits on the links:

$$\text{enlarge } \mathcal{H} = \otimes_j \mathcal{H}_j \quad \subset \quad \mathcal{H}_{\text{big}} \equiv (\otimes_j \mathcal{H}_j) \otimes \left(\otimes_{\bar{j}} \check{\mathcal{H}}_{\bar{j}}\right) \ .$$

Anticipating the answer, I will label the two states on a link $\bar{j}$ as $\left|\tau^z_{\bar{j}}\right\rangle = |\pm 1\rangle$. So as for any two-state system, associated with each link there are pauli operators

$$\boldsymbol{\tau}^z_{\bar{j}} \left|\tau^z_{\bar{j}}\right\rangle = \tau^z_{\bar{j}} \left|\tau^z_{\bar{j}}\right\rangle, \quad \boldsymbol{\tau}^x_{\bar{j}} \left|\tau^z_{\bar{j}}\right\rangle = \left|-\tau^z_{\bar{j}}\right\rangle \ , \ \boldsymbol{\tau}^z_{\bar{j}} \boldsymbol{\tau}^x_{\bar{j}'} = (-1)^{\delta_{\bar{j}\bar{j}'}} \boldsymbol{\tau}^x_{\bar{j}} \boldsymbol{\tau}^z_{\bar{j}'} \ .$$

We want to pick out a subspace of this larger fake Hilbert space which is the real one. One way to think about this is to declare that the following 'gauge transformation' is an equivalence:

$$\mathcal{G} : \tau^z_{\bar{j}} \to s_{\bar{j}} \tau^z_{\bar{j}}, \quad s_{\bar{j}} = \pm 1 \ . \tag{4.17}$$

Notice that this group $\mathcal{G}$ is big – we pick an element of $\mathbb{Z}_2$ (*i.e.* a sign) $s_{\bar{j}} = \pm 1$ for each link of the lattice, its order is $|\mathbb{Z}_2|^N = 2^N$. If a configuration of τ^zs is equivalent to its image under this map, we can just pick $s_{\bar{j}} = \tau^z_{\bar{j}}$ and this means that any configuration of τ^zs is equivalent to 1: we can just get rid of the τs and we are back at our original Hilbert space. This is called *unitary gauge*. Below we will implement this more concretely. But first we must decide how the gauge redundancy $\mathcal{G}$ acts on our original variables. It will be useful to choose

$$\mathcal{G} : \mathbf{X}_j \to s_{j-\frac{1}{2}} \mathbf{X}_j s_{j+\frac{1}{2}}, \quad \mathbf{Z}_j \to \mathbf{Z}_j. \tag{4.18}$$

This is the transformation law of a *link variable* in lattice gauge theory. The site of the original lattice is a link of the dual lattice. Let me use notation which emphasizes this:

$$\mathbf{X}_j \equiv \boldsymbol{\sigma}^x_{j-\frac{1}{2}, j+\frac{1}{2}}, \quad \mathbf{Z}_j \equiv \boldsymbol{\sigma}^z_{j-\frac{1}{2}, j+\frac{1}{2}} \ .$$

The *generator* of the gauge transformation is

$$\mathbf{G}_{\bar{j}} = \boldsymbol{\sigma}^z_{\bar{j}-1,\bar{j}} \boldsymbol{\tau}^x_{\bar{j}} \boldsymbol{\sigma}^z_{\bar{j},\bar{j}+1}.$$

What I mean by 'generator' is that the action of the symmetry on states is $|\psi\rangle \to \mathbf{G}\,|\psi\rangle$ and the action on operators is

$$\mathcal{O} \to \mathbf{G}\mathcal{O}\mathbf{G}^\dagger.$$

The operator $\mathbf{G}_{\bar{j}}$ generates the transformation where $s_{\bar{j}} = -1$ and all the other $s_{\bar{j}'} = +1$. Check that this reproduces (4.17) and (4.18).

The physical hilbert space is cut out of $\mathcal{H}_{\text{big}}$ by projecting to eigenstates of these operators with eigenvalue 1:

$$\mathcal{H}_{\text{phys}} = \{|\psi\rangle \in \mathcal{H}_{\text{big}} \text{ such that } \mathbf{G}_{\bar{j}}\,|\psi\rangle = |\psi\rangle\,, \forall j\} \subset \mathcal{H}_{\text{big}}.$$

This is just the statement that physical states don't change under our made-up redundancy[21].

In order for this construction to make sense, our hamiltonian acting on the big hilbert space must be *gauge invariant*. That is, we must have

$$[\mathbf{H}, \mathbf{G}_{\bar{j}}] = 0, \quad \forall \bar{j}.$$

This way, we can simultaneously diagonalize $\mathbf{H}$ and $\mathbf{G}_{\bar{j}}$. A gauge invariant hamiltonian acting on the big hilbert space is:

$$\mathbf{H}_{\text{big}} - J\sum_{\bar{j}} \left(\boldsymbol{\sigma}^z_{\bar{j}-1,\bar{j}}\boldsymbol{\sigma}^z_{\bar{j},\bar{j}+1} + g\boldsymbol{\tau}^z_{\bar{j}}\boldsymbol{\sigma}^x_{\bar{j},\bar{j}+1}\boldsymbol{\tau}^z_{\bar{j}+1}\right) \tag{4.19}$$

In unitary gauge, $\boldsymbol{\tau}^z = 1$, we can erase the $\boldsymbol{\tau}^z$s and this reduces to $\mathbf{H}_{\text{TFIM}}$. This is why I chose it.

On the other hand, using the local constraints

$$1 = \mathbf{G}_{\bar{j}} \;\Leftrightarrow\; \boldsymbol{\sigma}^z_{\bar{j}-1,\bar{j}}\boldsymbol{\sigma}^z_{\bar{j},\bar{j}+1} = \boldsymbol{\tau}^x_{\bar{j}} \quad \text{on} \quad \mathcal{H}_{\text{phys}},$$

we have

$$\mathbf{H}_{\text{big}} = -J\sum_{\bar{j}} \left(\boldsymbol{\tau}^x_{\bar{j}} + g\boldsymbol{\tau}^z_{\bar{j}}\boldsymbol{\sigma}^x_{\bar{j},\bar{j}+1}\boldsymbol{\tau}^z_{\bar{j}+1}\right)\ . \tag{4.20}$$

In this last expression, we still have the gauge freedom to play with – too many variables. Let's use it to get rid of the original $\boldsymbol{\sigma}$s. On an open chain, we can use the transformation

$$\boldsymbol{\sigma}^x_{\bar{j},\bar{j}+1} \to s_{\bar{j}}\boldsymbol{\sigma}^x_{\bar{j},\bar{j}+1}s_{\bar{j}+1} \tag{4.21}$$

to set $\boldsymbol{\sigma}^x = 1$ for each $\bar{j}$. In that case, the expression (4.20) precisely reduces to our dual description of the TFIM in terms of domain wall operators, (4.5).

[21] We can call this constraint the 'Gauss' Law constraint'. If you squint at this expression you can recognize it as a lattice version of the divergence of the electric flux coming out of the site $\bar{j}$.

But with periodic boundary conditions, the object

$$\mathbf{W} \equiv \prod_{\bar{j}=1}^{N} \boldsymbol{\sigma}^x_{\bar{j},\bar{j}+1}$$

is a sign which does not change under the transformation (4.21). It is called a *Wilson line* operator. We can't get rid of it. In fact it is the familiar object

$$\mathbf{W} = \prod_j \mathbf{X}_j = \mathbf{S},$$

the $\mathbb{Z}_2$ symmetry charge.

You can see from (4.19) that if $\mathbf{W} = -1$, we have APBCs on $\boldsymbol{\tau}^z$. We can choose a gauge where we get rid of all the $\boldsymbol{\sigma}$s except the first, $\boldsymbol{\sigma}^z_{\frac{1}{2},\frac{3}{2}} = \mathbf{W}$. (This is a manifestation of the fact that we had to label a configuration of the spins by the location of the domain walls plus the value of one spin, which we are again choosing to be the first one.) Then the last link in the chain has an extra factor:

$$\mathbf{H}_{\text{dual}} = -J \sum_{\bar{j}=\frac{1}{2}}^{N-\frac{3}{2}} \left(\boldsymbol{\tau}^x_{\bar{j}} + g\boldsymbol{\tau}^z_{\bar{j}} \boldsymbol{\tau}^z_{\bar{j}+1} \right) + \boldsymbol{\tau}^x_{N-\frac{1}{2}} + \mathbf{W} g \boldsymbol{\tau}_{N-\frac{1}{2}} \boldsymbol{\tau}_{\frac{1}{2}}$$

Gauge theory without metaphysics: Kitaev's toric code in 1d

Since there seemed to be some discontent about the sudden appearance of gauge theory in the previous discussion, I will try to make amends here by giving an example of whence this construction can come. It is a 1d version of Kitaev's toric code. The analogous construction in higher dimensions is extremely important as a demonstration of the emergence of gauge theory from a local lattice model. This silly reduction of it also does that, but gauge theory in 1d is not quite as exciting as it is in higher dimensions.

Suppose I tell you that the following is a high-energy description of a lattice quantum system: The Hilbert space is

$$\mathcal{H}_{\rm big} \equiv (\otimes_j \mathcal{H}_j) \otimes \left(\otimes_{\bar{j}} \check{\mathcal{H}}_{\bar{j}}\right)$$

(nothing is fake, there is no redundancy, no nonsense, this is really the Hilbert space), and the Hamiltonian is

$$\mathbf{H}_{\rm TC} = \mathbf{H}_{\mathbf{G}} + \mathbf{H}_{\rm big}$$

where $\mathbf{H}_{\rm big} = -J \sum_{\bar{j}} \left(\boldsymbol{\sigma}^z_{\bar{j}-1,\bar{j}} \boldsymbol{\sigma}^z_{\bar{j},\bar{j}+1} + g \boldsymbol{\tau}^z_{\bar{j}} \boldsymbol{\sigma}^x_{\bar{j},\bar{j}+1} \boldsymbol{\tau}^z_{\bar{j}+1} \right)$ as before, and the new term relative to the previous discussion is

$$\mathbf{H}_{\mathbf{G}} \equiv -E_{\rm giant} \sum_{\bar{j}} \mathbf{G}_{\bar{j}}$$

with $\mathbf{G}_{\bar{j}} = \boldsymbol{\sigma}^z_{\bar{j}-1,\bar{j}} \boldsymbol{\tau}^x_{\bar{j}} \boldsymbol{\sigma}^z_{\bar{j},\bar{j}+1}$ as before. Here $E_{\rm giant}$ is an energy scale which is a gajillion times bigger than any other energy in the problem ($E_{\rm giant} \gg J \gg gJ$).

A few relevant facts:

$$[\mathbf{H}_{\rm big}, \mathbf{G}_{\bar{j}}] = 0, \qquad [\mathbf{G}_{\bar{j}}, \mathbf{G}_{\bar{j}'}] = 0, \quad \forall \bar{j}, \bar{j}'.$$

This means that we can diagonalize $\mathbf{H}_{\rm big}$ by first diagonalizing $\mathbf{H}_{\mathbf{G}}$ (which is a good idea, since its coefficient is so giant), and further, we can do *that* by diagonalizing each $\mathbf{G}_{\bar{j}}$ at the same time.

So who are these $\mathbf{G}_{\bar{j}}$s? $\mathbf{G}_{\bar{j}}$ is unitary (so should be thought of as enacting a transformation on $\mathcal{H}_{\rm big}$), and, in this example, $\mathbf{G}^2_{\bar{j}} = 1$, it is an element of $\mathbb{Z}_2$, so its eigenvalues are ± 1. For each $\mathbf{G}_{\bar{j}}$, the eigenstate with eigenvalue $+1$ is the low-energy one.

Let us pause to note that this model actually has a *local symmetry*: the symmetry generated by $\mathbf{G}_{\bar{j}}$ acts only on the dofs at or next to $\bar{j}$, and this is a symmetry for any $\bar{j}$. *However*, with the sign of $E_{\rm giant}$ as indicated, all states on which this symmetry

actually acts nontrivially have an enormous energy $\sim E_{\text{giant}}$. This means that we may forget about them and focus on the low-energy subspace of the Hilbert space

$$\mathcal{H}_{\text{phys}} \equiv \{|\psi\rangle \in \mathcal{H}_{\text{big}} \text{ such that } \mathbf{G}_{\bar{j}} |\psi\rangle = |\psi\rangle \, , \forall j\} \subset \mathcal{H}_{\text{big}}.$$

You have seen this before, on the previous page: it is the Hilbert space of the $\mathbb{Z}_2$ gauge theory. We have emerged a gauge theory.

Now let me explain better what I was saying about "choosing a gauge". What I meant is the following. States of $\mathcal{H}_{\text{phys}}$ correspond to *orbits* of the action of $\{\mathbf{G}_{\bar{j}}\}$ on $\mathcal{H}_{\text{big}}$.[22] We can think of the projection to the $\mathbf{G} = 1$ subspace as *choosing a representative of each orbit* of the action of (each) $\mathbf{G}$.

Consider an (oversimplified) example of a single two-state system with $\mathbf{G} = \boldsymbol{\sigma}^z$. In the $\mathbf{X}$-basis, this symmetry action has two orbits:

$$\frac{1}{\sqrt{2}} (|\rightarrow\rangle + |\leftarrow\rangle) = |\uparrow\rangle \quad \text{and} \quad \frac{1}{\sqrt{2}} (|\rightarrow\rangle - |\leftarrow\rangle) = |\downarrow\rangle$$

which are determined by the eigenvalue $\sigma^z = \uparrow, \downarrow$. If we *know* that we only want to keep the state with eigenvalue $+1$ then we can label an orbit by the first term. In this case there is only one orbit, and the projection is one-dimensional. We can describe this by saying that we have used the gauge redundancy to set $\sigma^x = 1$, and simply set $\boldsymbol{\sigma}^x = 1$ wherever it appears in our Hamiltonian.

More generally, we will have some action of a (local in space) group $g \in G$ of the form $\mathbf{U}_g$ on our $\mathcal{H}_{\text{big}}$. This means that $\mathcal{H}_{\text{big}}$ forms a (in general highly reducible) representation of G. The orbits of G are

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} \mathbf{U}_g |g_0\rangle$$

where $|G|$ is the order of the group G, and $|g_0\rangle$ is a reference state (like a highest-weight state in the theory of Lie groups). We may label the orbits by the reference

[22] Given a unitary operator on a Hilbert space and a state $|\psi_0\rangle$ in that space, what I mean by 'the orbit of $|\psi_0\rangle$ under the action of $\mathbf{U}$' is the set of images under repeated action of $\mathbf{U}$:

$$\text{orbit of } |\psi_0\rangle \text{ under action of } \mathbf{U} \equiv \{|\psi_0\rangle \, , \mathbf{U} |\psi_0\rangle \, , \mathbf{U}^2 |\psi_0\rangle \, ...\}$$

(if $\mathbf{U}$ represents a finite group, this contains a finite number of elements; call this order($\mathbf{U}$)). The associated eigenvector of $\mathbf{U}$ with eigenvalue 1 is

$$\frac{1}{\sqrt{\text{order}(\mathbf{U})}} \sum_{l=0}^{\text{order}(\mathbf{U})} \mathbf{U}^l |\psi_0\rangle \, .$$

It is just the average over the group. (For non-abelian groups, we must speak about the orbit of $|\psi_0\rangle$ under the action of the whole group.)

state. And more to the point, states that lie in the *same* orbit correspond to the *same* unit eigenvector, and can be regarded as equivalent. This is the origin of the usual way of thinking about gauge redundancy.

There is much more to say about the toric code in $d > 1$, for which I will teleologically refer to these notes. We have shown that this model reduces at energies $E \ll E_{\text{giant}}$ to the gauge theory description of the TFIM, which we have solved by the Jordan-Wigner transformation. The situation is a little bit different in 2d in that the model is not solvable away from the zero-correlation length limit.

Boundary conditions on fermions

How does this arise in terms of the fermions? Just as for domain walls, the sign of the fermion operator is also not well-defined. With periodic boundary conditions on the spins ($\mathbf{Z}_{j+N} = \mathbf{Z}_j$), we can also define the fermions by putting a *branch cut* at $j = N$, that is, we'll write the Jordan-Wigner formula as $\boldsymbol{\chi}_j = \mathbf{Z}_j \prod_{N \geq i > j} \mathbf{X}_i$, where the string of $\mathbf{X}_i$s stops at N. This is a choice. The inverse map is then

$$\mathbf{Z}_j = \left(\mathbf{c}_j^\dagger + \mathbf{c}_j\right) \prod_{N \geq i > j} e^{\mathbf{i}\pi\mathbf{n}_i} \ . \tag{4.22}$$

The ferromagnetic term in the hamiltonian, for $j \neq N$, is

$$\mathbf{Z}_j\mathbf{Z}_{j+1} = \left(\mathbf{c}_j^\dagger + \mathbf{c}_j\right) e^{\mathbf{i}\pi\mathbf{n}_{j+1}} \left(\mathbf{c}_{j+1}^\dagger + \mathbf{c}_{j+1}\right) = \left(\mathbf{c}_j^\dagger + \mathbf{c}_j\right) \left(-\mathbf{c}_{j+1}^\dagger + \mathbf{c}_{j+1}\right)$$

where I used the identity (4.9). (This is as in our previous discussion.) But with periodic boundary conditions on $\mathbf{Z}$, the term at the branch cut is:

$$\mathbf{Z}_N\mathbf{Z}_1 = \left(\mathbf{c}_N^\dagger + \mathbf{c}_N\right) \left(\mathbf{c}_1^\dagger + \mathbf{c}_1\right) \prod_{N \geq j > 1} e^{\mathbf{i}\pi\mathbf{n}_j} \ .$$

This last red factor includes *all* of the sites except the first:

$$\prod_{N \geq j > 1} e^{\mathbf{i}\pi\mathbf{n}_j} = \prod_{j=1}^{N} e^{\mathbf{i}\pi\mathbf{n}_j} e^{\mathbf{i}\pi\mathbf{n}_1} = e^{\mathbf{i}\pi\mathbf{N}} e^{\mathbf{i}\pi\mathbf{n}_1} \equiv (-1)^{\mathbf{N}} e^{\mathbf{i}\pi\mathbf{n}_1} \ .$$

Here $\mathbf{N} \equiv \sum_j \mathbf{c}_j^\dagger \mathbf{c}_j$ operator which counts the total number of fermions, and $(-1)^{\mathbf{N}}$ is called the 'fermion parity'[23]. This is a symmetry generator: $[\mathbf{H}, (-1)^{\mathbf{N}}] = 0$, since $\mathbf{H}$ only contains terms with an even number of fermions. Returning to the outlying ferromagnetic coupling,

$$\begin{aligned} \mathbf{Z}_N\mathbf{Z}_1 &= \left(\mathbf{c}_N^\dagger + \mathbf{c}_N\right) \underbrace{\left(\mathbf{c}_1^\dagger + \mathbf{c}_1\right) e^{\mathbf{i}\pi\mathbf{n}_1}}_{=\mathbf{c}_1^\dagger - \mathbf{c}_1} (-1)^{\mathbf{N}} \\ &= \left(\mathbf{c}_N^\dagger + \mathbf{c}_N\right) \left(\mathbf{c}_1^\dagger - \mathbf{c}_1\right) (-1)^{\mathbf{N}}. \end{aligned} \tag{4.23}$$

This is the same as the other terms, but for a factor of $-(-1)^{\mathbf{N}}$. This means that we get periodic boundary conditions on $\mathbf{c}$, *i.e.* $\mathbf{c}_{N+1} = +\mathbf{c}_1$ if the total number of fermions is *odd* and antiperiodic boundary conditions, $\mathbf{c}_{N+1} = -\mathbf{c}_1$, in sectors where $\mathbf{N}$ is even. In summary[24] :

$$\boxed{\mathbf{c}_{N+1} = -\mathbf{c}_1(-1)^{\mathbf{N}}} \quad .$$

[23] please don't confuse the number operator $\mathbf{N}$ with the number of sites N – sorry about that

[24] This fact is perhaps simpler in terms of the majoranas:

$$\text{for } j < N, \ -\mathbf{i}\tilde{\boldsymbol{\chi}}_{j+1}\boldsymbol{\chi}_j = \mathbf{Z}_j\mathbf{Z}_{j+1}, \quad \text{but} \quad -\mathbf{i}\tilde{\boldsymbol{\chi}}_N\boldsymbol{\chi}_1 = \mathbf{Z}_N\mathbf{Z}_1\,(-\mathbf{S})\,.$$

So the full Hamiltonian is

$$\frac{1}{J}\mathbf{H}_{\text{TFIM}}^{\text{PBC on }\mathbf{z}} = -\sum_{j=1}^{N-1}\left(\mathbf{c}_j^\dagger + \mathbf{c}_j\right)\left(\mathbf{c}_{j+1}^\dagger - \mathbf{c}_{j+1}\right) - \left(\mathbf{c}_N^\dagger + \mathbf{c}_N\right)\left(\mathbf{c}_1^\dagger - \mathbf{c}_1\right)(-1)^{\mathbf{N}} - 2g\underbrace{\sum_j \mathbf{c}_j^\dagger \mathbf{c}_j}_{=\mathbf{N}}.$$

We can diagonalize this by fourier transformation

$$\mathbf{c}_j = \frac{1}{\sqrt{N}}\sum_k e^{\mathbf{i}kja}\mathbf{c}_k$$

where the allowed values of k depend on the boundary conditions on $\mathbf{c}$:

$$\begin{aligned} \text{PBC,}\quad (-1)^{\mathbf{N}} = -1: &\qquad k \in \frac{2\pi}{Na}\{1...N\} \\ \text{APBC,}\quad (-1)^{\mathbf{N}} = +1: &\qquad k \in \frac{2\pi}{Na}\left(\frac{1}{2} + \{1...N\}\right) \end{aligned} \tag{4.24}$$

And the Hamiltonian in each case is:

$$\mathbf{H}/J = \sum_k \left(\mathbf{c}_k^\dagger \mathbf{c}_k\left(2g - 2\cos ka\right) + \left(\mathbf{c}_k \mathbf{c}_{-k}\mathbf{i}\sin ka + h.c.\right)\right) \equiv \sum_k \mathbf{h}_k$$

The dynamics determine the boundary conditions: is it less energy to have an odd number of fermions or an even number? First notice that the dispersion has an inversion symmetry $k \to -k$, so each k value is degenerate in energy with $-k$, unless it is fixed by this map. The fixed points are $k = 0, \pi/a$ (recall that $ka = -\pi \simeq -\pi + 2\pi = \pi$). These only occur for PBC.

The twofold (in particular even) degeneracy for most k means that the occupation $\mathbf{N}_{k\neq -k}$ only changes by two, and we don't care about the associated $\mathbf{h}_k$. More algebraically, for PBC,

$$\mathbf{N} = \sum_j \mathbf{c}_j^\dagger \mathbf{c}_j = 2\sum_{k\neq 0,\pi} \mathbf{c}_k^\dagger \mathbf{c}_k + \mathbf{c}_0^\dagger \mathbf{c}_0 + \mathbf{c}_\pi^\dagger \mathbf{c}_\pi$$

(for APBC we can simply omit the last term and conclude that $\mathbf{N}$ is even, so $(-1)^{\mathbf{N}} = 1$.) So for determining $(-1)^{\mathbf{N}}$, we only care about $\mathbf{h}_{ka=0}$ and $\mathbf{h}_{ka=\pi}$.

$$\mathbf{h}_{ka=\pi} = \mathbf{c}_\pi^\dagger \mathbf{c}_\pi\left(2g + 2\right) > 0 \quad \Longrightarrow \quad \text{empty in the groundstate}, \forall g \geq 0$$

As usual, the zero-momentum mode is the hero of the story:

$$\mathbf{h}_{ka=0} = \mathbf{c}_0^\dagger \mathbf{c}_0\left(2g - 2\right)$$

which *changes sign* at $g = g_c = 1$:

- For $g > g_c$,, $\mathbf{h}_0 > 0$ is a positive energy cost for filling the zeromode. So the groundstate for $g > g_c$ has this odd mode empty and hence $(-1)^{\mathbf{N}} = 1$, and APBCs.

- For $g < g_c$,, $\mathbf{h}_0 < 0$ is an incentive for filling the zeromode, of which there is odd number (namely, 1), so the system can lower its energy by having PBC and filling the $k = 0$ mode. In the thermodynamic limit $L = Na \to \infty$, these two groundstates become degenerate.

Comments on the splitting between the groundstates

Comments about the L-dependence of the splitting between these states:

(1) We would like to identify them with (linear combinations of) the two ferromagnetic groundstates, which become the fully-polarized (in $\mathbf{Z}$) states $|+\rangle$ and $|-\rangle$ as $g \to 0$. As we've argued from many other points of view, these states are split by an amount which is exponentially small in L: $\Delta E \sim e^{-L/\xi}$.

(2) *At the critical point*, where the fermions are massless, the energy splitting between the groundstates with PBC and APBC (in the scaling limit, where $a \to 0$), is an oft-used result in string theory, and is $\frac{1}{16}\frac{\hbar c}{L}$ (PBC is higher energy). The L-dependence is determined by dimensional analysis, since there is no other scale. Away from the critical point I don't know the answer at the moment.

(3) Note that these two sectors with different $(-1)^{\mathbf{N}}$ do not mix via $\mathbf{H}_{\text{TFIM}}$, since $(-1)^{\mathbf{N}}$ is a symmetry generator; you can't create a single fermion using $\mathbf{H}_{\text{TFIM}}$. There is *no* tunneling between these vacua. So these two states must in fact be the cat states $\frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$ which are eigenvectors of the $\mathbb{Z}_2$ symmetry of the TFIM, which, after all, is

$$\mathbf{S} = \prod_{j=1}^{N} \mathbf{X}_j = \prod_{j=1}^{N} (-1)^{\mathbf{c}_j^\dagger \mathbf{c}_j} = (-1)^{\mathbf{N}}.$$

(Recall that the symmetric cat state $\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ is the ground state in finite volume, so this must be the fermion state with antiperiodic boundary conditions.) Being forced to superpose states with different boundary conditions may be discomfiting, but it is part of the resolution of the puzzle of an odd number of domain walls on a circle.

4.2 Ferromagnets and antiferromagnets.

[Zee §6.5] Now we'll try a spin system in $D \geq 1+1$. Consider a chain of spins, each of spin $s \in \mathbb{Z}/2$, interacting via the Heisenberg hamiltonian:

$$\mathbf{H} = \sum_{\langle jj' \rangle} J \vec{\mathbf{S}}_j \cdot \vec{\mathbf{S}}_{j'}.$$

This hamiltonian is invariant under global spin rotations, $\mathbf{S}^a_j \to \mathcal{R}\mathbf{S}^a_j\mathcal{R}^{-1} = R^a_b \mathbf{S}^b_j$ for all j. For $J < 0$, this interaction is ferromagnetic, so it favors a state like $\left\langle \vec{\mathbf{S}}_j \right\rangle = s\hat{z}$. For $J > 0$, the neighboring spins want to anti-align; this is an antiferromagnet: $\left\langle \vec{\mathbf{S}}_j \right\rangle = (-1)^j s\hat{z}$. Note that I am lying about there being spontaneous breaking of a continuous symmetry in 1+1 dimensions. Really there is only short-range order because of the Coleman-Mermin-Wagner theorem. But that is enough for the calculation we want to do.[25]

We can write down the action that we get by coherent-state quantization – it's just many copies of the above, where each spin plays the role of the external magnetic field for its neighbors:

$$L = \mathbf{i}s \sum_j z_j^\dagger \partial_t z_j - Js^2 \sum_{\langle jj' \rangle} \vec{n}_j \cdot \vec{n}_{j'}.$$

Spin waves in ferromagnets. Let's use this to find the equation of motion for small fluctuations $\delta\vec{n}_i \equiv \vec{n}_i - s\hat{z}$ about the ferromagnetic state. Once we recognize the existence of the Berry phase term, this is the easy case. In fact the discussion is not restricted to $D = 1+1$.

[End of Lecture 14]

Assume the system is translation invariant, so we should Fourier transform. The condition that $\vec{n}_j^2 = 1$ means that $\delta n_z(k) = 0$.[26] The equations of motion are (using (3.21))

$$0 = \frac{\delta S}{\delta \vec{n}_j(t)} = s\vec{n}_j \times \partial_t \vec{n}_j - s^2 \sum_{\langle j|l \rangle} \vec{n}_l$$

[25] Even more generally, the consequence of short-range interactions of some particular sign for the groundstate is not so obvious. For example, antiferromagnetic interactions may be *frustrated*: If I want to disagree with both Kenenisa and Lasse, and Kenenisa and Lasse want to disagree with each other, then some of us will have to agree, or maybe someone has to withhold their opinion, $\langle S \rangle = 0$.

[26] $1 = n_j^2 \forall j \implies n_j \cdot \delta n_j = 0, \forall j$ which means that for any k,

$$0 = \sum_j e^{\mathbf{i}kja} n_j \cdot \delta n_j = \sum_j e^{\mathbf{i}kja}(s\hat{z} + \delta n_j) \cdot \delta n_j = s\delta n^z_k + \mathcal{O}(\delta n^2).$$

where $\sum_{\langle j|l\rangle}$ is an instruction to sum over the neighbors l of the fixed site j. Taking $s^{-1}\vec{n}_j \times$ (BHS) gives

$$-\partial_t \vec{n}_j + \vec{n}_j \times \sum_{\langle j|l\rangle} sJ\vec{n}_l.$$

Linearizing in $\delta\vec{n}$ and fourier transforming, we find

$$0 = \begin{pmatrix} h(k) & -\frac{\mathbf{i}}{2}\omega \\ \frac{\mathbf{i}}{2}\omega & h(k) \end{pmatrix} \begin{pmatrix} \delta n_x(k) \\ \delta n_y(k) \end{pmatrix}$$

with $h(k)$ determined by the exchange (J) term. It is the lattice laplacian in k-space. For example for the square lattice, it is $h(k) = 4s|J|\left(2 - \cos k_x a - \cos k_y a\right) \overset{k\to 0}{\simeq} 2s|J|a^2k^2$, with a the lattice spacing. For small k, the eigenvectors have $\omega \sim k^2$, a $z = 2$ dispersion (meaning that there is scale invariance near $\omega = k = 0$, but space and time scale differently: $k \to \lambda k, \omega \to \lambda^2\omega$. The two spin polarizations have their relative phases locked $\delta n_x(k) = \mathbf{i}\delta n_y(k)/h_k$, and so these modes describe precession of the spin about the ordering vector. These low-lying spin excitations are visible in neutron scattering and they dominate the low-temperature thermodynamics. Their thermal excitations produce a version of the blackbody spectrum with $z = 2$. We can determine the generalization of the Stefan-Boltzmann law by dimensional analysis: the free energy (or the energy itself) is extensive, so $F \propto L^d$, but it must have dimensions of energy, and the only other scale available is the temperature. With $z \neq 1$, temperature scales like $[T] = [L^{-z}]$. Therefore $F = cL^dT^{\frac{d+z}{z}}$. (For $z = 1$ this is the ordinary Stefan-Boltzmann law).

Notice that a ferromagnet is a bit special because the order parameter $Q^z = \sum_i \mathbf{S}_i^z$ is actually *conserved*, $[Q^z, \mathbf{H}] = 0$. This is the origin of the funny $z = 2$ dispersion of the goldstones, and the fact that although the groundstate breaks *two* generators Q^x and Q^y, there is only one gapless mode. If you are impatient to understand this connection, take a look at this paper.

Antiferromagnets. [Fradkin, 2d ed, p. 203] Now, let's study instead the equation of motion for small fluctuations about the antiferromagnetic (Neel) state, $S(\vec{x}) \sim (-1)^{\sum_i x_i} m$. The conclusion will be that there is a linear dispersion relation. This would be the conclusion if we simply erased the WZW/Berry phase term and replaced it with an ordinary kinetic term

$$\frac{1}{2g^2}\sum_j \partial_t \vec{n}_j \cdot \partial_t \vec{n}_j \ .$$

How this comes about is actually a bit more involved! An important role will be played[27] by the *ferro*magnetic fluctuation $\vec{\ell}_j$ in

$$\vec{n}_j = (-1)^j \vec{m}_j + a\vec{\ell}_j \ .$$

[27] A pointer to the past: this story is very similar to the origin of the second order kinetic term

$\vec{m}_j$ is the AF order parameter; a is the lattice spacing; $s \in \mathbb{Z}/2$ is the spin. The constraint $\vec{n}^2 = 1$ tells us that $\vec{m}^2 = 1$ and $\vec{m} \cdot \vec{\ell} = 0$.

Why do we have to include both variables? Because $\vec{m}$ are the AF order-parameter fluctuations, but the total spin is conserved, and therefore its local fluctuations $\vec{\ell}$ still constitute a slow mode. This is an illustration of a general point: amongst the low-energy modes in our effective field theory, we should make sure we keep track of the conserved quantities, which can often move around but can never disappear. The name for this principle is *hydrodynamics*.

The exchange (J) term in the action is

$$S_J[\vec{n}_j = (-1)^j \vec{m}_j + a\vec{\ell}_j] = -aJs^2 \int \mathrm{d}x\mathrm{d}t \left(\frac{1}{2} (\partial_x \vec{m})^2 + 2\ell^2 \right) \ .$$

Here we used $\vec{n}_i \cdot \vec{n}_j = \frac{1}{2}(n_i + n_j)^2 - 1$ and

$$\vec{n}_{2r} + \vec{n}_{2r-1} \approx a \left(\partial_x \vec{m}_{2r} + 2\ell_{2r} \right) + \mathcal{O}(a^2). \tag{4.25}$$

Low let's focus on $D = 1 + 1$. The WZW terms evaluate to[28]

$$S_W = 4\pi s \sum_{j=1}^{N} W_0[(-1)^j m_j + \ell_j] \overset{N\to\infty, a\to 0, Na \text{ fixed}}{\simeq} \int \mathrm{d}x\mathrm{d}t \ \left(\frac{s}{2} \vec{m} \cdot (\partial_t \vec{m} \times \partial_x \vec{m}) + s\vec{\ell} \cdot (\vec{m} \times \partial_t \vec{m}) \right). \tag{4.26}$$

Altogether, we find that ℓ is an auxiliary field with no time derivative:

$$L[m, \ell] = -2aJs^2 \vec{\ell}^2 + s\vec{\ell} \cdot (\vec{m} \times \partial_t \vec{m}) + L[m]$$

for the Goldstone mode in a superfluid arises, which we discussed as an application of coherent state quantization of bosons in §3.1. The role of $\vec{\ell}$ there is played by ρ, the density.

[28] The essential ingredient is

$$\delta W_0[n] = \frac{1}{4\pi} \int \mathrm{d}t \delta\vec{n} \cdot (\vec{n} \times \partial_t \vec{n}) \ .$$

So

$$4\pi W_0 \left[\underbrace{n_{2r}}_{=-n_{2r-1}+\Delta n} \right] + 4\pi W_0[n_{2r-1}] = 4\pi W_0[-n_{2r-1}] + \int dt \frac{\delta W_0}{\delta n^i_{2r-1}(t)} \underbrace{\Delta n}_{\overset{(4.25)}{=} \partial_x \hat{n}^i_{2r-1} a} + W_0[n_{2r-1}] = \int dt a \hat{n} \times \partial_t \hat{n} \cdot \partial_x \hat{n}.$$

The factor of $\frac{1}{2}$ in (4.26) comes from $a \sum_r = \frac{1}{2} \int dx$.

so we can integrate out ℓ (this is the step analogous to what we did for ρ in the EFT of SF in §3.1) to find

$$S[\vec{m}] = \int \mathrm{d}x\mathrm{d}t \left(\frac{1}{2g^2} \left(\frac{1}{v_s} \left(\partial_t \vec{m}\right)^2 - v_s \left(\partial_x \vec{m}\right)^2 \right) + \frac{\theta}{8\pi} \epsilon_{\mu\nu} \vec{m} \cdot \left(\partial_\mu \vec{m} \times \partial_\nu \vec{m}\right) \right), \quad (4.27)$$

with $g^2 = \frac{2}{s}$ and $v_s = 2aJs$, and $\theta = 2\pi s$. The equation of motion for small fluctuations of $\vec{m}$ therefore gives linear dispersion with velocity v_s. Notice that there are two independent gapless modes. Some of these fluctuations have wavenumber k close to π, since they are fluctuations of the AF order ($k = \pi$ means changing sign between each site), that is, $\omega \sim |k - \pi|$. (For a more microscopic treatment, see the book by Auerbach.)

The last ('theta') term in (4.27) is a total derivative. This means it doesn't affect the EOM, and it doesn't affect the Feynman rules. It is even more topological than the WZW term – its value only depends on the topology of the field configuration, and not on local variations. It is like the $\theta F \wedge F$ term in 4d gauge theory. You might think then that it doesn't matter. Although it doesn't affect small fluctuations of the fields, it does affect the path integral. Where have we seen this functional before? The integrand is the same as in our 2d representation of the WZW term in 0+1 dimensions: the object multiplying theta counts the winding number of the field configuration $\vec{m}$, the number of times Q the map $\vec{m} : \mathbb{R}^2 \to S^2$ covers its image (we can assume that the map $\vec{m}(|x| \to \infty)$ approaches a constant, say the north pole). We can break up the path integral into sectors, labelled by this number $Q \equiv \frac{1}{8\pi} \int \mathrm{d}x\mathrm{d}t \; \epsilon_{\mu\nu} \vec{m} \cdot \left(\partial_\mu \vec{m} \times \partial_\nu \vec{m}\right)$:

$$Z = \int [D\vec{m}] e^{\mathbf{i}S} = \sum_{Q \in \mathbb{Z}} \int [D\vec{m}]_Q e^{\mathbf{i}S_{\theta=0}} e^{\mathbf{i}\theta Q} \; .$$

θ determines the relative phase of different topological sectors (for $\theta = \pi$, this a minus sign for odd Q).

Actually, the theta term makes a huge difference. (Perhaps it is not so surprising if you think about the quantum mechanics of a particle constrained to move on a ring with magnetic flux through it?) The model with even s flows to a trivial theory in the IR, while the model with odd s flows to a nontrivial fixed point, called the $\mathrm{SU}(2)_1$ WZW model. It can be described in terms of one free relativistic boson. If you are impatient to understand more about this, the 2nd edition of the book by Fradkin continues this discussion. Perhaps I can be persuaded to say more.

Nonlinear sigma models in perturbation theory. Let us discuss what happens in perturbation theory in small g. A momentum-shell calculation integrating out fast modes (see the next subsection, §4.3) shows that

$$\frac{dg^2}{d\ell} = (D-2)g^2 + (n-2)K_D g^4 + \mathcal{O}(g^5) \quad (4.28)$$

where ℓ is the logarithmic RG time, and $\ell \to \infty$ is the IR. n is the number of components of $\hat{n}$, here $n = 3$, and $K_D = \frac{\Omega_{D-1}}{(2\pi)^D}$ as usual. Cultural remark: the second term is proportional to the *curvature* of the target space, here S^{n-1}, which has positive curvature for $n > 1$. For $n = 2$, we get S^1 which is one-dimensional and hence flat and there is no perturbative beta function. In fact, for $n = 2$, it's a free massless scalar. (But there is more to say about this innocent-looking scalar!)

The fact that the RHS of (4.28) is positive at small g in $D = 2$ says that this model is *asymptotically free* – the coupling is weak in the UV (though this isn't so important if we are starting from a lattice model) and becomes strong in the IR. This is opposite what happens in QED; the screening of the charge in QED makes sense in terms of polarization of the vacuum by virtual charges. Why does this antiscreening happen here? There's a nice answer: the effect of the short-wavelength fluctuations is to make the spin-ordering vector $\vec{n}$ effectively *smaller*. It is like what happens when you do the block spin procedure, by just averaging the spins. But rescaling the variable $\vec{n} \to a\vec{n}$ with $a \lesssim 1$ is the same as rescaling the coupling $g \to g/a$ – the coupling gets bigger. (Beware Peskin's comments about the connection between this result and the Coleman-Mermin-Wagner theorem: it's true that the logs in 2d enhance this effect, but in fact the model can reach a fixed point at finite coupling; in fact, this is what happens when $\theta = \pi$.)

Beyond perturbation theory. Like in QCD, this *infrared slavery* (the dark side of asymptotic freedom) means that we don't really know what happens in the IR from this calculation. From other viewpoints (Bethe ansatz solutions, many other methods), we know that (for integer s) there is an energy gap above the groundstate (named after Haldane) of order

$$\Lambda_H \sim \Lambda_0 e^{-\frac{c}{g_0^2}},$$

analogous to the QCD scale. Here g_0 is the value of g at the scale Λ_0; so Λ_H is roughly the energy scale where g becomes large. This is dimensional transmutation again.

For $s \in \mathbb{Z}$, for studying bulk properties like the energy gap, we can ignore the theta term since it only appears as $e^{2\pi \mathbf{i} n}$, with $n \in \mathbb{Z}$ in the path integral. [29] For half-integer s, there is destructive interference between the topological sectors. Various results (such as the paper by Read and Shankar, Nuclear Physics B336 (1990) 457-474, which contains an amazingly apt Woody Allen joke) show that this destroys the gap. This last sentence was a bit unsatisfying; more satisfying would be to understand the origin of the gap in the $\theta = 2\pi n$ case, and show that this interference removes that mechanism.

[29] $\theta = 2\pi n$ does, however, affect other properties, such as the groundstate wavefunction and the behavior in the presence of a boundary. $\theta = 2\pi$ is actually a different phase of matter than $\theta = 0$. It is an example of a SPT (symmetry-protected topological) phase, the first one discovered. See the homework for more on this.

This strategy is taken in this paper by Affleck.

In the section on large-N, we'll get some intuition for these claims.

4.3 The beta function for 2d non-linear sigma models

[Polyakov §3.2; Peskin §13.3; Auerbach chapter 13] I can't resist explaining the result (4.28). Consider this action for a $D = 2$ non-linear sigma model with target space S^{n+1}, of radius R:

$$S = \int \mathrm{d}^2x R^2 \partial_\mu \hat{n} \cdot \partial^\mu \hat{n} \equiv \int \mathrm{d}^2x R^2 \mathrm{d}n^2.$$

Notice that R is a coupling constant (it's what I called $1/g$ earlier). In the second step I just made some compact notation.

Since not all of the components of $\hat{n}$ are independent (recall that $\hat{n} \cdot \hat{n} = 1$!), the expansion into slow and fast modes here is a little trickier than in our previous examples. Following Polyakov, let

$$n^i(x) \equiv n^i_<(x)\sqrt{1-\phi_>^2} + \sum_{a=1}^{n-1} \phi_a^>(x) e^i_a(x). \tag{4.29}$$

Here the slow modes are represented by the unit vector $n^i_<(x)$, $\hat{n}_< \cdot \hat{n}_< = 1$; the variables e^i_a are a basis of unit vectors spanning the $n-1$ directions perpendicular to $\vec{n}_<(x)$

$$n_< \cdot \hat{e}_a = 0, \hat{e}_a \cdot \hat{e}_a = 1; \tag{4.30}$$

they are not dynamical variables and how we choose them does not matter.

The fast modes are encoded in $\phi_a^>(x) \equiv \int_{\Lambda/s}^{\Lambda} đk e^{\mathbf{i}kx}\phi_k$, which only has fourier modes in a shell of momenta, and $\phi_>^2 \equiv \sum_{a=1}^{n-1} \phi_a^> \phi_a^>$. Notice that differentiating the relations in (4.30) gives

$$\hat{n}_< \cdot \mathrm{d}\hat{n}_< = 0, \quad \hat{n}_< \cdot \mathrm{d}\hat{e}_a + \mathrm{d}\hat{n}_< \cdot \hat{e}_a = 0. \tag{4.31}$$

Below when I write ϕs, the $>$ symbol is implicit.

We need to plug the expansion (4.29) into the action, whose basic ingredient is

$$\mathrm{d}n^i = \mathrm{d}n^i_< \left(1-\phi^2\right)^{\frac{1}{2}} - n^i_< \frac{\phi \cdot \mathrm{d}\phi}{\sqrt{1-\phi^2}} + \mathrm{d}\phi \cdot e^i + \phi \cdot \mathrm{d}e^i.$$

So $S = \int d^2x\ \mathcal{L}$ with

$$\mathcal{L} = \frac{1}{2g^2} \left(\mathrm{d}\vec{n}\right)^2$$

$$= \frac{1}{2g^2}\left((\mathrm{d}n_<)^2 \left(1-\phi^2\right) + \underbrace{\mathrm{d}\phi^2}_{\text{kinetic term for } \phi} + 2\phi_a \mathrm{d}\phi_b \vec{e}_a \cdot \mathrm{d}\vec{e}_b \right.$$
$$\left. + \underbrace{\mathrm{d}\phi_a \mathrm{d}\vec{n}_< \cdot \vec{e}_a}_{\text{source for } \phi} + \phi_a\phi_b \mathrm{d}\vec{e}_a \cdot \mathrm{d}\vec{e}_b + \mathcal{O}(\phi^3) \right) \tag{4.32}$$

So let's do the integral over ϕ, by treating the $\mathrm{d}\phi^2$ term as the kinetic term in a gaussian integral, and the rest as perturbations:

$$e^{-S_{\text{eff}}[n_<]} = \int [D\phi_>]^{\Lambda}_{\Lambda/s} e^{-\int L} = \int [D\phi_>]^{\Lambda}_{\Lambda/s} e^{-\frac{1}{2g^2}\int (\mathrm{d}\phi)^2} \,(\text{all the rest}) \equiv \langle \text{all the rest} \rangle_{>,0} \, Z_{>,0} \ .$$

The $\langle ... \rangle_{>,0}$s that follow are with respect to this measure.

$$\implies L_{\text{eff}}[n_<] = \frac{1}{2g^2} (\mathrm{d}n_<)^2 \left(1 - \langle \phi^2 \rangle_{>,0}\right) + \langle \phi_a \phi_b \rangle_{>,0} \, \mathrm{d}\vec{e}_a \cdot \mathrm{d}\vec{e}_b + \text{terms with more derivatives}$$

$$\langle \phi_a \phi_b \rangle_{>,0} = \delta_{ab} g^2 \int_{\Lambda/s}^{\Lambda} \frac{\mathrm{d}^2 k}{k^2} = g^2 K_2 \log(s) \delta_{ab}, \qquad K_2 = \frac{1}{2\pi}.$$

What to do with this $\mathrm{d}\vec{e}_a \cdot \mathrm{d}\vec{e}_b$ nonsense? Remember, $\vec{e}_a$ are just some arbitrary basis of the space perpendicular to $\hat{n}_<$; its variation can be expanded in our ON basis at x, $(n_<, e_c)$ as

$$\mathrm{d}\vec{e}_a = \underbrace{(\mathrm{d}e_a \cdot \hat{n}_<)}_{\overset{(4.31)}{=} -\mathrm{d}\hat{n}_< \cdot \vec{e}_a} \vec{e}_c \hat{n}_< + \sum_{c=1}^{n-1} (\mathrm{d}\vec{e}_a \cdot \vec{e}_c)$$

Therefore

$$\mathrm{d}\vec{e}_a \cdot \mathrm{d}\vec{e}_a = + (\mathrm{d}n_<)^2 + \sum_{c,a} (\vec{e}_c \cdot \mathrm{d}\vec{e}_a)^2$$

where the second term is a higher-derivative operator that we can ignore for our present purposes. Therefore

$$L_{\text{eff}}[n] = \frac{1}{2g^2} (\mathrm{d}\hat{n}_<)^2 \left(1 - ((N-1)-1) g^2 K_2 \log s\right) + ...$$
$$\simeq \frac{1}{2}\left(g^2 + \frac{g^4}{2\pi}(N-2)\log s + ...\right)^{-1} (\mathrm{d}\hat{n}_<)^2 + ... \tag{4.33}$$

Differentiating this running coupling with respect to s gives the one-loop term in the beta function quoted above. The tree-level (order g^2) term comes from engineering dimensions.

[End of Lecture 15]

4.4 $\mathbb{CP}^1$ representation and large-N

[Auerbach, *Interacting Electrons and Quantum Magnetism*, Polyakov, *Gauge fields and strings*] Above we used large spin as our small parameter to try to control the contributions to the path integral. Here we describe another route to a small parameter, which can be just as useful if we're interested in small spin (like spin-$\frac{1}{2}$).

Recall the relationship between the coherent state vector $\check{n}$ and the spinor components z: $n^a = z^\dagger \sigma^a z$. Imagine doing this at each point in space and time:

$$n^a(x) = z^\dagger(x)\sigma^a z(x). \tag{4.34}$$

We saw that the Berry phase term could be written nicely in terms of z as $\mathbf{i}z^\dagger \dot{z}$, what about the rest of the path integral?

First, some counting: $1 = \check{n}^2 \Leftrightarrow 1 = z^\dagger \cdot z = \sum_{m=\uparrow,\downarrow} |z_m|^2$. But this leaves only two components of n, and three components of z_m. The difference is made up by the fact that the rephasing

$$z_m(x) \to e^{\mathbf{i}\chi(x)} z_m(x) \tag{4.35}$$

doesn't change $\check{n}$. So it can't act on the physical Hilbert space. This is a (local, since $\chi(x)$ depends on x) $\mathsf{U}(1)$ gauge redundancy of the description in terms of z.

There two ways to proceed from here. One is via exact path integral tricks which are relatively straightforward in this case, but generally unavailable. The second is by the Landau method of knowing the answer: what else could it be.

Path integral manipulations. [Auerbach, chapter 14] First notice that the AF kinetic term is

$$\partial_\mu n^a \partial^\mu n^a = 4\left(\partial_\mu z^\dagger \partial^\mu z - \mathcal{A}_\mu \mathcal{A}^\mu\right) = 4\left(\partial_\mu z^\dagger \partial^\mu z - \mathcal{A}_\mu \mathcal{A}^\mu z^\dagger z\right). \tag{4.36}$$

where $\mathcal{A}_\mu \equiv -\frac{\mathbf{i}}{2}\left(z^\dagger \partial_\mu z - \partial_\mu z^\dagger z\right)$ is a connection one-form made from z itself. Notice that $\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial\chi$ and the BHS of (4.36) is gauge invariant under (4.35). We must impose the constraint $|z(x)|^2 = 1$ at each site, which let's do it by a lagrange muliptlier $\delta[|z|^2 - 1] = \int D\lambda\; e^{\mathbf{i}\int d^d x \lambda(x)\left(|z|^2 - 1\right)}$. In the action, the $\mathcal{A}^2$ term is a self-interaction of the zs, which makes it difficult to do the integral. The standard trick for ameliorating this problem is the Hubbard-Stratonovich identity:

$$e^{c\mathcal{A}_\mu^2} = \sqrt{\frac{c}{\pi}} \int dA_\mu\; e^{-cA_\mu^2 + 2cA_\mu \mathcal{A}^\mu}.$$

The saddle point value of A is $\mathcal{A}$. This gives

$$e^{-\#\int dn^2} = \int [dA] e^{-\#\int |(\partial - \mathbf{i}A)z|^2}.$$

Finally, let's think about the measure at each point: $\int d^3n\delta(n^2-1)... = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi....$ Compare this to the integral over zs, parametrized as $z = \begin{pmatrix}\rho_1 e^{\mathbf{i}\phi_1}\\ \rho_2 e^{\mathbf{i}\phi_2}\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}e^{\mathbf{i}\varphi/2}e^{\mathbf{i}\chi/2}\\ \sin\frac{\theta}{2}e^{-\mathbf{i}\varphi/2}e^{\mathbf{i}\chi/2}\end{pmatrix}$:

$$\begin{aligned}\int dzdz^\dagger\delta(|z|^2-1)... &= \int \prod_{m=1,2}\rho_m d\rho_m d\phi_m \delta(\rho_1+\rho_2-1)...\\ &= c\int d\rho\rho\sqrt{1-\rho^2}d\varphi d\chi... = c'\int \sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta d\varphi d\chi...\end{aligned}$$

which is the same as $\int dn$ except for the extra integral over χ: that's the gauge direction. The integral over χ is just a number at each point, as long as we integrate invariant objects (otherwise, it gives zero). Thinking of z as parametrizing an arbitrary normalized spinor $z = \mathcal{R}(\theta,\varphi,\chi)\begin{pmatrix}1\\0\end{pmatrix}$, so that $\mathcal{R}$ is an arbitrary element of $\mathsf{SU}(2)$, we've just shown the geometric equivalence between the round S^2 and $\mathbb{CP}^1 = \mathsf{SU}(2)/\mathsf{U}(1)$.

Therefore, we can rewrite the nonlinear sigma model (NLSM) on S^2 path integral as

$$Z_{S^2} \simeq \int [dzdz^\dagger dAd\lambda] e^{-\int d^Dx\left(\frac{2\Lambda^{D-2}}{g^2}|(\partial-\mathbf{i}A)z|^2-\mathbf{i}\lambda\left(|z|^2-1\right)\right)}. \tag{4.37}$$

This is a $\mathsf{U}(1)$ gauge theory with $N=2$ charged scalars. It is called the $\mathbb{CP}^1$ sigma model. There are two slightly funny things: (1) the first is that the gauge field A lacks a kinetic term: in the microscopic description we are making here, it is infinitely strongly coupled. We'll see what the interactions with matter have to say about the coupling in the IR. (2) The second funny thing is that the scalars z have a funny interaction with this field λ which only appears linearly. If we add a $\lambda^2/(4\kappa)$ quadratic term, we can do the lambda integral and find $V(|z|^2) = \kappa(|z|^2-1)^2$, an ordinary quartic potential for $|z|$. This has the effect of replacing the delta function imposition with an energetic recommendation that $|z|^2=1$. This is called a *soft* constraint, and it shouldn't change the universal physics.

Alternatively, we could have arrived at this point

$$Z_{S^2} \simeq \int [dzdz^\dagger dA] e^{-\int d^Dx\left(\frac{2\Lambda^{D-2}}{g^2}|(\partial-\mathbf{i}A)z|^2-\kappa\left(|z|^2-1\right)^2\right)}$$

by regarding (4.34) as a *slave-particle* or *parton* ansatz for a new set of variables. The demand of gauge invariance (4.35) is a strong constraint on the form of the interactions, and requires the inclusion of the gauge field A. Relative to the most general 2-derivative Lagrangian consistent with the symmetries and gauge redundancy, we are missing kinetic terms for A and σ; we'll see below that these are generated by the fluctuations of the zs.

Other such ansatze are possible, such as one in terms of slave fermions $\vec{S} = \psi^\dagger \vec{\sigma} \psi$. In this case, this turns out to be *also* correct – more later. More generally, any given change of variables may or may not be useful to describe the relevant physics.

Large N. This representation allows the introduction of another possible small parameter (besides the number of components of $\vec{n}$ (which gives $\mathsf{SO}(n)$ symmetry) or the size of the spin (which takes a large representation of $\mathsf{SO}(3)$), namely the number of components of z (which will give a theory with $\mathsf{SU}(N)$ symmetry). Suppose instead of two components, it has N

$$\sum_{m=1}^{N} |z_m|^2 = \frac{N}{2},$$

and let's think about the resulting $\mathbb{CP}^{N-1}$ sigma model (notice that $\mathbb{CP}^{N-1}$ and S^N are different generalizations of S^2, in the sense that for $N \to 2$ they are both S^2):

$$\begin{aligned} Z_{\mathbb{CP}^{N-1}} &= \int [dz dz^\dagger dA d\lambda] e^{-\int d^D x \left(\frac{2\Lambda^{D-2}}{g^2} |(\partial - \mathbf{i}A)z|^2 - \mathbf{i}\lambda \left(|z|^2 - N/2 \right) \right)} \\ &= \int [dA d\lambda] e^{-NS[A,\lambda]} \overset{N \gg 1}{\simeq} Z' e^{-NS[\underline{A},\underline{\lambda}]}. \end{aligned}$$

The z-integral is gaussian in the representation (4.37) even for $N = 2$, but the resulting integrals over A, λ are then horrible, with action

$$S[A,\lambda] = \mathrm{Tr}\ \ln\left(-(\partial - \mathbf{i}A)^2 + \mathbf{i}\lambda \right) - \frac{\Lambda^{D-2}}{g^2} \int \mathbf{i}\lambda. \tag{4.38}$$

(In the last step, I absorbed some factors into a redefinition of λ.) The role of large N is to make those integrals well-peaked about their saddle point. The saddle point equations are solved by $\underline{A} = 0$ (though there may sometimes be other saddles where $\underline{A} \neq 0$, which break various discrete symmetries). This leaves us with

$$S[0,\lambda] = V \int \text{đ}^D k \ln(k^2 + \mathbf{i}\lambda) - \frac{\Lambda^{D-2}}{g^2} V \mathbf{i}\lambda$$

(where V is the number of sites, the volume of space, and I've assumed constant λ), which is solved by $\lambda = -\mathbf{i}\underline{\lambda}$ satisfying

$$\int \frac{\text{đ}^D k}{k^2 + \underline{\lambda}} = \frac{\Lambda^{D-2}}{g^2}.$$

This is an equation to determine $\underline{\lambda}$, which in turn appears as the coefficient of $z^\dagger z$ in the action – a mass for the charged fields (so this equation is sometimes called the *gap equation*). The solution of this equation depends on the number of dimensions D.

$$D = 1: \quad \frac{1}{g^2 \Lambda} = \int \frac{\text{đ}k}{k^2 + \underline{\lambda}} = \frac{1}{\sqrt{\underline{\lambda}}} \underbrace{\int \frac{\text{đ}\underline{k}}{\underline{k}^2 + 1}}_{= \frac{1}{2}} \quad \Longrightarrow \ \underline{\lambda} = \frac{g^4 \Lambda^2}{4}\ .$$

Since the action for z contains a term $\lambda|z|^2$, this says that the mass of the excitations is $m = \Lambda g^2/\sqrt{2}$. Where did that come from? $D = 1$ means we are studying the quantum mechanics of a particle contrained to move on $\mathbb{CP}^{N-1}$:

$$H = \frac{g^2\Lambda}{2}\partial_z\partial_{\bar{z}} + \infty\left(|z|^2 - N/2\right)^2.$$

The groundstate is the uniform state $\langle z|\text{groundstate}\rangle = \Psi(z) = \frac{1}{\sqrt{\text{vol}}}$. QM of finite number of degrees of freedom on a compact space has a gap above the groundstate. This gap is determined by the kinetic energy and naturally goes like $g^2\Lambda$.

$$D = 2: \quad g^{-2} = \int \frac{\text{đ}^2k}{k^2 + \underline{\lambda}} = -\frac{1}{4\pi}\ln\frac{\underline{\lambda}}{\Lambda^2} \implies \underline{\lambda} = \Lambda^2 e^{-\frac{4\pi}{g^2}}.$$

This is the case with asymptotic freedom; here we see again that asymptotic freedom is accompanied by dimensional transmutation: the interactions have generated a mass scale

$$m = \Lambda e^{-\frac{2\pi}{g^2}}$$

which is parametrically (in the bare coupling g) smaller than the cutoff. This is the promised (Haldane) gap.

$$D = 3: \quad \frac{\Lambda}{g^2} = \int \frac{\text{đ}^3k}{k^2 + \underline{\lambda}} = \frac{1}{2\pi^2}\left(\Lambda - \sqrt{\lambda}\arctan\frac{\Lambda}{\sqrt{\lambda}}\right) \implies 1 - \frac{2\pi^2}{g^2} = x\arctan\frac{1}{x} \overset{x\ll 1}{\simeq} \frac{\pi}{2}x$$

where $x \equiv \frac{\sqrt{\lambda}}{\Lambda}$. Notice that for $D \geq 3$ there is a critical value of g below which there is no solution of this equation. That means symmetry breaking: the saddle point is at $\underline{\lambda} = m^2 = 0$, and the z-fields are gapless Goldstone modes. This doesn't happen in $D \leq 2$. The critical coupling occurs when $g_c^{-2} = \int \frac{\text{đ}^D k}{k^2} \simeq \frac{\Lambda^{D-2}}{D-2}$. The rate at which the mass goes to zero as $g \to g_c$ from above is

$$m^2 \simeq \Lambda^2\left(\frac{g^2 - g_c^2}{g_c^2}\right)^{\frac{2}{D-2}}.$$

This is a universal exponent. (For more on critical exponents from large-N calculations, see Peskin p. 464-465.)

Correlation functions. A quantity we'd like to be able to compute for $N = 2$ is $S^{+-}(x) \equiv \langle S^+(0)S^-(x)\rangle$. For example, this will allow us to see more quantitatively how the saddle point value for λ produces a gap. We can write this in terms of the coherent state variables using the identity

$$\mathbf{S}^a = \mathcal{N}_s \int dn\, |\check{n}\rangle\, \langle\check{n}|\, n^a, \quad (\mathcal{N}_s = \frac{(s+1)(2s+1)}{4\pi}).$$

(Up to the constant factor, this identity follows from $\mathsf{SU}(2)$ invariance. The constant can be checked by looking at a convenient matrix element of the BHS.) Then:

$$S^{+-}(x) = \langle (n^x + \mathbf{i}n^y)(0)(n^x - \mathbf{i}n^y)(x) \rangle \,.$$

Recalling that $n^x + \mathbf{i}n^y = z^\dagger \sigma^+ z = z_1^\star z_2$, we can generalize this to large N as the four-point function

$$S^{m\neq m'}(x) = \left\langle z_m^\star(0) z_{m'}(0) z_m(x) z_{m'}^\star(x) \right\rangle \overset{N\gg 1}{\simeq} |G(x)|^2$$

which *factorizes* at leading order in large N. This phenomenon (large-N factorization) that at large-N the correlations are dominated by the disconnected bits is general. (We'll see a diagrammatic argument momentarily.) The factors are correlators of zs in the fixed saddle-point configuration of A, λ:

$$\begin{aligned} G(x) &= \frac{1}{Z} \int [dz] z^\dagger(0) z(x) e^{-\frac{2\Lambda^{D-2}}{g^2} \int \mathrm{đ}^D k \left(|k|^2 + \underline{\lambda}\right) z_k^\dagger z_k - \frac{NV\Lambda^{D-2}}{g^2}\underline{\lambda}} \\ &\propto \int \mathrm{đ}^d k \frac{e^{-\mathbf{i}kx}}{|k|^2 + \underline{\lambda}} \simeq \frac{1}{|x|^{\frac{D-1}{2}}} e^{-|x|\sqrt{\underline{\lambda}}}. \end{aligned}$$

This says that the correlation length for the spins in $S^{m\neq m'}(x) \overset{x>\xi}{\simeq} \frac{1}{|x|^{D-1}} e^{-|x|/\xi}$ is $\xi = \frac{1}{\sqrt{\underline{\lambda}}}$ depends variously on D. In $D = 1$, it is $\xi = \frac{1}{\Lambda g^2}$, so large-$N$ predicts a gap, growing with g. In $D = 2$, the correlation length is $\xi = \Lambda^{-1} e^{+\frac{2\pi}{g^2}}$ In $D = 3$, the correlation length diverges as $g \to g_c$: $2\xi = \Lambda^{-1} \left(\frac{2}{\pi} - \frac{4\pi}{g^2}\right)^{-1}$, signaling the presence of gapless modes, which we interpret as Goldstones.

Exercise. Check that the other components of the spin such as $S^z = |z^m|^2 - |z^{m'}|^2$ have the same falloff, as they must by $\mathsf{SU}(N)$ symmetry.

A dynamical gauge field emerges. Finally, let me show you that a gauge field emerges. Let's expand the action $S_{\text{eff}}[A, \lambda]$ in (4.38) about the saddle point at $A = 0, \lambda = \underline{\lambda} \equiv m^2$:

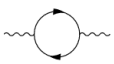
$$S[A = 0 + a, \lambda = m^2 + v] = W_0 + \underbrace{W_1}_{=0 \text{ by def}} + W_2 + \mathcal{O}(\delta^3)$$

where the interesting bit is the terms quadratic in the fluctuations:

$$W_2 = \frac{N}{2} \int \mathrm{đ}^D q \left(v(q) \Pi(q) v(-q) + A_\mu(q) \Pi_{\mu\nu}(q) A_\nu(-q) \right)$$

where

$$\Pi(q) = \ldots = \int \mathrm{đ}^D k \frac{1}{(k^2 + m^2)((k+q)^2 + m^2)} \tag{4.39}$$

$\Pi_{\mu\nu} =$  +diamagnetic diagram $= \int \text{đ}^D k \frac{(2k+q)_\mu (2k+q)_\nu}{(k^2+m^2)((k+q)^2+m^2)} - 2g_{\mu\nu} \int \frac{\text{đ}^D k}{k^2+m^2}.$ (4.40)

Familiarly, gauge invariance implies that $q^\mu \Pi_{\mu\nu}(q) = 0$ – it prevents a mass for the gauge field. For example, in $D = 2$, the long wavelength behavior is

$$\Pi_{\mu\nu}(q) \overset{q\to 0}{\sim} \frac{c}{m^2}\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)$$

which means that the effective action for the gauge fluctuation is

$$W_2 \sim \frac{N}{m^2}\int d^2x F_{\mu\nu}F^{\mu\nu} + \text{more derivatives.}$$

It is a dynamical gauge field.

Another term we can add to the action for a 2d gauge field is

$$\theta \int \frac{F}{2\pi}$$

where we regard $F = dA$ as a two-form. This is the 2d theta term, analogous to $\int F \wedge F$ in $D = 4$ in that $F = dA$ is locally a total derivative, it doesn't affect the equations of motion, and it integrates to an integer on smooth configurations (we will show this when we study anomalies). This integer is called the *Chern number* of the gauge field configuration. What integer is it? On the homework you'll show that $F \propto \epsilon^{abc} n^a dn^b dn^c$. It's the skyrmion number! So the coefficient is $\theta = 2\pi s$.

[End of Lecture 16]

4.4.1 Large-N diagrams.

I think it will help to bring home some of the previous ideas by rederiving them using diagrams in a familiar context. So let's study the $\mathsf{O}(N)$ model:

$$L = \frac{1}{2}\partial\vec{\varphi}\cdot\partial\vec{\varphi} + \frac{g}{4N}\left(\vec{\varphi}\cdot\vec{\varphi}\right)^2 + \frac{m^2}{2}\vec{\varphi}\cdot\vec{\varphi}. \tag{4.41}$$

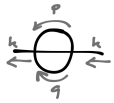
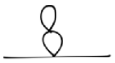
Let's do euclidean spacetime, D dimensions. The bare propagator is

$$\langle\varphi_b(x)\varphi_a(0)\rangle = \delta_{ab}\int \text{đ}^D k\ \frac{e^{-\mathbf{i}kx}}{k^2+m^2} \equiv \delta_{ab}\int \text{đ}^D k\ \Delta_0(k).$$

The bare vertex is $-\frac{2g}{N}\left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right)$. With this normalization, the leading correction to the propagator is

 $= -\frac{g}{4N}(4N+8)\delta_{ab}\int \frac{\text{đ}q}{q^2+m^2} \overset{N\gg 1}{\simeq} -g\delta_{ab}\int \text{đ}q\Delta_0(q)$

of order N^0. This is the motivation for the normalization of the coupling in (4.41).

Which diagrams dominate at large N (and fixed g)? Compare two diagrams at the same order in λ with different topology of the index flow: eyeball  and cactus  . The former has one index loop, and the latter has two, and therefore dominates. The general pattern is that: at large N cacti dominate the 1PI self-energy. Each extra pod we add to the cactus costs a factor of g/N but gains an index loop N. So the sum of cacti is a function of gN^0.

The full propagator, by the usual geometric series, is then

$$\Delta_F(k) = \frac{1}{k^2 + m^2 + \Sigma(k)}. \tag{4.42}$$

We can sum all the cacti by noticing that cacti are self-similar: if we replace Δ_0 by Δ_F in the propagator:

$$\Sigma(p) = g \int \mathrm{đ}^D k \Delta_F(k) + \mathcal{O}(1/N). \tag{4.43}$$

The equations (4.42), (4.43) are integral equations for Δ_F; they are called Schwinger-Dyson equations,

OK, now notice the p-dependence in (4.43): the RHS is independent of p to leading order in N, so $\Sigma(p) = \delta m^2$ is just a mass shift.

Look at the position-space propagator

$$\langle \varphi_b(x) \varphi_a(y) \rangle = \delta_{ab} \int \mathrm{đ}^D k e^{-\mathbf{i}k(x-y)} \Delta_F(k). \tag{4.44}$$

Let

$$y^2 \equiv \left\langle \frac{\sum_a \varphi_a(x)\varphi_a(x)}{N} \right\rangle = \left\langle \frac{\varphi^2}{N} \right\rangle;$$

it is independent of x by translation invariance. Now let $y \to x$ in (4.44):

$$y^2 = \int \mathrm{đ}^D k \Delta_F(k) \overset{(4.43)}{=} g^{-1}\Sigma.$$

Now integrate the BHS of (4.42):

$$\int \mathrm{đ}^D p \Delta_F(p) = \int \mathrm{đ}^D \frac{1}{p^2 + m^2 + \Sigma}$$

$$y^2 = \int \mathrm{đ}^D p \frac{1}{p^2 + m^2 + g y^2}.$$

This is an equation for the positive number y^2. Notice its similarity to the gap equation for λ we found from saddle point.

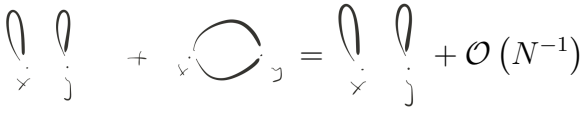
Large-N factorization. [Halpern] The fact that the fluctuations about the saddle point are suppressed by powers of N has consequences for the structure of the correlation functions in a large-N field theory. A basic example is

$$\langle \mathcal{I}(x)\mathcal{I}(y)\rangle = \langle \mathcal{I}(x)\rangle \langle \mathcal{I}(y)\rangle + \mathcal{O}\left(N^{-1}\right)$$

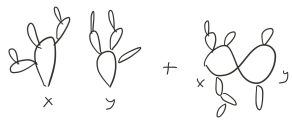
where $\mathcal{I}$ are any invariants of the large-N group (*i.e.* $\mathsf{O}(N)$ in the $\mathsf{O}(N)$ model (naturally) and $\mathsf{SU}(N)$ in the $\mathbb{CP}^{N-1}$ model), and $\langle ...\rangle$ denotes either euclidean vacuum expectation value or time-ordered vacuum expectation value. Consider, for example, in the $\mathsf{O}(N)$ model, normalized as above

$$\left\langle \frac{\varphi^2(x)}{N}\frac{\varphi^2(y)}{N}\right\rangle .$$

In the free theory, $g=0$, there are two diagrams

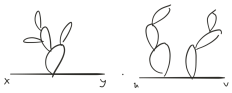
$$\left\langle \frac{\varphi^2(x)}{N}\frac{\varphi^2(y)}{N}\right\rangle_{\text{free}} =$$  $$+\mathcal{O}\left(N^{-1}\right)$$

– the disconnected diagram dominates, because it has one more index loop and the same number of interactions (zero). With interactions, representative diagrams are

$$\left\langle \frac{\varphi^2(x)}{N}\frac{\varphi^2(y)}{N}\right\rangle =$$  $$= \left\langle \frac{\varphi^2(x)}{N}\right\rangle \left\langle \frac{\varphi^2(y)}{N}\right\rangle + \mathcal{O}\left(N^{-1}\right) = y^4 + \mathcal{O}\left(N^{-1}\right)$$

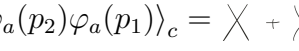
– it is independent of $x-y$ to leading order.

The same phenomenon happens for correlators of non-local singlet operators:

$$\left\langle \frac{\varphi(x)\cdot\varphi(y)}{N}\frac{\varphi(u)\cdot\varphi(v)}{N}\right\rangle =$$  $$= \left\langle \frac{\varphi(x)\cdot\varphi(y)}{N}\right\rangle \left\langle \frac{\varphi(u)\cdot\varphi(v)}{N}\right\rangle + \mathcal{O}\left(N^{-1}\right)$$

The basic statement is that mean field theory works for singlets. At large N, the entanglement follows the flavor lines.

We can still ask: what processes dominate the connected (small) bit at large N? And what about non-singlet operators? Consider (no sum on b, a):

$$G^{b\neq a}_{4,c} = \langle \varphi_b(p_4)\varphi_b(p_3)\varphi_a(p_2)\varphi_a(p_1)\rangle_c =$$  $$+\mathcal{O}\left(N^{-2}\right)$$

The answer is: *bubbles*. More specifically chains of bubbles, propagating in the s-channel. What's special about the s-channel, here? It's the channel in which we can

make $\mathsf{O}(N)$ singlets. In the other channels, we don't get loops with free flavor indices. The diagrams I've displayed all go like $f(g)/N$. Other candidates are eyeballs:  and ladders:  but as you can see, these go like N^{-2}. However, bubbles can have cactuses growing on them, like this:  These all go like $1/N$. To sum all of these, we just use the full propagator in the internal lines of the bubbles, $\Delta_0 \to \Delta_F$.

I claim that the bubble sum is a geometric series:

$$G_{4,c}^{b\neq a} = -\left(\Delta_0(\text{external})\right)^4 \frac{2}{N}\frac{g}{1+gL(p_1+p_2)} + \mathcal{O}\left(N^{-2}\right) \tag{4.45}$$

where L is the loop integral $L(p) \equiv \int \mathrm{d}^D k \Delta_F(k)\Delta_F(p+k)$. You can see this by being careful about the symmetry factors.

 $= \Delta_0\,(\text{external})^4 \left(\frac{g}{4N}\right)\cdot 2\cdot 4$

 $= \Delta_0\,(\text{external})^4 \left(\frac{g}{4N}\right)^2 \cdot 2\cdot 4\cdot 8\cdot \underbrace{\frac{1}{2!}}_{Dyson} L = \Delta_0\,(\text{external})^4 \frac{2}{N}(g)^2 L.$

Similarly, the chain of two bubbles is $\frac{2}{N}g^3L^2$, etc.

Here's how we knew this had to work without worrying about the damn symmetry factors: the bubble chain *is* the σ propagator! At the saddle, $\sigma \simeq \varphi^a\varphi^a$, which is what is going in and out of this amplitude. And the effective action for sigma (after integrating out φ) is

$$S_{\text{eff}}[\sigma] = \int \frac{\sigma^2}{g} + \operatorname{tr}\ln\left(\partial^2 + m^2 + \sigma\right).$$

The connected two-point function means we subtract off $\langle\underline{\sigma}\rangle\langle\underline{\sigma}\rangle$, which is the same as considering the two point function of the deviation from saddle value. This is

$$\langle\sigma_1\sigma_2\rangle = \left(\frac{\delta^2}{\delta\sigma_1\delta\sigma_2}S_{\text{eff}}[\sigma]\right)^{-1} = \left(\frac{1}{g^{-1}+\left(\frac{1}{\partial^2+m^2+\sigma}\right)^2}\right)^{-1}$$

which becomes exactly the expression above if we write it in momentum space.

Two comments: (1) We were pretty brash in integrating out all the φ variables and keeping the σ variable: how do we know which are the slow ones and which are the

fast ones? This sort of non-Wilsonian strategy is common in the literature on large-N, where physicists are so excited to see an integral that they can actually do that they don't pause to worry about slow and fast. But if we did run afoul of Wilson, at least we'll know it, because the action for σ will be nonlocal.

(2) $\sigma \sim \varphi^2$ is a composite operator. Nevertheless, the sigma propagator we've just derived can have poles at some $p^2 = m^2$ (likely with complex m). These would produce particle-like resonances in a scattering experiment (such as $2-2$ scattering of φs of the same flavor) which involved sigmas propagating in the s-channel. Who is to say what is fundamental.

Now that you believe me, look again at (4.45); it is of the form

$$G_{4,c}^{b\neq a} = -\left(\Delta_0(\text{external})\right)^4 \frac{2}{N} g_{\text{eff}}(p_1+p_2) + \mathcal{O}\left(N^{-2}\right)$$

where now

$$g_{\text{eff}}(p) = \frac{g}{1 + g\int \mathrm{d}^D k \Delta_F(k)\Delta_F(p+k))}$$

is a momentum-dependent effective coupling, just like one dreams of when talking about the RG.

5 Duality

5.1 XY transition from superfluid to Mott insulator, and T-duality

In this subsection and the next we're going to study ways to think about bosonic field theories with a $\mathsf{U}(1)$ symmetry, and dualities between them, in $D = 1{+}1$ and $D = 2{+}1$.

[This discussion is from Ashvin Vishwanath's lecture notes.] Consider the Bose-Hubbard model (in any dimension, but we'll specify to $D = 1 + 1$ at some point)

$$\mathbf{H}_{BH} = -\tilde{J}\sum_{\langle ij\rangle}\left(\mathbf{b}_i^\dagger \mathbf{b}_j + h.c.\right) + \frac{U}{2}\sum_i \mathbf{n}_i\left(\mathbf{n}_i - 1\right) - \mu\sum_i \mathbf{n}_i$$

where the $\mathbf{b}^\dagger$s and $\mathbf{b}$ are bosonic creation and annihilation operators at each site: $[\mathbf{b}_i, \mathbf{b}_j^\dagger] = \delta_{ij}$. $\mathbf{n}_i \equiv \mathbf{b}_i^\dagger \mathbf{b}_i$ counts the number of bosons at site i. The second Hubbard-U term is zero if $\mathbf{n}_j^b = 0, 1$, but exacts an energetic penalty $\Delta E = U$ if a single site j is occupied by two bosons.

The Hilbert space which represents the boson algebra has a useful *number-phase* representation in terms of

$$[\mathbf{n}_i, \phi_j] = -\mathbf{i}\delta_{ij}, \quad \phi_i \equiv \phi_i + 2\pi, \quad \mathbf{n}_i \in \mathbb{Z}$$

(where the last statement pertains to the eigenvalues of the operator). The bosons are

$$\mathbf{b}_i = e^{-\mathbf{i}\phi_i}\sqrt{\mathbf{n}_i}, \quad \mathbf{b}_i^\dagger = \sqrt{\mathbf{n}_i}e^{+\mathbf{i}\phi_i} \ ;$$

these expressions have the same algebra as the original $\mathbf{b}$s. In terms of these operators, the hamiltonian is

$$\mathbf{H}_{BH} = -\tilde{J}\sum_{\langle ij\rangle}\left(\sqrt{\mathbf{n}_i}e^{\mathbf{i}(\phi_i - \phi_j)}\sqrt{\mathbf{n}_j} + h.c.\right) + \frac{U}{2}\sum_i \mathbf{n}_i\left(\mathbf{n}_i - 1\right) - \mu\sum_i \mathbf{n}_i.$$

If $\langle \mathbf{n}_i\rangle = n_0 \gg 1$, so that $\mathbf{n}_i = n_0 + \Delta\mathbf{n}_i, \Delta\mathbf{n}_i \ll n_0$ then $\mathbf{b}_i = e^{-\mathbf{i}\phi}\sqrt{\mathbf{n}_i} \simeq e^{-\mathbf{i}\phi_i}\sqrt{n_0}$ and

$$\mathbf{H}_{BH} \simeq -\underbrace{2\tilde{J}n_0}_{\equiv J}\sum_{\langle ij\rangle}\cos\left(\phi_i - \phi_j\right) + \frac{U}{2}\sum_i \left(\Delta\mathbf{n}_i\right)^2 \equiv \mathbf{H}_{\text{rotors}}$$

where we set $n_0 \equiv \mu/U \gg 1$. This is a *rotor model*.

This model has two phases:
$\boxed{U \gg J}$: then we must satisfy the U term first and the number is locked, $\Delta\mathbf{n} = 0$ in

the groundstate. This is a Mott insulator, with a gap of order U. Since $\mathbf{n}$ and ϕ are conjugate variables, definite number means wildly fluctuating phase.

$\boxed{U \ll J}$: then we must satisfy the J term first and the phase is locked, $\phi =$ constant in the groundstate, or at least it will try. This is the superfluid (SF). That is, we can try to expand the cosine potential[30]

$$\mathbf{H}_{\text{rotors}} = U\sum_i \mathbf{n}_i^2 - J\sum_{\langle ij\rangle}\cos\left(\phi_i-\phi_j\right) \simeq U\sum_i \mathbf{n}_i^2 - J\sum_{\langle ij\rangle}\left(1-\frac{1}{2}\left(\phi_i-\phi_j\right)^2+...\right) \tag{5.1}$$

which is a bunch of harmonic oscillators and can be solved by Fourier: $\phi_i = \frac{1}{\sqrt{N^d}}\sum_i e^{-\mathbf{i}k\cdot x_i}\phi_k$, so

$$\mathbf{H} \simeq \sum_k \left(U\pi_k\pi_{-k} + J\left(1-\cos ka\right)\phi_k\phi_{-k}\right)$$

This has gapless *phonon* modes at $k=0$, whose existence is predicted by Nambu-Goldstone. I have written the hamiltonian in 1d notation but nothing has required it so far. The low energy physics is described by the continuum lagrangian density

$$L_{\text{eff}} = \frac{\rho_s}{2}\left(\frac{\left(\partial_\tau\phi\right)^2}{c} + c\left(\vec{\nabla}\phi\right)^2\right) \tag{5.2}$$

with $\rho_s = \sqrt{J/U}, c = \sqrt{JU}$. ρ_s is called the superfluid stiffness. This is a free massless scalar theory. The demand of the $\mathsf{U}(1)$ symmetry $\phi\to\phi+\alpha$ forbids interactions which would be relevant; the only allowed interactions are derivative interactions (as you can see by keeping more terms in the Taylor expansion (5.1)) such as $(\partial\phi)^4$.

Now 1d comes in: In $d>1$, there is long range order – the bosons condense and spontaneously break the phase rotation symmetry $\phi\to\phi+\alpha$; the variable ϕ is a Goldstone boson. In 1d there is no long-range order. The two phases are still distinct however, since one has a gap and the other does not. The correlators of the boson operator $b_i\sim e^{\mathbf{i}\phi_i}$ diagnose the difference. In the Mott phase they have exponential decay. In the "SF" they have

$$\left\langle e^{\mathbf{i}\phi(x)}e^{-\mathbf{i}\phi(y)}\right\rangle = \frac{c_0}{r^\eta},\quad \eta = \frac{1}{2\pi\rho_s} = \frac{1}{2K}.$$

This is *algebraic long range order*. This is a sharp distinction between the two phases we've discussed, even though the IR fluctuations destroy the $\langle b\rangle$.

[30] From now on the background density n_0 will not play a role and I will write $\mathbf{n}_i$ for $\Delta\mathbf{n}_i$.

Massless scalars in $D = 1 + 1$ and T-duality-invariance of the spectrum. A lot of physics is hidden in the innocent-looking theory of the superfluid goldstone boson. Consider the following (real-time) continuum action for a free massless scalar field in 1+1 dimensions:

$$S[\phi] = \frac{T}{2} \int dt \int_0^L dx \left((\partial_0 \phi)^2 - (\partial_x \phi)^2 \right) = 2T \int dx dt \partial_+ \phi \partial_- \phi \ . \tag{5.3}$$

I have set the velocity of the bosons to $c = 1$ by rescaling t. Here $x^\pm \equiv t \pm x$ are lightcone coordinates; the derivatives are $\partial_\pm \equiv \frac{1}{2}(\partial_t \pm \partial_x)$. Space is a circle: the point labelled x is the same as the point labelled $x + L$. It will sometimes be useful to call actual space the 'base space', to distinguish it from the field space, aka the 'target space'. This denotation is common in the study of nonlinear sigma models, which are field theories of maps from the base space to the target space.

We will assume that the field space of ϕ itself is periodic:

$$\phi(x,t) \equiv \phi(x,t) + 2\pi, \quad \forall x, t \ .$$

So the *field* space is a circle S^1 with (angular) coordinate ϕ. It can be useful to think of the action (5.3) as describing the propagation of a string, since a field configuration describes an embedding of the real two dimensional space into the *target space*, which here is a circle. This is a simple special case of a nonlinear sigma model. The name T-duality comes from the literature on string theory. The worldsheet theory of a string propagating on a circle of radius $R = \sqrt{\rho_s}$ is governed by the Lagrangian (5.2). To see this, recall that the action of a 2d nonlinear sigma model with target space metric $g_{\mu\nu}\phi^\mu\phi^\nu$ is $\frac{1}{\alpha'}\int d^2\sigma g_{\mu\nu}\partial\phi^\mu\partial\phi^\nu$. Here $\frac{1}{\alpha'}$ is the tension (energy per unit length) of the string; work in units where this disappears from now on. Here we have only one dimension, with $g_{\phi\phi} = \rho_s$.

Notice that we could rescale $\phi \to \lambda\phi$ and change the radius; but this would change the periodicity of $\phi \equiv \phi + 2\pi$. The proper length of the period is $2\pi R$ and is invariant under a change of field variables. This proper length distinguishes different theories because the operators $: e^{\alpha\phi} :$ (and all good operators of definite scaling dimension in the theory of the free boson (unlike ϕ itself)) must be periodic; this determines the allowed values of α.

First a little bit of classical field theory. The equations of motion for ϕ are

$$0 = \frac{\delta S}{\delta \phi(x,t)} \propto \partial^\mu \partial_\mu \phi \propto \partial_+ \partial_- \phi$$

which is solved by

$$\phi(x,t) \equiv \phi_L(x^+) + \phi_R(x^-) \ .$$

In euclidean time, $\phi_{L,R}$ depend (anti-)holomorphically on the complex coordinate $z \equiv \frac{1}{2}(x + \mathbf{i}\tau)$ and the machinery of complex analysis becomes useful.

Symmetries: Since $S[\phi]$ only depends on ϕ through its derivatives, there is a simple symmetry $\phi \to \phi + \epsilon$. By the Nöther method the associated current is

$$j_\mu = T\partial_\mu \phi \ . \tag{5.4}$$

This symmetry is translations in the target space, and so I will sometimes call the associated conserved charge 'momentum'.

There is another symmetry which is less obvious. It comes about because of the topology of the target space. Since $\phi(x,t) \equiv \phi(x,t) + 2\pi m, m \in \mathbb{Z}$ describe the same point (it is a redundancy in our description, in fact a discrete gauge redundancy), we don't need $\phi(x+L,t) \stackrel{?}{=} \phi(x,t)$. To make the field configuration single-valued, it is enough to have

$$\phi(x+L,t) = \phi(x,t) + 2\pi m, \quad m \in \mathbb{Z}$$

The number m cannot change without the string breaking: it is a topological charge, a winding number:

$$m = \frac{1}{2\pi}\phi(x,t)|_{x=0}^{x=L} \stackrel{\text{FTC}}{=} \frac{1}{2\pi}\int_0^L dx \partial_x \phi \ . \tag{5.5}$$

The associated current whose charge density is $\frac{1}{\pi}\partial_x\phi$ (which integrates over space to the topological charge) is

$$\tilde{j}_\mu = \frac{1}{2\pi}(\partial_x\phi, -\partial_0\phi)_\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi. \tag{5.6}$$

This is conserved because of the equality of the mixed partials: $\epsilon^{\mu\nu}\partial_\mu\partial_\nu = 0$.

[End of Lecture 17]

Let's expand in normal modes: $\phi = \phi_L + \phi_R$ with

$$\begin{aligned}
\phi_L(t+x) &= q_L + \underbrace{(p+w)}_{\equiv \frac{1}{2T}p_L}(t+x) - \mathbf{i}\sqrt{\frac{L}{4\pi T}}\sum_{n\neq 0}\frac{\rho_n}{n}e^{\mathbf{i}n(t+x)\frac{2\pi}{L}}, \\
\phi_R(t-x) &= q_R + \underbrace{(p-w)}_{\equiv \frac{1}{2T}p_R}(t-x) - \mathbf{i}\sqrt{\frac{L}{4\pi T}}\sum_{n\neq 0}\frac{\tilde{\rho}_n}{n}e^{\mathbf{i}n(t-x)\frac{2\pi}{L}},
\end{aligned} \tag{5.7}$$

The factor of $\frac{1}{n}$ is a convention whose origin you will appreciate below, as are the other normalization factors. Real ϕ means $\rho_n^\dagger = \rho_{-n}$ (If we didn't put the $\mathbf{i}$ it would have been $-\rho_{-n}$).

Here $q \equiv \frac{1}{L}\int_0^L dx\phi(x,t) = q_L + q_R$ is the center-of-mass position of the string. The canonical momentum for ϕ is $\pi(x,t) = T\partial_0\phi(x,t) = T\left(\partial_+\phi_L + \partial_-\phi_R\right)$.

QM. Now we'll do quantum mechanics. Recall that a quantum mechanical particle on a circle has momentum quantized in units of integers over the period. Since ϕ is periodic, the wavefunction(al)s must be periodic in the center-of-mass coordinate q with period 2π, and this means that the total (target-space) momentum must be an integer

$$\mathbb{Z} \ni j = \pi_0 \equiv \int_0^L dx \pi(x,t) = T \int_0^L dx \partial_t \phi \overset{(5.7)}{=} LT2p$$

So our conserved charges are quantized according to

$$p = \frac{j}{2LT}, \qquad w \overset{(5.7)(5.5)}{=} \frac{\pi m}{L}, \qquad j, m \in \mathbb{Z}\ .$$

(Don't confuse the target-space momentum j with the 'worldsheet momentum' n!)

(Note that this theory is scale-free. We could use this freedom to choose units where $L = 2\pi$.)

Now I put the mode coefficients in boldface:

$$\begin{aligned}
\boldsymbol{\phi}_L(x^+) &= \mathbf{q}_L + \frac{1}{2T}\mathbf{p}_L x^+ - \mathbf{i}\sqrt{\frac{L}{4\pi T}} \sum_{n\neq 0} \frac{\boldsymbol{\rho}_n}{n} e^{\mathbf{i}\frac{2\pi}{L}nx^+}, \\
\boldsymbol{\phi}_R(x^-) &= \mathbf{q}_R + \frac{1}{2T}\mathbf{p}_R x^- - \mathbf{i}\sqrt{\frac{L}{4\pi T}} \sum_{n\neq 0} \frac{\tilde{\boldsymbol{\rho}}_n}{n} e^{\mathbf{i}\frac{2\pi}{L}nx^-},
\end{aligned} \tag{5.8}$$

The nonzero canonical equal-time commutators are

$$[\boldsymbol{\phi}(x), \boldsymbol{\pi}(x')] = \mathbf{i}\delta(x - x')$$

which determines the commutators of the modes (this was the motivation for the weird normalizations)

$$[\mathbf{q}_L, \mathbf{p}_L] = [\mathbf{q}_R, \mathbf{p}_R] = \mathbf{i}, \qquad [\boldsymbol{\rho}_n, \boldsymbol{\rho}_{n'}^\dagger] = n\delta_{n,n'}, \text{or} \quad [\boldsymbol{\rho}_n, \boldsymbol{\rho}_{n'}] = n\delta_{n+n'}\ ,$$

and the same for the rightmovers with twiddles (and $[\boldsymbol{\rho}, \tilde{\boldsymbol{\rho}}] = 0$). This is one simple harmonic oscillator for each $n \geq 1$ (and each chirality); the funny normalization is conventional.

$$\begin{aligned}
\mathbf{H} &= \int dx \left(\boldsymbol{\pi}(x)\dot{\boldsymbol{\phi}}(x) - \mathcal{L}\right) = \frac{1}{2}\int dx \left(\frac{\boldsymbol{\pi}^2}{T} + T\left(\partial_x \boldsymbol{\phi}\right)^2\right) \\
&= L \underbrace{\frac{1}{4T}\left(\mathbf{p}_L^2 + \mathbf{p}_R^2\right)}_{\frac{\boldsymbol{\pi}_0^2}{2T} + \frac{T}{2}\mathbf{w}^2} + \pi \sum_{n=1}^{\infty} \left(\boldsymbol{\rho}_{-n}\boldsymbol{\rho}_n + \tilde{\boldsymbol{\rho}}_{-n}\tilde{\boldsymbol{\rho}}_n\right) + \mathfrak{a} \\
&= \frac{1}{2L}\left(\frac{j^2}{T} + T\left(2\pi m\right)^2\right) + \pi \sum_{n=1}^{\infty} n\left(\mathbf{N}_n + \tilde{\mathbf{N}}_n\right) + \mathfrak{a}
\end{aligned} \tag{5.9}$$

Here $\mathfrak{a}$ is a (UV sensitive) constant which will not be important for us (it is very important in string theory), which is the price we pay for writing the hamiltonian as a sum of normal-ordered terms – the modes with negative indices are to the right and they annihilate the vacuum:

$$\boldsymbol{\rho}_n |0\rangle = 0, \quad \tilde{\boldsymbol{\rho}}_n |0\rangle = 0, \quad \text{for } n > 0 \ .$$

Energy eigenstates can be labelled by a target-momentum j and a winding m. Notice that there is an operator $\mathbf{w}$ whose eigenvalues are w, and it has a conjugate momentum $\mathbf{p}_L - \mathbf{p}_R$ which increments its value. So when I write $|0\rangle$ above, I really should label a vacuum of the oscillator modes with p, w.

$\mathbf{N}_n \equiv \frac{1}{n}\boldsymbol{\rho}_{-n}\boldsymbol{\rho}_n$ is the number operator; if we redefine $\mathbf{a}_n \equiv \sqrt{n}^{-1}\boldsymbol{\rho}_n (n > 0)$, we have $[\mathbf{a}_n, \mathbf{a}_m^\dagger] = \delta_{nm}$ and $\mathbf{N}_n = \mathbf{a}_n^\dagger \mathbf{a}_n$ is the ordinary thing.

Notice that the separate conservation of (5.4) and (5.6) means that there are separately-conserved left-moving and right-moving currents:

$$(j_L)^\mu = (j_L^z, j_L^{\bar{z}})^\mu \equiv (j_+, 0)^\mu$$

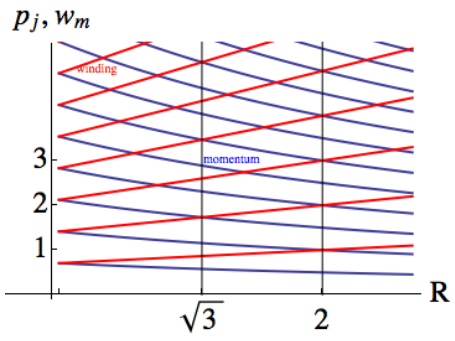
$$(j_R)^\mu = (j_R^z, j_R^{\bar{z}})^\mu \equiv (0, j_-)^\mu$$

Here j_L only depends on the modes $\boldsymbol{\rho}_n$, and j_R only depends on the modes $\tilde{\boldsymbol{\rho}}_n$:

$$j_+ = \partial_+\phi = \partial_+\phi(x^+) = \mathbf{p} + \mathbf{w} + \sqrt{\frac{\pi}{LT}} \sum_{n\neq 0} \boldsymbol{\rho}_n e^{\mathbf{i}\frac{2\pi}{L}nx^+}$$

$$j_- = \partial_-\phi = \partial_-\phi(x^-) = \mathbf{p} - \mathbf{w} + \sqrt{\frac{\pi}{LT}} \sum_{n\neq 0} \tilde{\boldsymbol{\rho}}_n e^{\mathbf{i}\frac{2\pi}{L}nx^-}$$

Here's an **Observation (T-duality)**: At large T (think of this as a large radius of the target space), the momentum modes are closely-spaced in energy, and exciting the winding modes is costly, since the string has a tension, it costs energy-per-unit-length T to stretch it. But the spectrum (5.9) is invariant under the operation



$$m \leftrightarrow j, \quad T \leftrightarrow \frac{1}{(2\pi)^2 T}$$

which takes the radius of the circle to its inverse and exchanges the momentum and winding modes. This is called *T-duality*. The required duality map on the fields is

$$\phi_L + \phi_R \leftrightarrow \phi_L - \phi_R \ .$$

(The variable R in the plot is $R \equiv \sqrt{\pi T}$.)

T-duality says string theory on a large circle is the same as string theory on a small circle. On the homework you'll get to see a derivation of this statement in the continuum which allows some generalizations.

Vertex operators. It is worthwhile to pause for another moment and think about the operators which create the winding modes. They are like vortex creation operators. Since ϕ has logarithmic correlators, you might think that exponentiating it is a good idea. First let's take advantage of the fact that the ϕ correlations split into left and right bits to write $\phi(z,\bar{z}) = \phi_L(z) + \phi_R(\bar{z})$:

$$\langle \phi_L(z)\phi_L(0)\rangle = -\frac{1}{\pi T}\log\frac{z}{a}, \quad \langle \phi_R(\bar{z})\phi_R(0)\rangle = -\frac{1}{\pi T}\log\frac{\bar{z}}{a}\ , \quad \langle \phi_L(z)\phi_R(0)\rangle = 0\ . \tag{5.10}$$

A set of operators with definite scaling dimension is:

$$\mathcal{V}_{\alpha,\beta}(z,\bar{z}) =: e^{\mathbf{i}(\alpha\phi_L(z)+\beta\phi_R(\bar{z}))} :\ .$$

This is a composite operator which we have defined by normal-ordering. The normal ordering prescription is: $\boxed{q,p,-,+}$, that is: positive-momentum modes (lowering operators) go on the right, and p counts as a lowering operator, so in particular using the expansion (please beware my factors here): $\phi_L(z) = \mathbf{q}_L + \mathbf{p}_L z + \mathbf{i}\sum_{n\neq 0}\frac{\boldsymbol{\rho}_n}{n}w^n$, we have

$$: e^{\mathbf{i}\alpha\phi_L(z)} :\equiv e^{\mathbf{i}\alpha\mathbf{q}_L}e^{\mathbf{i}\alpha\mathbf{p}_L z}e^{\mathbf{i}\alpha\sum_{n<0}\frac{\boldsymbol{\rho}_n}{n}w^n}e^{\mathbf{i}\alpha\sum_{n>0}\frac{\boldsymbol{\rho}_n}{n}w^n}$$

(I used the definition $w \equiv e^{2\pi i z/L}$.)

How should we think about this operator ? In the QM of a free particle, the operator $e^{\mathbf{i}p\mathbf{x}}$ inserts momentum p – it takes a momentum-space wavefunction $\psi(p_0) = \langle p_0|\psi\rangle$ and gives

$$\langle p_0|\, e^{\mathbf{i}p\mathbf{x}}\,|\psi\rangle = \psi(p_0+p).$$

It's the same thing here, with one more twist.

In order for $\mathcal{V}_{\alpha,\beta}$ to be well-defined under $\phi \to \phi + 2\pi$, we'd better have $p = \frac{\alpha+\beta}{2} \in \mathbb{Z}$ – momentum is quantized, just like for the particle (the center of mass is just a particle). Let's consider what the operator $\mathcal{V}_{\alpha,\beta}$ does to a winding and momentum eigenstate $|w,p\rangle$ (for simplicity, take one with no oscillator excitations, $\boldsymbol{\rho}_n\,|p,w\rangle = 0, n<0$):

$$\mathcal{V}_{\alpha\beta}(0)\,|w,p\rangle = e^{\mathbf{i}\left(\frac{\alpha+\beta}{2}\right)\mathbf{q}_0}e^{\mathbf{i}\left(\frac{\alpha-\beta}{2}\right)\tilde{\boldsymbol{\phi}}_0}e^{\mathbf{i}\alpha\sum_{n<0}\boldsymbol{\rho}_n}e^{\mathbf{i}\alpha\sum_{n>0}\boldsymbol{\rho}_n}\,|w,p\rangle = e^{\mathbf{i}\alpha\sum_{n<0}\boldsymbol{\rho}_n}\left|w+\frac{\alpha-\beta}{2}, p+\frac{\alpha+\beta}{2}\right\rangle \tag{5.11}$$

The monster in front here creates oscillator excitations. I wrote $\mathbf{q}_0 \equiv \mathbf{q}_L + \mathbf{q}_R$ and $\tilde{\boldsymbol{\phi}}_0 \equiv \mathbf{q}_L - \mathbf{q}_R$. The important thing is that the winding number has been incremented

by $\alpha - \beta$; this means that $\alpha - \beta$ must be an integer, too. We conclude that

$$\alpha + \beta \in 2\mathbb{Z}, \quad \alpha - \beta \in 2\mathbb{Z} \tag{5.12}$$

so they can both be odd integers or they can both be even integers.

By doing the gaussian integral (or moving the annihilation operators to the right) their correlators are

$$\langle \mathcal{V}_{\alpha,\beta}(z,\bar{z}) \mathcal{V}_{\alpha',\beta'}(0,0)\rangle = \frac{D_0}{z^{\frac{\alpha^2}{\pi T}} \bar{z}^{\frac{\beta^2}{\pi T}}} \ . \tag{5.13}$$

The zeromode prefactor D_0 is:

$$D_0 = \left\langle e^{\mathbf{i}((\alpha+\alpha')\mathbf{q}_L + (\beta+\beta')\mathbf{q}_R)} \right\rangle_0 = \delta_{\alpha+\alpha'} \delta_{\beta+\beta'}.$$

This is charge conservation.

We conclude that the operator $\mathcal{V}_{\alpha,\beta}$ has scaling dimension $\Delta = h_L + h_R$ with

$$(h_L, h_R) = \frac{1}{2\pi T}(\alpha^2, \beta^2).$$

($h_L - h_R$ is the spin.) Notice the remarkable fact that the exponential of a dimension-zero operator manages to have nonzero scaling dimension. This requires that the multiplicative prefactor depend on the cutoff a to the appropriate power (and it is therefore nonuniversal). We could perform a multiplicative renormalization of our operators $\mathcal{V}$ to remove this cutoff dependence from the correlators.

The values of α, β allowed by single-valuedness of ϕ and its wavefunctional are best understood in terms of the integers j, m that we introduced above. We see (at least) three special values of the parameter T:

- **The SU(2) radius**: When $2\pi T = 1$, T-duality maps the theory to itself. The operators with $(\alpha, \beta) = (1, 1)$ are *marginal*. Also, the operators with $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$ have the scaling behavior of currents ($\langle j_-(z) j_-(0)\rangle \propto \frac{1}{z^2}$) and by holomorphicity are in fact conserved.

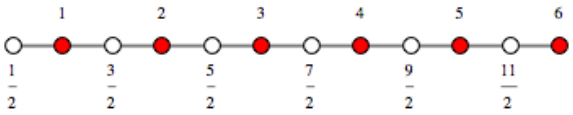
- **The free fermion radius**: when $2\pi T = 2$, $\mathcal{V}_{1,0}$ in (5.13) looks like $\langle \mathcal{V}_{1,0}(z)\mathcal{V}_{1,0}(0)\rangle = \frac{D}{z^{\frac{1}{2}}}$ which is the behavior of a left-moving free fermion, with action $\int dt dx \bar{\psi}\partial_+\psi$. In fact the scalar theory with this radius is (locally) equivalent to a massless Dirac fermion! This equivalence is an example of *bosonization*. In particular, the radius-changing deformation of the boson maps to a marginal four-fermion interaction: by studying free bosons we can learn about interacting fermions. (More precisely, just like in the Jordan-Wigner description of the TFIM, the scalar theory is equivalent to a Dirac fermion with the fermion number gauged.)

- **The supersymmetric radius**: when $2\pi T = \frac{2}{3}$, $\mathcal{V}_{1,0}$ has dimension $(\frac{3}{2}, 0)$ and represents a supersymmetry current.

After this detour, let's turn to the drama of the bose-Hubbard model. Starting from large J/U, where we found a superfluid, what happens as U grows and makes the phase fluctuate more? Our continuum description in terms of harmonic oscillators hides (but does not ignore) the fact that $\phi \simeq \phi + 2\pi$. The system admits vortices, aka winding modes.

Lattice T-duality. To see their effects let us do T-duality on the lattice.

The dual variables live on the bonds, labelled by $\bar{i} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$....



Introduce

$$\mathbf{m}_{\bar{i}} \equiv \frac{\phi_{i+1} - \phi_i}{2\pi}, \quad \Theta_{\bar{i}} \equiv \sum_{j<\bar{i}} 2\pi \mathbf{n}_j \tag{5.14}$$

which together imply

$$[\mathbf{m}_{\bar{i}}, \Theta_{\bar{j}}] = -\mathbf{i}\delta_{\bar{i}\bar{j}}.$$

To understand where these expressions come from, notice that the operator

$$e^{\mathbf{i}\Theta_{\bar{i}}} = e^{\mathbf{i}\sum_{j<\bar{i}} 2\pi \mathbf{n}_j}$$

rotates the phase of the boson on all sites to the left of $\bar{i}$ (by 2π). It inserts a vortex in between the sites i and $i+1$. The rotor hamiltonian is

$$\begin{aligned} \mathbf{H}_{\text{rotors}} &= \frac{U}{2} \sum_{\bar{i}} \left(\frac{\Theta_{\bar{i}+1} - \Theta_{\bar{i}}}{2\pi} \right)^2 - J \sum_{\bar{i}} \cos 2\pi \mathbf{m}_{\bar{i}} \\ &\overset{SF}{\simeq} \sum_{\bar{i}} \left(\frac{U}{2} \left(\frac{\Delta\Theta}{2\pi} \right)^2 + \frac{J}{2} (2\pi \mathbf{m}_{\bar{i}})^2 \right) \end{aligned} \tag{5.15}$$

where in the second step, we assumed we were in the SF phase, so the phase fluctuations and hence $\mathbf{m}_{\bar{i}}$ are small. This looks like a chain of masses connected by springs again, but with the roles of kinetic and potential energies reversed – the second term should be regarded as a π^2 kinetic energy term. BUT: we must not forget that $\Theta \in 2\pi\mathbb{Z}$! It's oscillators with discretized positions. We can rewrite it in terms of continuous Θ at the expense of imposing the condition $\Theta \in 2\pi\mathbb{Z}$ energetically by adding a term $-\lambda \cos \Theta$[31].

[31] This step seems scary at first sight, since we're adding degrees of freedom to our system, albeit gapped ones. $\Theta_{\bar{i}}$ is the number of bosons to the left of $\bar{i}$ (times 2π). An analogy that I find useful is

The resulting model has the action

$$L_{\text{eff}} = \frac{1}{2(2\pi)^2\rho_s}\left(\partial_\mu\Theta\right)^2 - \lambda\cos\Theta. \tag{5.16}$$

Ignoring the λ term, this is the T-dual action, with ρ_s replaced by $\frac{1}{(2\pi)^2\rho_s}$. The coupling got inverted here because in the dual variables it's the J term that's like the π^2 inertia term, and the U term is like the restoring force. This $\Theta = \phi_L - \phi_R$ is therefore T-dual variable, with ETCRs

$$[\phi(x), \Theta(y)] = 2\pi\mathbf{i}\mathsf{sign}(x-y). \tag{5.17}$$

This commutator follows directly from the definition of Θ (5.14). (5.17) means that the operator $\cos\Theta(x)$ jumps the SF phase variable ϕ by 2π – it inserts a 2π vortex, as we designed it to do. So λ is like a chemical potential for vortices.

This system has two regimes, depending on the scaling dimension of the vortex insertion operator:

• If λ is an irrelevant coupling, we can ignore it in the IR and we get a superfluid, with algebraic LRO.

• If the vortices are relevant, $\lambda \to \infty$ in the IR, and we pin the dual phase, $\Theta_{\bar{i}} = 0, \forall\bar{i}$. This is the Mott insulator, since $\Theta_{\bar{i}} = 0$ means $\mathbf{n}_i = 0$ – the number fluctuations are frozen.

When is λ relevant? Expanding around the free theory,

$$\left\langle e^{\mathbf{i}\Theta(x)}e^{-\mathbf{i}\Theta(0)}\right\rangle = \frac{c}{x^{2\pi\rho_s}}$$

this has scaling dimension $\Delta = \pi\rho_s$ which is relevant if $2 > \Delta = \pi\rho_s$. Since the bose correlators behave as $\left\langle b^\dagger b\right\rangle \sim x^{-\eta}$ with $\eta = \frac{1}{2\pi\rho_s}$, we see that only if $\eta < \frac{1}{4}$ do we have a stable SF phase. (Recall that $\rho_s = \sqrt{J/U}$.) If $\eta > \frac{1}{4}$, the SF is unstable to proliferation of vortices and we end up in the Mott insulator, where the quantization of particle number matters. A lesson: we can think of the Mott insulator as a condensate of vortices.

Note: If we think about this euclidean field theory as a 2+0 dimensional stat-mech problem, the role of the varying ρ_s is played by temperature, and this transition we've found of the XY model, where by varying the radius the vortices become relevant, is the Kosterlitz-Thouless transition. Most continuous phase transitions occur by tuning

to the fact that the number of atoms of air in the room is an integer. This constraint can have some important consequences, for example, were they to solidify. But in our coarse-grained description of the fluid phase, we use variables (the continuum number density) where the number of atoms (implicitly) varies continuously. The nice thing about this story (both for vortices and for air) is that the system tells us when we can't ignore this quantization constraint.

the coefficient of a relevant operator to zero (recall the general $\mathsf{O}(n)$ transition, where we have to tune $r \to r_c$ to get massless scalars). This is *not* what happens in the 2d XY model; rather, we are varying a *marginal* parameter and the dimensions of other operators depend on it and become relevant at some critical value of that marginal parameter. This leads to very weird scaling near the transition, of the form $e^{-\frac{a}{\sqrt{K-K_c}}}$ (for example, in the correlation length, the exponential arises from inverting expressions involving $G_R(z) = -\frac{1}{4\pi K}\log z$) – it is sometimes called an 'infinite order' phase transition, because all derivatives of such a function are continuous.

[End of Lecture 18]

If you are eager to learn more about duality, I recommend these new notes, the first section of which offers a useful parallel to the discussion above (part of which I actually followed in lecture).

6 Conformal field theory

I haven't been very explicit about it so far, but we've been studying CFT for a while now. I decided that it would be useful to understand some important examples well before saying anything about the more formal aspects of the structure of CFT and constraints on physics from conformal invariance.

So far we have been studying examples of CFT which have a free-field description (maybe not always an obvious one). When this crutch is absent, what do we do? Remarkably, the answer is not 'nothing'. In particular we can define what we mean by CFT without recourse to perturbation theory, and sometimes we can even *solve* such a thing (I will try to make clear what 'solve' means here).

Operator product expansion (OPE).

In QFT, if we act with a bunch of local operators close together (compared to their separation from other operators), their net effect is that of another local operator. If we have a *basis* of operators in our CFT, we may expand this net effect in terms of this basis. So far this is true in any QFT. In a scale-invariant theory, we can choose a basis of operators of definite scaling dimension[32],

$$\mathcal{O}_a \to \lambda^{-\Delta_a}\mathcal{O}_a, \qquad \left\langle \mathcal{O}_a^\dagger(z)\mathcal{O}_b(0)\right\rangle \sim \frac{\delta_{ab}}{z^{2\Delta_a}}.$$

[32]Some fine print: actually, this requires that the dilatation operator is hermitian. This fails to be the case in some non-unitary theories, for example in *logarithmic CFTs* where the dilatation operator involves Jordan blocks, and where therefore the OPE involves also some logarithms (which shift under scale transformations).

We can organize this expansion (the *operator product expansion*) in order of increasing dimension, $\{\mathcal{O}_a, \Delta(a) \leq \Delta(b) \text{ if } a < b\}$. Then the operators with the longest range of influence come first:

$$\begin{aligned}\mathcal{O}_A(z)\mathcal{O}_B(0) &\overset{z\to 0}{\sim} C_{AB1}(z)\mathcal{O}_1(0) + C_{AB2}(z)\mathcal{O}_2(0) + \cdots \\ &= \frac{c_{AB1}}{z^{\Delta_A+\Delta_B-\Delta_1}}\mathcal{O}_1(0) + \frac{c_{AB2}}{z^{\Delta_A+\Delta_B-\Delta_2}}\mathcal{O}_2(0) + \cdots \end{aligned} \tag{6.1}$$

In general, there will be power-law singularities in the coefficients, produced by the collisions of the operators; the powers are determined by scaling. But at some point in the expansion, the effects start to vanish as $z \to 0$, *i.e.* $\Delta_A + \Delta_B - \Delta_n < 0$ for Δ_n big enough. The terms in the OPE after this are *regular terms* and can often be ignored, for example because they don't effect correlation functions without more insertions nearby. In the future, we will be writing expressions like

$$\partial\phi(z)\partial\phi(0) \sim \frac{1}{z^2} + \text{regular terms} .$$

After this, "plus regular terms" will be implicit.

Free massless fermions in 1+1d have groundstate correlators that go like

$$\left\langle \mathbf{\Psi}^\dagger(x)\mathbf{\Psi}(0)\right\rangle \propto \frac{1}{x}. \tag{6.2}$$

In terms of the OPE, this can be expressed as $\psi_R^\dagger(x,t)\psi_R(0) \sim -\frac{1}{t-x}$ or in euclidean coords

$$\psi_R^\dagger(\bar z)\psi_R(0) \sim \frac{1}{\bar z}.$$

The operators $e^{\mathbf{i}\phi(x)}$ satisfy

$$: e^{\mathbf{i}n\phi_L(z)} :: e^{\mathbf{i}m\phi_L(0)} :=: e^{\mathbf{i}(n+m)\phi_L(0)} : z^{\frac{2nm}{2\pi T}} + ... \tag{6.3}$$

Note that the power of z agrees with our general OPE formula since $n^2 + m^2 - (n+m)^2 = -2nm$. When we set $n = -m = 1$ and $\pi T = 1$, this looks like the fermion expression (6.2).

In general the ... in (6.3) can contain further singular terms coming from the Taylor expansion of $\phi_L(z) = \phi_L(0) + z\partial\phi_L(0) + \cdots$. That is, going more slowly, we have

$$\begin{aligned} : e^{\mathbf{i}n\phi_L(z)} :: e^{\mathbf{i}m\phi_L(0)} : &= : e^{\mathbf{i}n\phi_L(z)+\mathbf{i}m\phi_L(0)} : z^{\frac{2nm}{2\pi T}} \\ &= : e^{\mathbf{i}(n+m)\phi_L(0)} : z^{\frac{2nm}{2\pi T}} + \mathbf{i}n\partial\phi_L(0) : e^{\mathbf{i}(n+m)\phi_L(0)} : z^{\frac{2nm}{2\pi T}+1} + . \end{aligned} \tag{6.4}$$

[Simmons-Duffin, $D > 2$: Rychkov. $D = 2$: Ginsparg.] Here is the way to think about CFT non-perturbatively: A CFT is a list of operators with definite scaling dimensions $\{\mathcal{O}_a, \Delta_a\}$ and their OPE structure constants, c_{abc}, appearing in (6.1). From

this information you can compute any correlation function of local operators by successive uses of the OPE. So the Ising model is[33]:

$$\{\underbrace{\mathbb{1}}_{(0,0)}, \underbrace{\chi_L}_{(\frac{1}{2},0)}, \underbrace{\chi_R}_{(0,\frac{1}{2})}, \underbrace{\varepsilon \equiv \mathbf{i}\chi_L\chi_R}_{(\frac{1}{2},\frac{1}{2})}, \underbrace{\sigma}_{(\frac{1}{16},\frac{1}{16})}, \underbrace{\mu}_{(\frac{1}{16},\frac{1}{16})}, \underbrace{T \equiv \chi_L\partial\chi_L}_{(2,0)}, \underbrace{\bar{T} \equiv \chi_R\bar{\partial}\chi_R}_{(0,2)}, \cdots\}$$

and $c_{\chi_L\chi_R\epsilon} = \mathbf{i}, c_{\sigma,\psi,\mu} = 1, ...,$ or more explicitly:

$$\sigma \times \sigma = \mathbb{1} + c_{\sigma\sigma\varepsilon}\varepsilon, \varepsilon \times \varepsilon = \mathbb{1} + \underbrace{c_{\varepsilon\varepsilon\varepsilon}}_{=0}\varepsilon, \sigma \times \varepsilon = c_{\sigma\sigma\varepsilon}\sigma.$$

Notice that I include the identity operator in this list; it has dimension zero for sure. In this way of enumerating the operators the list goes on forever: *e.g.* we can keep appending more derivatives. We'll learn to how do better soon by listing only the *primary operators* at the top of each representation of the conformal group. But in fact, as in effective field theory, one can often get far just knowing about the few lowest-dimension operators[34]. A special role is played by the operator $T(z)$ in this list, the *stress tensor*.

6.1 The stress tensor and conformal invariance (abstract CFT)

Let's think a bit about relativistic field theory in the continuum. Any continuum QFT has a stress tensor[35]; there are two useful perspectives on this operator. The simpler, but worse, one is to regard it as the Nöther current for spacetime translations. The stress tensor $T_{\mu\nu}$ is conserved if the action is translation-invariant: that is, invariant under the replacement $\phi(x^\mu) \to \tilde{\phi}(\tilde{x}) \equiv \phi(x^\mu + a^\mu)$. This is $d+1$ symmetries, so we have $d+1$ conserved currents: $T^\mu_\nu, \partial_\mu T^\mu_\nu = 0$. The associated charges are the energy and momentum

$$\mathbf{H} = \int_{\text{space}} T_{00}, \quad \mathbf{P}_i = \int_{\text{space}} T_{0i} \ .$$

[33] Actually, I should leave out the fermions. As this model arises from a spin system, a single fermion operator is not allowed. From the point of view advocated in §4.1.3, it is not gauge invariant.

[34] Naturally, this strategy is generally called 'effective conformal field theory'. A recent victory in this direction can be found in this paper, which solves QCD in $D = 2$ by diagonalizing a 5×5 matrix (!).

[35] Maybe this is not true. The following is a very technical comment which you should ignore if you want. It is certainly true if the field theory has a definition in terms of a lagrangian or a local hamiltonian. Possible exceptions come from: (1) field theory in a fixed AdS geometry; the absence of dynamical gravity means no stress tensor. This is called 'generalized free field theory'. (An attempt to take such a thing seriously as a CFT is in this paper by Heemskerk et al.)

(2) In his CFT notes, Rychkov discusses a long-range Ising system with a fixed point with no stress tensor; he claims without further discussion that this is the same as the first example. (3) Weird things 'defined' by scaling limits of string theory. Does a 2d CFT without a stress tensor have a central charge?

These generate translations in time and space by commutators (or classically by Poisson brackets):

$$\delta_a \phi(x) = \tilde{\phi}(\tilde{x}) - \phi(x) = a^\mu [\mathbf{P}_\mu, \phi(x)].$$

The finite solution of this equation is

$$\phi(x+a) = e^{\mathbf{i}\mathbf{P}_\mu a^\mu} \phi(x) e^{-\mathbf{i}\mathbf{P}_\mu a^\mu}.$$

The current we get from the Nöther method is not symmetric in its indices.

The better way to think about the stress tensor is as the (linear) response of the system to a local, small perturbation of the metric of spacetime:

$$\delta S = \int \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu} \ , \quad \text{aka} \quad T^{\mu\nu}(x) = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}(x)}. \tag{6.5}$$

Here $\sqrt{g} \equiv \sqrt{\det g}$ is the covariant measure.[36] Consider in particular making a *scale transformation*, which we can accomplish by changing the metric by:

$$\delta g_{\mu\nu} = 2\lambda g_{\mu\nu} \quad \to \quad \delta S = \int T^\mu_\mu 2\lambda \tag{6.6}$$

where λ is a constant. Therefore we see that if $T^\mu_\mu = 0$ then the theory is scale invariant. In the other direction, scale invariance (vanishing of (6.6)) actually only implies that $T^\mu_\mu = \partial^\mu K_\mu$ is a total derivative. But the object K is a vector operator whose scaling dimension must be $D-1$ (since that of the stress tensor is D). On general grounds, a vector operator of dimension $D-1$ is a conserved current (at least this is the only way I know to protect its dimension). And a conserved current has $\partial^\mu K_\mu = 0$, so that $T^\mu_\mu = 0$ anyway.

If $T^\mu_\mu = 0$ then nothing we said above depended on the fact that λ is a constant, and we should also consider the following transformation:

$$\delta g_{\mu\nu} = 2\delta\Omega(x) g_{\mu\nu} \quad \to \quad \delta S = \int T^\mu_\mu 2\delta\Omega(x) \tag{6.7}$$

Such a transformation, which rescales the whole metric (and therefore preserves angles between vectors at the same point) in a position-dependent way, is a *Weyl transformation*, closely related to a conformal transformation. The difference is that for arbitrary $\Omega(x)$, the new metric will be curved ($\mathcal{R} \propto \nabla^2 \log \Omega$). If we are not interested in QFT in curved spacetime, we should restrict ourselves to choices of Ω which preserve the

[36] The $T^{\mu\nu}$ from the metric variation can be related to the Nöther current for translations by "improvement," which means adding boundary terms to the action; this modifies the Nöther current.

initial choice of curvature; this means that they can be undone by a coordinate transformation. Infinitesimally, such a transformation is $x^\mu \to (x')^\mu = x^\mu + \xi^\mu(x)$, and the metric change is

$$\delta g_{\mu\nu}(x) = \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) \stackrel{!}{=} 2\delta\Omega g_{\mu\nu}. \tag{6.8}$$

This a set of PDEs for ξ_μ and $\delta\Omega$.

For the case of $g_{\mu\nu} = \eta_{\mu\nu}$, Minkowski spacetime, the stage of special relativity, the constraint (6.8) is solved by the following formulae[37] (and by translations and rotations and boosts, which don't change the Minkowski metric at all). The conserved currents and charges of the transformations above (in flat spacetime) are:

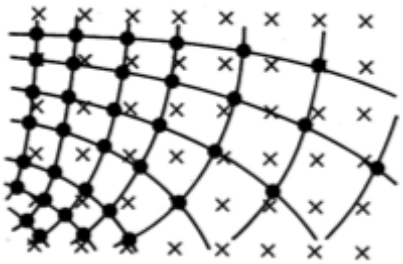
$$S_\mu = x^\nu T_{\mu\nu} \qquad \to \quad D \equiv \int S_0 d^d x = \int x^\mu \hat{p}_\mu \tag{6.9}$$

$$C_{\mu\nu} = (2x_\mu x_\lambda - x^2 g_{\mu\lambda}) T^\lambda_\nu \qquad \to \quad C_\mu \equiv \int C_{0\mu} d^d x \tag{6.10}$$

since both $\partial^\mu S_\mu$ and $\partial^\mu C_{\mu\nu}$ are proportional to T^μ_μ. In the last equation of the first line, $\hat{p}_\mu$ is the momentum *density*; the action of the integrand on fields is via $x^\mu \partial_\mu$, a rescaling.

Here is the right way to think about this condition on finite transformations. We are demanding a change of coordinates which accomplishes the following:

$$\eta_{\mu\nu}(dx')^\mu (dx')^\nu \stackrel{!}{=} \Omega^2(x) dx^\mu dx^\nu \eta_{\mu\nu}.$$



[Fig credit: Rychkov]

The jacobian for this change of coordinates must therefore satisfy

$$J^\mu_\nu \equiv \frac{\partial (x')^\mu}{\partial x^\nu} = \Omega(x) \mathsf{O}^\mu_\nu(x)$$

where $\mathsf{O}^\mu_\nu(x)$ is a spacetime-dependent Lorentz transformation ($\mathsf{O}\eta\mathsf{O} = \eta$). This is a spacetime-dependent rescaling and rotation; you should think of it as a position-dependent RG transformation.

We conclude from the above discussion that, at least classically, if $T^\mu_\mu = 0$ (and $T^{\mu\nu}$ is symmetric) then the theory has both scale invariance and conformal invariance. The precise logical relation between scale invariance and conformal invariance is a subject of a lot of discussion. There is no example of an interacting relativistic unitary fixed point without conformal invariance.

The resulting set of transformations forms an extension of the Poincare group. There are $d+2$ extra generators $\mathbf{C}_\mu$ and $\mathbf{D}$. In fact the algebra is $\mathsf{so}(d+1,2)$.

[37] The details are at the beginning of the Ginsparg notes, or on page 3 here.

Conformal algebra. Here is how to think about this algebra. Most of the commutators just say that vectors (like $\mathbf{C}_\mu$ and $\mathbf{P}_\mu$) transform as vectors and D is a scalar. The important ones are:

$$[\mathbf{D}, \mathbf{P}_\mu] = \mathbf{i}\mathbf{P}_\mu, \quad \mathbf{P}_\mu \text{ is a raising operator for } \mathbf{D}$$

$$[\mathbf{D}, \mathbf{C}_\mu] = -\mathbf{i}\mathbf{C}_\mu, \quad \mathbf{C}_\mu \text{ is a lowering operator for } \mathbf{D}$$

$$[\mathbf{C}_\mu, \mathbf{P}_\nu] = 2\mathbf{i}\left(\eta_{\mu\nu}\mathbf{D} - \mathbf{M}_{\mu\nu}\right)$$

which says you can recover $\mathbf{D}$ and the spin from $\mathbf{C}_\mu$.

The right way to think about the transformation $\mathbf{C}$ generates is:

$$\text{inversion} \circ \text{translation} \circ \text{inversion:} \qquad \frac{x'^\mu}{\vec{x'}.\vec{x'}} = \frac{x^\mu}{\vec{x}.\vec{x}} + b^\mu \quad . \tag{6.11}$$

Inversion is $x^\mu \to -\frac{x^\mu}{x^\nu x_\nu}$; inversion plus poincaré implies conformal.

So if we want to study representations of this algebra, we can diagonalize $\mathbf{D}$ and some of the rotation generators. (Notice that the familiar Poincaré casimir $\mathbf{P}^\mu\mathbf{P}_\mu$ is *not* a casimir of the conformal group.) Then we can build representation starting with states that have been lowered as much as possible, so $\mathbf{C}\,|\text{primary}\rangle = 0$; the rest of the representation is obtained by acting with derivatives, *i.e.* $\mathbf{P}_\mu$ (and $\mathsf{SO}(d)$ rotations).

Fields also form representations of the spacetime symmetry group. A primary field is one which is local enough that its transformation rule under a conformal transformation is the same as if it were just a scale transformation:

$$\Phi(x) \to \Phi'(x') = \Omega(x)^{-\Delta}\Phi(x)$$

for a scalar field, or

$$\Phi(x) \to \Phi'(x') = \Omega(x)^{-\Delta}\mathcal{R}[\mathsf{O}(x)^\mu_\nu]\Phi(x')$$

for a field with spin. $\mathcal{R}$ is the representation matrix for the rotation O in the Φ representation; for a vector field, this is just $\mathsf{O}(x)^\mu_\nu$.

Fields which are not primaries (for example, the derivative of a primary) are called descendants and have more complicated transformation rules. Fields can be organized into eigenstates of $\mathbf{D}$, of definite scaling; every such field is a primary or a descendant. The algebra determines the behavior of descendants from that of primaries.

6.1.1 Geometric interpretation of conformal group

Here is a geometric interpretation of the conformal group, called the *projective null cone* construction (due to Dirac (!)). The conformal group in $\mathbb{R}^{d,1}$ is isomorphic to

$\mathsf{SO}(d+1,2)$, the Lorentz group of a space with two extra dimensions. This space $\mathbb{R}^{d+1,2}$ has metric

$$\eta_{ab} = \text{diag}(-++...+-)_{ab} \tag{6.12}$$

where the last two dimensions are the 'extra' ones. So we can find linear representations of the conformal group by temporarily adding some extra coordinates. A light ray in this space can be parameterized by $d+1$ dimensional coordinates x^μ in the following way:

$$\zeta^a = \kappa(x^\mu, \frac{1}{2}(1-x^2), \frac{1}{2}(1+x^2))^a \tag{6.13}$$

where κ is some arbitrary constant. The group $\mathsf{SO}(d+1,2)$ moves these light rays around. We can interpet these transformations as maps on the x^μ. In fact these transformations (combined by rescalings to get back to the original slice) are precisely the conformal transformations.

Regard the expression (6.13) as an embedding of $\mathbb{R}^{d,1} \subset \mathbb{R}^{d+1,2}$; this is an *isometric embedding*, *i.e.* the induced metric is again $ds^2 = -dt^2 + d\vec{x}^2$. To see that Lorentz boosts in the embedding space are conformal transformations on the null slice, note that on the slice, $\xi^a\xi_a = 0 \implies \xi^a d\xi_a = 0$, and so the transformation $\xi^a \to \lambda(x)\xi^a$ (which preserves a null subspace) takes $d\xi^a d\xi_a \to \lambda(x)^2 d\xi^a d\xi_a$, a conformal transformation.

Invariants in $\mathbb{R}^{d+1,2}$ should therefore be conformal invariants. Consider the object:

$$\zeta_1 \cdot \zeta_2 = \eta_{ab}\zeta_1^a\zeta_1^b = \frac{1}{2}\kappa_1\kappa_2(x_1-x_2)^2. \tag{6.14}$$

ζ^a and $\lambda\zeta^a$ are identified with the same x^μ, so κ is a redundant variable. So conformal invariants actually are cross ratios of invariants in $\mathbb{R}^{d+1,2}$, for example

$$\frac{\zeta_1 \cdot \zeta_2 \zeta_3 \cdot \zeta_4}{\zeta_1 \cdot \zeta_3 \zeta_2 \cdot \zeta_4}. \tag{6.15}$$

An extremely useful consequence of this is the statement that $r_{12}^2 \equiv (x_1-x_2)^\mu(x_1-x_2)_\mu$ transforms under a conformal transformation by a rescaling

$$x_{12}^2 \to \frac{x_{12}^2}{\Omega(x_1)\Omega(x_2)}$$

where $\Omega(x_i) \equiv 1 + 2b\cdot x_i + b^2 x_i^2$ is the rescaling of the metric at x_i under the special conformal transformation $x^\mu \to \frac{x^\mu + b^\mu}{1+2b\cdot x + b^2x^2}$.

6.1.2 Infinite conformal algebra in $D = 2$.

Here there is an important division between $D = 1 + 1$ and $D > 1 + 1$. In $D = 2$, in lightcone or holomorphic coordinates, tracelessness of T says

$$T^\mu_\mu \propto T_{z\bar{z}} = 0 \ .$$

Conservation of the stress tensor $0 = \partial^\mu T^\nu_\mu$ then says

$$\bar{\partial}_z T_{zz} = 0, \quad \partial_z T_{\bar{z}\bar{z}} = 0.$$

That is $T(z) \equiv T_{zz}$ is holomorphic (and $\tilde{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}$ is antiholomorphic). In the quantum theory, these statements are true as operator equations; that is: they are exactly true away from other operator insertions in the path integral (the lingo for this is 'up to contact terms'). This holomorphic factorization has the following dramatic consequence: Given an arbitrary holomorphic function[38] $\xi(z)$, the current

$$j^{(\xi)}_\mu = (j_z, j_{\bar{z}})_\mu \equiv (\xi(z)T(z), 0)_\mu$$

is also conserved (!):

$$\partial^\mu j^{(\xi)}_\mu = \bar{\partial}_z j_z - \partial j_{\bar{z}} = 0.$$

This is infinitely many conserved currents! Basically, just from scale invariance.

What are these transformations? Recall that the current $T_{\mu\nu}$ generates translations, which by abuse of notation[39] we can write as $x^\mu \to x^\mu + a^\mu$. Accordingly, the current $T(z)$ generates 'holomorphic translations': $z \to z + \xi$ with ξ constant. So it's not too shocking that $j^{(\xi)}_\mu$ generates the *local* transformation $z \to z + \xi(z)$. The finite version of this transformation is just an arbitrary holomorphic map:

$$z \to z'(z), \quad \bar{z} \to \bar{z}'(\bar{z}).$$

(The important thing here is that z' does not depend on $\bar{z}$!) What does this transformation do to the metric? The flat metric in holomorphic coordinates is

$$ds^2 = d\tau^2 + dx^2 = dzd\bar{z} \to \frac{\partial z}{\partial z'}dz\frac{\partial \bar{z}}{\partial \bar{z}'}d\bar{z} = f(z, \bar{z})dzd\bar{z}.$$

[38] More precisely, since there is one for z and one for $\bar{z}$, $\xi^z, \xi^{\bar{z}}$ are components of a holomorphic vector field.

[39] This abuse of notation is both very tempting and very confusing. We are not merely relabeling our coordinates; that doesn't do anything – physics is coordinate-independent. We are transforming our fields by

$$\phi \to \phi'(x') = \phi(x) \ .$$

If you promise to keep this in mind, then we can use the less cumbersome expressions below.

The metric has only changed by an overall function. This means that the *angle* between any two vectors has not changed. This is the definition of a *conformal transformation*. The conformal group is infinite dimensional in $D = 2$.

For the example of the free massless scalar in $D = 2$ (with curved-worldsheet action $S[\phi] = \frac{1}{2\pi K}\int dxdt\sqrt{g}g^{\mu\nu}(x)\partial_\mu\phi\partial_\nu\phi$) the stress tensor is

$$T_{\mu\nu} = \frac{1}{2\pi K}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\partial\phi\right)^2\right).$$

Notice that it is traceless: $g^{\mu\nu}T_{\mu\nu} \equiv T^\mu_\mu = 0$. In holomorphic coordinates, $ds^2 = dzd\bar{z}$, the nonzero components are

$$\begin{aligned} T_{zz} \equiv T(z) &= \frac{1}{2\pi K}\partial_z\phi\partial_z\phi = \frac{1}{2\pi K} :\partial_z\phi_L\partial_z\phi_L: , \\ \bar{T}_{\bar{z}\bar{z}} \equiv T(\bar{z}) &= \frac{1}{2\pi K}\bar{\partial}_z\phi\bar{\partial}_z\phi = \frac{1}{2\pi K} :\bar{\partial}_z\phi_R\bar{\partial}_z\phi_R: \end{aligned} \tag{6.16}$$

In the last step I've emphasized the factorization into L and R parts, and the fact that quantumly we must define this composite operator somehow, and we are doing it by normal ordering. This involves a choice of additive constant, about which there is a bit more to say.

Conformal invariance constrains the operator algebra of a CFT, and since (various moments of) the stress tensor components are generators of conformal transformations, their OPEs are highly constrained.

6.2 Radial quantization

Here is an important example of a conformal transformation: Consider a cylinder with complex coordinate $w = x + \mathbf{i}\tau$; I call it a cylinder because $x \simeq x + L$. Let's set $L = 2\pi$ for convenience. Consider the map

$$w \mapsto z = e^{-\mathbf{i}w} = e^{\tau - \mathbf{i}x}.$$

Since it's holomorphic ($z(w, \bar{w})$ doesn't depend on $\bar{w}$), this is a conformal transformation. The image is the complex z-plane. Equal-τ surfaces are circles. The spatial momentum operator was $\partial_\sigma = z\partial_z - \bar{z}\bar{\partial}_z$, which acts by $z \to e^{\mathbf{i}b}z$ – rotations about the origin. The hamiltonian on the cylinder was $\partial_\tau = z\partial_z + \bar{z}\bar{\partial}_z$. So time translations become rescaling about the origin of the z-plane: the dilatation operator (the operator $\mathbf{S}$ that generates scale transformations) is the hamiltonian in radial quantization. Time-ordered correlators on the cylinder are *radially-ordered*. One reason this is nice is that it provides an IR cutoff even when studying infinite-volume physics. (Notice that

the operators $z\partial_z \pm \bar{z}\bar{\partial}_z$ commute and so we can label states by their scaling dimension Δ and *spin*.

So far, this all goes through for CFT in any D. In $D > 2$, the equal-radial-time-slices are $D-1$-spheres, and therefore the single quantum number for spin would be replaced by a representation of $SO(D)$.

Example. The mode expansion of the free boson (about the origin of the z plane) is now[40]

$$\partial_z \phi(z) = \sum_{n \in \mathbb{Z}} \boldsymbol{\rho}_n z^{-n-1}$$

(with $\boldsymbol{\rho}_0 \equiv p$), and we can extract them by a contour integral:

$$\boldsymbol{\rho}_n = \oint_{C_0} \frac{dz}{2\pi \mathbf{i}} z^n \partial\phi(z)$$

where C_0 is a contour encircling $z = 0$ (and no other operators!).

Primaries and quasiprimaries.

Def: A primary operator (or field) $\Phi_{h,\bar{h}}(z,\bar{z})$ of weight $(h,\bar{h})$ transforms under the conformal transformation[41]

$$(z,\bar{z}) \to (f(z),\bar{f}(\bar{z})) \qquad \text{by} \qquad \Phi_{h,\bar{h}}(z,\bar{z}) \to (\partial_z f)^h \left(\bar{\partial}_z \bar{f}\right)^{\bar{h}} \Phi(f(z),\bar{f}(\bar{z})).$$

(The way to remember this is that $\Phi(z) dz^h d\bar{z}^{\bar{h}}$ is a scalar.) For example,

$$\begin{array}{rlll} \text{scaling:} & z \to e^{\lambda} z & \implies \Phi \to e^{\lambda\Delta}\Phi, & \Delta = h + \bar{h} \ \text{ (scaling dimension)} \\ \text{rotations:} & z \to e^{\mathbf{i}\theta} z & \implies \Phi \to e^{\mathbf{i}s\theta}\Phi, & s = h - \bar{h} \ \text{ (spin)} \end{array} \tag{6.17}$$

The infinitesimal transformation $f(z) = z + \xi(z)$ results in

$$\delta_\xi \Phi(z) = (\xi\partial_z + h\partial_z\xi)\,\Phi + \text{antiholomorphic bits.}$$

This transformation is *generated* by $\int \frac{dz}{2\pi\mathbf{i}} \xi(z) T(z) \equiv \mathbf{L}[\xi]$ (as in 6.1.2) in the sense that $\delta_\xi \Phi = \mathbf{i}[\mathbf{L}[\xi], \Phi]$.

Consider for a while a holomorphic operator, with $\bar{h} = 0$. It has a mode expansion

$$\Phi_h(z) = \sum_{n \in \mathbb{Z}} \Phi_n z^{-n-h}.$$

[40] Allow me to use $\phi_{\text{here}} = \sqrt{\pi T}\phi_{\text{before}}$, so that $\phi_{\text{here}} \simeq \phi_{\text{here}} + 2\pi R$ and the action is $S = -\frac{1}{4\pi}\int d^2z \partial\phi\bar{\partial}\phi$.

[41] In general dimension $D \geq 2$, the transformation of a primary operator of dimension Δ is

$$\Phi_\Delta(x) \to \left|\frac{\partial x'}{\partial x}\right|^{\Delta} \Phi'_\Delta(x')$$

where $|\frac{\partial x'}{\partial x}|$ is the Jacobian of the conformal map.

The shift by h in the moding on the plane comes from the conformal transformation from the cylinder:

$$\Phi_h(w = -\mathbf{i}\ln z) = \sum_n \Phi_n e^{-\mathbf{i}wn}$$

where this is just fourier expansion, and the conformal factor is $\left(\frac{\partial z}{\partial w}\right)^h = z^h$. Note that $n < 0$ is positive energy.

The modes of the stress tensor are called Virasoro operators

$$T(z) = \sum_n \mathbf{L}_n z^{-n-2}, \quad \mathbf{L}_n = \oint_{C_0} \frac{dz}{2\pi\mathbf{i}} z^{n+1} T(z).$$

The definition of primary implies that

$$[\mathbf{L}_0, \Phi_{h,\bar{h}}(0)] = h\Phi_{h,\bar{h}}(0), \qquad [\mathbf{L}_n, \Phi_{h,\bar{h}}(0)] = 0, \ \forall n > 0.$$

This in turn implies that the state

$$\left|\Phi_{h,\bar{h}}\right\rangle \equiv \Phi_{h,\bar{h}}(0)\left|0\right\rangle$$

is a *highest weight state of the Virasoro operators*, in the sense that

$$\mathbf{L}_0\left|\Phi_{h,\bar{h}}\right\rangle = h\left|\Phi_{h,\bar{h}}\right\rangle, \qquad \mathbf{L}_n\left|\Phi_{h,\bar{h}}\right\rangle = 0, \ \forall n > 0.$$

Note that the modes with $n > 0$ *raise* the value of $\mathbf{L}_0$. They include the ordinary special conformal generators $C_\mu = (\mathbf{L}_1, \bar{\mathbf{L}}_1)_\mu$.

In $D = 1 + 1$ it is important to distinguish between Vir primary and ordinary conformal primary, which is just killed by $\mathbf{L}_1$ and not $\mathbf{L}_{n\geq 2}$.

Contours and commutators. You may be bothered by the connection between the algebra in terms of OPEs on the complex plane

$$(\text{like} \quad \partial\phi(z)\partial\phi(w) \sim \frac{1}{(z-w)^2} + \cdots \quad)$$

and the perhaps-more-familiar algebra of mode operators.

$$(\text{like} \quad [\boldsymbol{\rho}_n, \boldsymbol{\rho}_m] = n\delta_{n+m} \quad).$$

The very direct connection between the two comes from radial quantization. Recall that the path integral on the plane produces *radially ordered* correlators:

$$\int [D\phi] \qquad \underbrace{e^{-S[\phi]}...A(z)B(w).....}_{\text{these are numbers, order doesn't matter}} \qquad = \langle T(....\mathbf{A}(z)\mathbf{B}(w)...)\rangle$$

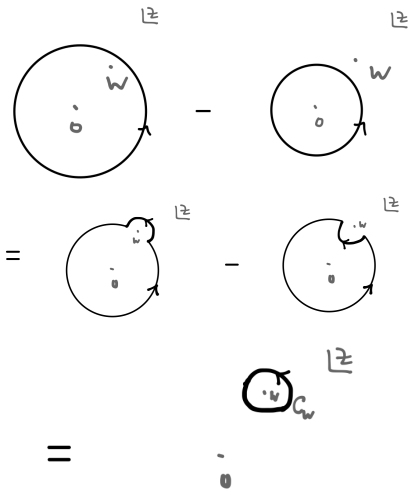
with

$$T(\mathbf{A}(z)\mathbf{B}(w)) = \begin{cases} \mathbf{A}(z)\mathbf{B}(w), & |z| > |w| \\ \mathbf{B}(w)\mathbf{A}(z), & |z| < |w| \end{cases}$$

So consider, for example, the commutator

$$[L[\xi],\Phi_h(w)] = \left(\oint_{|z|>|w|} - \oint_{|z|<|w|}\right)\frac{dz}{2\pi\mathbf{i}}\xi(z)T(z)\Phi_h(w)$$

$$= \oint_{C_w}\frac{dz}{2\pi\mathbf{i}}\xi(z)T(z)\Phi_h(w).$$



This is a general rule: the commutator of modes of two fields is given by the contour integral of one about the other.

The previous expression is of interest since it describes the conformal transformation of the primary Φ_h parameterized by the holomorphic vector field ξ:

$$\mathbf{i}[\mathbf{L}[\xi],\Phi_h(w)] = \delta_\xi\Phi_h(w) = (\xi\partial_z + h\partial_z\xi)\Phi.$$

Compare this expression to the general OPE

$$T(z)\Phi_h(0) = \sum_n \frac{1}{z^{n+1}}\mathcal{O}^{(n)}(0)$$

and we determine

$$\begin{cases}\mathcal{O}^{(1)} = & h\Phi_h \\ \mathcal{O}^{(0)} = & \partial\Phi_h \\ \mathcal{O}^{(\text{else})} = & 0\end{cases}.$$

That is, we have shown that the OPE of the stress tensor with a *primary operator* of dimension Δ is

$$T(z)\mathcal{O}(0) \sim \frac{\Delta\mathcal{O}}{z^2} + \frac{\partial\mathcal{O}}{z}. \tag{6.18}$$

Notice that L_0 is scaling, L_{-1} is translations, and L_1 is special conformal.

Some examples to check: (1) $T(z)\partial\phi(0)$ for the free boson theory. (2) $T(z)T(0)$ for the free boson theory (this one is a trick question as we'll see in 6.2.1).

Not all operators of definite scaling dimension behave this way, and this can be taken as a definition of a primary operator. It implies that under a conformal transformation $(z,\bar{z}) \to (w(z),\bar{w}(\bar{z}))$,

$$\mathcal{O}_{h,\bar{h}}(z,\bar{z}) \to \left(\frac{\partial w}{\partial z}\right)^h\left(\frac{\partial\bar{w}}{\partial\bar{z}}\right)^{\bar{h}}\mathcal{O}_{h,\bar{h}}(w(z),\bar{w}(\bar{z}))$$

State-operator correspondence. (any D)

To get a CFT state $|\Phi\rangle$ from a local operator $\Phi(z)$ just insert that operator at the origin acting on the CFT vacuum in radial quantization about the origin. It's easier in equations:

$$\Phi(0)\,|0\rangle \equiv |\Phi\rangle\,.$$

The state $|0\rangle$ is then the image of the identity operator under this map.

To get an operator from a state, let's think about the path integral on the ball Σ. Think of it as a functional of the boundary conditions on the fields (which I'll call ϕ; θ is a coordinate on $\partial\Sigma$):

$$\int_{\phi|_{\partial\Sigma}(\theta)=\phi_0(\theta)} [D\phi]e^{-S} = \Psi_0[\phi_0] = \langle\phi_0|0\rangle$$

This is an integral representation of the groundstate wavefunctional. If instead we consider the path integral with a local operator insertion, we get a wavefunctional for a different state:

$$\int_{\phi|_{\partial\Sigma}(\theta)\phi_0(\theta)} [D\phi]e^{-S}\Phi(0) = \Psi_\Phi[\phi_0] = \langle\phi_0|\Phi\rangle\ .$$

In a CFT this is related by a conformal transformation $z = e^{-\mathbf{i}w}$ to the path integral on the cylinder with the state $|\Phi\rangle$ inserted in the far past. So to get the operator corresponding to an arbitrary state just glue this path integral around where you want to put the operator. To move it to a different place, just act with translation generators: $\Phi(x) = e^{-\mathbf{i}\mathbf{P}^\mu x_\mu}\Phi(0)e^{\mathbf{i}\mathbf{P}^\mu x_\mu}$.

For more on the beauty of the state-operator correspondence in D-dimensional CFT see the notes here Rychkov.

Convergence of OPE. With this realization in mind, it is clear that the OPE in CFT is a *convergent* expansion: we are simply inserting a resolution of the identity on the hilbert space in a particular basis of eigenstates of the dilatation operator.

Adjoint in radial quantization. The adjoint is a bit weird: it is just the adjoint on the cylinder, but two things. First, the state in the far future on the cylinder gets mapped to $z = \infty$. Second, we must remember that the map from the cylinder to the sphere produces a Jacobian. Since the adjoint operation involves an *inversion* (a particular element of the conformal group), the primary Φ acquires an extra factor from the Jacobian. This produces an extra factor in the state:

$$|\Phi\rangle^\dagger = \lim_{z\to\infty} \langle\Phi|\, z^{-h}\ .$$

This is sometimes called the BPZ adjoint after its discoverers.

6.2.1 The Virasoro central charge in 2d CFT

The OPE of the stress tensor with itself is

$$T(z)T(0) \sim \frac{c_L/2}{z^4} + \frac{2T}{z^2} + \frac{\partial T}{z}. \tag{6.19}$$

(A good way to discover this is to evaluate it for the free scalar or the free fermion case.) The quantity c appearing here is called the *Virasoro central charge*. The word 'central' is because it is a c-number, not an operator. It is a crucial piece of data about the CFT. It can be extracted from

$$\langle T(z)T(0)\rangle = \frac{c_L/2}{z^4} \ .$$

There is also a right-moving central charge which appears in the OPE $\bar{T}\bar{T}$.

For free bosons, with $T(z) = \frac{1}{4\pi K}(\partial\phi)^2$ it is equal to $c_L = 1$ (note that the Ks all cancel out). Notice that it is additive: with N free bosons, the answer is $c_L = N$.

For a majorana fermion, with $T(z) = \chi_L \partial\chi_L$, it is equal to $c_L = \frac{1}{2}$. A basic check of bosonization is that a complex fermion has central charge $\frac{1}{2} + \frac{1}{2} = 1$.

The associated algebra of modes is called the Virasoro algebra,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}.$$

Notice that $\{L_{-1}, L_0, L_1\} = \{P, D, C\}$ form a closed subalgebra (from which c drops out); this is the global conformal algebra that generalizes to higher dimensions.

[End of Lecture 19]

It is monotonic under RG flows between CFTs: $c_{UV} > c_{IR}$. Therefore it is a useful measure of the number of degrees of freedom. In fact, using the definition

$$c = \lim_{z\to 0} 2z^4 \langle T(z)T(0)\rangle$$

it can be extended away from RG fixed points; Zamolodchikov proved that this quantity is monotonic along any RG flow.

Comparing (6.18) and (6.19) we see that nonzero c means that the stress tensor itself is not a primary. It transforms weirdly under conformal transformations, in a very definite way. This means, for example, that under the map from the cylinder to the plane, the additive normalization of the stress tensor changes. For an application of this to Casimir energy of CFT on a sphere, see the homework.

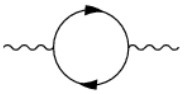
Relatedly, c can be interpreted as a *conformal anomaly* or *Weyl anomaly*.

$$T^\mu_\mu = \frac{c}{24\pi}\mathcal{R}.$$

The relation between this statement and the stress tensor OPE is the same as that between the chiral anomaly in the form $\partial^\mu j^A_\mu = k\frac{F}{2\pi}$ the current-current OPE

$$j(z)j(0) \sim \frac{k}{z^2} \ .$$

In the realization by free fermions, both of these effects come from the 'diangle' diagram:



This singularity in the OPE implies the anomalous nonconservation when coupling to a background gauge field. In the presence of a source coupling to the current (A for j, or curvature for $T_{\mu\nu}$), the statement about the two-point function implies a statement about the one-point function.

6.3 Back to general dimensions.

The conformal group in $D > 2$ is finite dimensional, but it still produces powerful constraints.

So we can make our list of operators specifying the CFT much shorter by simply enumerating the primaries (and their dimensions, spins, and structure constants). All the structure constants for the descendants are determined from this data by conformal invariance.

6.3.1 Constraints on correlation functions from CFT

(any D) We consider correlators of primaries; to get correlators of descendants just take derivatives of those of their primaries. We'll also focus for simplicity on scalar operators. A correlation function of N primaries transforms under a conformal transformation as

$$\left\langle \prod_i \Phi'_i(x'_i) \right\rangle = \prod_i \Omega(x_i)^{-\Delta_i} \left\langle \prod_i \Phi_i(x_i) \right\rangle .$$

You should regard the operators Φ' on the LHS as the images of Φ under a spacetime-dependent RG transformation.

Suppose we have in our possession a conformal *invariant* I which depends on N spacetime positions.

- Translations imply that I depends only on differences $x_i - x_j$ (of which there are $D(N-1)$).

- Rotations imply that I depends only on distances $r_{ij} \equiv |x_i - x_j|$ of which there are $\frac{N(N-1)}{2}$.
- Scale invariance implies that I depends only on ratios of differences $\frac{r_{ij}}{r_{kl}}$.
- Special conformal transformations act on these distances by

$$(r'_{12})^2 = \frac{r_{12}^2}{\Omega_1 \Omega_2}$$

where

$$\Omega_i \equiv 1 + 2b \cdot x_i + b^2 x_i^2$$

is the factor by which the metric at x_i rescales under the associated transformation. (The easiest way to see this is by doing it for an inversion first, and then using the construction of special conformal transformations as $I \circ T \circ I$. Alternatively, this is where the projective null cone comes into its own.) Only *cross-ratios*

$$\frac{r_{ij} r_{kl}}{r_{ik} r_{jl}}$$

are invariant under this. There are $\frac{N(N-3)}{2}$ of these (and none at all for $N < 4$).

This discussion of invariants has the following implications.

1. One-point functions of primaries of nonzero dimension vanish if there is a conformally invariant vacuum, $\mathbf{D}\,|0\rangle = 0$. That this is so can be seen as follows. The infinitesimal scale transformation of a primary is

$$\delta\Phi(x) = \left(\Omega^{-\Delta}\Phi(x') - \Phi(x)\right)|_{\Omega = e^{-\lambda}, x' = e^{-\lambda}x} = \lambda\left(\Delta + x^\mu \partial_\mu\right)\Phi(x) \;\; = \mathbf{i}\lambda[\mathbf{D}, \Phi(x)].$$

The last step is the statement that the dilatation operator $\mathbf{D}$ generates scale transformations. Therefore:

$$\langle 0|\,\Phi_\Delta(0)\,|0\rangle = \frac{\mathbf{i}}{\Delta}\,\langle 0|\,[\mathbf{D}, \Phi_\Delta(0)]\,|0\rangle = 0.$$

2. In CFT, two point functions of primaries are only nonzero if the primaries have the same dimension. This requires conformal invariance; the rest is determined by scale, translation and rotation:

$$\langle 0|\,\phi_1(z)\phi_2(w)\,|0\rangle = \frac{\delta_{h_1,h_2}}{r_{12}^{2h_1}}.$$

3.
$$\left\langle \prod_{i=1}^{3} \phi_i(z_i) \right\rangle = C_{123} r_{12}^{2(-h_1-h_2+h_3)} r_{23}^{2(-h_2-h_3+h_1)} r_{31}^{2(-h_3-h_1+h_2)}$$
C_{123} is the OPE coefficient between the three operators. Notice that this is more than scaling, which just says that the sum of the powers should be $h_1 + h_2 + h_3$.

4.
$$\left\langle \prod_{i=1}^{4} \phi_i(z_i) \right\rangle = f(x_1, x_2) \prod_{i<j} r_{ij}^{-2(h_i+h_j+\sum_k h_k/3)}$$
where x_{12} are the two cross-ratios. In two dimensions, these are
$$x = \frac{z_{12} z_{34}}{z_{13} z_{24}}$$
and $\bar{x}$.

It seems like $N \geq 4$-point functions are underconstrained. However, we have see that the successive use of the OPE relates them (via *known* functions) to sums of 3-point functions. The unknown functions of cross-ratios are actually determined by this process! The general name for such objects is *conformal blocks*.

6.3.2 Thermodynamics of a CFT

Thermodynamics of scale-invariant theories is very constrained. The partition function is defined as
$$Z_{CFT} = \mathrm{Tr}_{CFT} \left(\exp(-H/T) \right).$$
In the thermodynamic limit, $\ln Z$ is *extensive*, *i.e.* proportional to the volume of the space. But $\ln Z$ is a dimensionless quantity. Hence, we must have $\ln Z \sim V T^d$ (d is the number of spatial dimensions) in the absence of any other energy scales (such as a chemical potential for some conserved charge). The free energy then will be
$$F = -T \ln Z = c V T^{d+1}.$$
where this c should be regarded as a rough measure of the number of degrees of freedom of the CFT. In $d = 2$ it is proportional to the Virasoro central charge. In other cases, this statement is less sharp.

As with any conservation law, $T^\mu_\mu = 0$ is an operator equation in the full quantum theory. What happens if we put it inside $\mathrm{Tr}(e^{-H/T})$, with no other insertions? The operator equation then translates into the following equation
$$0 = \mathrm{Tr}(T^\mu_\mu e^{-H/T}) = \langle T_{00} \rangle - \langle T_{ii} \rangle = \mathcal{E} - dP.$$

This last relation gives the speed of the sound

$$c_s = \sqrt{\left(\frac{\partial P}{\partial \mathcal{E}}\right)_S} = \sqrt{\frac{1}{d}} \tag{6.20}$$

6.3.3 High-energy density of states of CFT

Here is a consequence of conformal invariance for the spectrum of a CFT in $D = 1+1$.

Consider the thermal partition function of a $D = 1+1$ CFT on a circle of radius L. This can be computed by a path integral on $S^1 \times S^1$ (a 2-torus) where $x \equiv x + L$ and $\tau \equiv \tau + \beta$.

1) Conformal invariance implies that $Z(\beta, L)$ actually only depends on the ratio: $Z = Z(\beta/L)$.[42]

2) The path integral doesn't know which direction is time and which is space. Therefore

$$Z(\beta/L) = Z(L/\beta). \tag{6.21}$$

Notice that this requires a rescaling to get back to the same volume, so isn't true in a QFT without scale invariance. Also, we used the euclidean rotation invariance, *i.e.* Lorentz invariance so that time and space are equivalent.

This condition of *modular invariance* relates the high-temperature ($T \gg L^{-1}$) behavior to the low-temperature ($T \ll L^{-1}$) behavior.

A brief comment on the fancy name: we can define a more general partition sum where we include a chemical potential for angular momentum:

$$Z(\tau, \bar{\tau}) \equiv \mathrm{tr} e^{-\beta(L_0 + \bar{L}_0 - \frac{\alpha c}{L})} e^{\mu(L_0 - \bar{L}_0)} \equiv \mathrm{tr} q^{L_0 - a} \bar{q}^{\bar{L}_0 - \bar{a}}$$

[42] There is actually some danger in this statement. The conformal anomaly $T^\mu_\mu = \frac{c}{24\pi}\mathcal{R}$ implies that on a general Riemann surface, the partition sum *does* depend on the overall volume, in a way determined by the curvature. More generally, the partition sum on a surface Σ of genus g and overall scale R (this is the radius in the case where $\Sigma = S^2$) satisfies

$$R\partial_R Z_\Sigma(R) = \left\langle \int_\Sigma T^\mu_\mu \right\rangle = \left\langle \int_\Sigma \frac{c}{24\pi}\mathcal{R} \right\rangle \overset{\text{Gauss-Bonnet}}{=} \frac{c}{24\pi}(2-2g).$$

which we can integrate to find the volume dependence. But in the case of a torus ($g = 1$), the curvature integrates to zero and the answer does not depend on the volume.

where $q \equiv e^{2\pi \mathbf{i} \tau}$. This complex parameter τ specifies the shape of a more general torus, where the complex spacetime coordinate is identified by

$$z \equiv z + 1, z \equiv z + \tau.$$

You get the same torus if τ is replaced according to

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & c \end{pmatrix} \in \mathsf{SL}(2, \mathbb{Z}).$$

---

3) On the homework, you showed that the funny transformation law of the stress tensor under a conformal transformation determines the vacuum energy on a circle in terms of the central charge:

$$H_{\text{CFT on } S^1} = L_0 + \bar{L}_0 - \alpha \frac{c}{L},$$

where the constant $\alpha = \frac{\pi^2}{3}$. The groundstate is the state $|0\rangle = |\mathbb{1}\rangle$ corresponding to the identity, which has $L_0 |\mathbb{1}\rangle = \bar{L}_0 |\mathbb{1}\rangle = 0$, so the groundstate energy is $E_0 = -\alpha \frac{c}{L}$.

This means that

$$Z(x = \beta/L) \overset{x \to \infty}{\simeq} e^{-\beta E_0} + \cdots = e^{+\frac{\beta \alpha c}{L}} + \cdots$$

Using (6.21), we learn that the free energy at high temperature is then

$$Z \overset{\beta/L \to 0}{\simeq} e^{\frac{\pm L \alpha c}{\beta}}.$$

That is, the free energy at high temperature is

$$F(T) = -T \log Z \simeq \alpha c L T^2. \tag{6.22}$$

The fact that it goes like T^2 is determined by dimensional analysis as we saw above, but here the exact coefficient is determined in terms of the central charge, c.

The result is usually stated in terms of the microcanonical entropy, $S(E)$, related to (6.22) by a Legendre transformation. If $F(T) = \gamma L^d T^{d+1}$, then

$$E = F + TS|_{S=-\partial_T F} = d\gamma L^d T^d \implies T = \left(\frac{E}{d\gamma L^d}\right)^{1/d}$$

which says

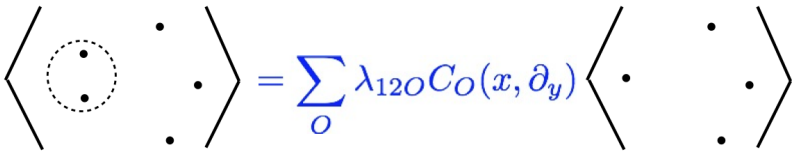
$$S(E) = -\partial_T F = \frac{(d+1)\gamma L^d}{(d\gamma L^d)^{1/d}} E^{\frac{d}{d+1}}.$$

In $D = 2$ γ is determined by the central charge.

This result is due to Cardy, and was recently made https://arxiv.org/abs/1904.06359.

6.3.4 Too few words about the Conformal Bootstrap

Earlier I said that given a 'solution' to a CFT in the form of a list of primaries and their dimensions and OPE structure constants, you could compute any n-point correlator by repeated use of the OPE to reduce it to a two-point function, by the following kind of operation:



[from Rychkov's CFT notes]

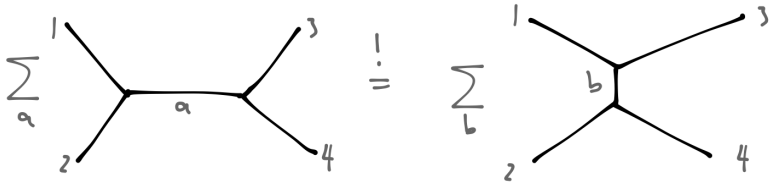
Some interesting questions and previews of the answers:

(1) Which sets of scaling dimensions arise from CFTs which actually exist?

The general answer is not known. One set of constraints follows from unitarity in the form of positivity of the inner product on the CFT Hilbert space. For example, the dimension of a scalar operator (other than $\mathbb{1}$) must be larger than the free field dimension $(\frac{D-2}{2})$[43].

(2) In implementing the reduction to three-point functions and structure constants, you have a choice about the order in which you group the operators. Do you get the same answer independent of the order?

The condition that you do – the associativity of the OPE – imposes significant constraints on the structure constants and dimensions.



(3) Are there more constraints from higher-order diagrams?

[43] This follows from the fact that $\mathbf{D}$ appears on the RHS of $[\mathbf{C}_\mu, \mathbf{P}_\nu]$. A crucial ingredient comes from the fact that the dagger of a lowering operator is a raising operator (obvious, right?). In radial quantization this implies the weird-looking formula:

$$\mathbf{P}_\mu^\dagger = \mathbf{C}_\mu \ .$$

For more details, I recommend the discussion leading up to eqn (5.57) of Jared Kaplan's AdS/CFT notes.

Claim: no. The analog of the 'pentagon identity' is automatic.

To make use of the (many!) associativity constraints, a further ingredient is required, which is called *conformal blocks*. For more in this direction, a good starting point is Rychkov and the longer review by Simmons-Duffin.

In $D = 2$ it really works [BPZ]. This approach has led to complete solutions of all unitary CFTs with $c < 1$ (this is explained in Ginsparg's notes), and a number of interesting examples (*e.g.* Liouville theory) with $c > 1$. More recently, this program has had success in $2 + 1$ dimensions: the 3d Ising model has been cornered.

7 Duality, part 2

7.1 (2+1)-d XY is dual to (2+1)d electrodynamics

7.1.1 Mean field theory

Earlier (during our discussion of boson coherent states) I made some claims about the phase diagram of the Bose-Hubbard model

$$H_{BH} = \sum_i \left(-\mu n_i + U n_i(n_i - 1)\right) + \sum_{ij} b_i^\dagger w_{ij} b_j$$

which I would like to clarify.

[Sachdev] Consider a variational approach to the BH model. We'll find the best product-state wavefunction $|\Psi_{\rm var}\rangle = \otimes_i |\psi_i\rangle$, and minimize the BH energy $\langle\Psi_{\rm var}| H_{BH} |\Psi_{\rm var}\rangle$ over all ψ_i. We can parametrize the single-site states as the groundstates of the mean-field hamiltonian:

$$H_{\rm MF} = \sum_i h_i = \sum_i \left(-\mu n_i + U n_i(n_i - 1) - \Psi^\star b_i - \Psi b_i^\dagger\right).$$

Here Ψ is an effective field which incorporates the effects of the neighboring sites. Notice that nonzero Ψ breaks the $\mathsf{U}(1)$ boson number conservation: particles can hop out of the site we are considering. This also means that nonzero Ψ will signal SSB.

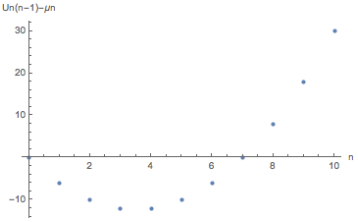
What does this simple approximation give up? For one, it assumes the ground-state preserves the lattice translation symmetry, which doesn't always happen. More painfully, it also gives up on any entanglement at all in the groundstate. Phases for which entanglement plays an important role will not be found this way.

We want to minimize over Ψ the quantity

$$\begin{aligned}\mathcal{E}_0 \equiv \frac{1}{M}\langle\Psi_{\rm var}| H_{BH} |\Psi_{\rm var}\rangle &= \frac{1}{M}\left(\langle\Psi_{\rm var}| \left(\underbrace{H_{BH} - H_{MF}}_{=w\sum b^\dagger b + \Psi b + h.c.} + H_{MF}\right)|\Psi_{\rm var}\rangle\right) \\ &= \frac{1}{M}E_{MF}(\Psi) - zw\left\langle b^\dagger\right\rangle\langle b\rangle + \langle b\rangle\Psi^\star + \left\langle b^\dagger\right\rangle\Psi. \end{aligned} \tag{7.1}$$

Here z is the coordination number of the lattice (the number of neighbors of a site, which we assume is the same for every site), and $\langle..\rangle \equiv \langle\Psi_{\rm var}|..|\Psi_{\rm var}\rangle$.

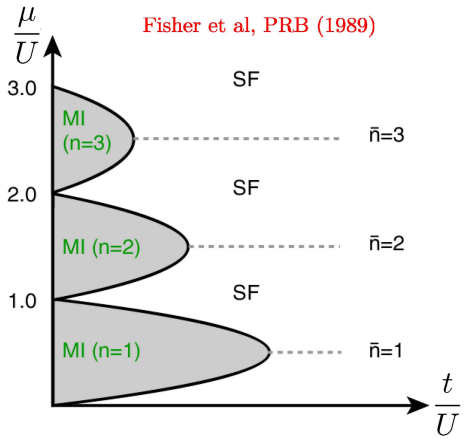
First consider $w = 0$, no hopping. Then $\Psi_B = 0$ (neighbors don't matter), and the single-site state is a number eigenstate $|\psi_i\rangle = |n_0(\mu/U)\rangle$, where $n_0(x) = 0$ for $x < 0$, and $n_0(x) = \lceil x\rceil$, (the ceiling of x, *i.e.* , the next integer larger than x), for $x > 0$. Precisely when μ/U is an integer, there is a twofold degeneracy per site.



This degeneracy is broken by a small hopping term. Away from the degenerate points, within a single Mott plateau, the hopping term does very little (even away from mean field theory). This is because there is an energy gap, and $[N, H_{BH}] = 0$, which means that a small perturbation has no other states to mix in which might have other eigenvalues of N. Therefore, within a whole open set, the particle number remains fixed. This means $\partial_\mu \langle N \rangle = 0$, the system is incompressible.

We can find the boundaries of this region by expanding $\mathcal{E}_0$ in Ψ, following Landau: $\mathcal{E}_0 = \mathcal{E}_0^0 + r|\Psi|^2 + \mathcal{O}(|\Psi|^4)$. We can compute the coefficients in perturbation theory, and this produces the following picture.

Mean field theory gives the famous picture at right, with lobes of different Mott insulator states with different (integer!) numbers of bosons per site. (The hopping parameter w is called t in the figure.)



(Fig credit: Roman Lutchyn)

7.1.2 Coherent state path integral

Actually we can do a bit better; some of our hard work will pay off. Consider the coherent state path integral for the Euclidean partition sum

$$Z = \int [d^2 b] e^{-\int_0^{1/T} d\tau \mathcal{L}_b}$$

$$\text{with} \quad \mathcal{L}_b = \sum_i \left(b_i^\dagger \partial_\tau b_i - \mu b_i^\dagger b_i + U b_i^\dagger b_i^\dagger b_i b_i \right) - \sum_{ij} b_i^\dagger w_{ij} b_j$$

where we introduced the hopping matrix $w_{ij} = w$ if $\langle ij \rangle$ share a link, otherwise zero. Here the bs are numbers, coherent state eigenvalues. Here is another application of the Hubbard-Stratonovich transformation:

$$Z = \int [d^2 b][d^2 \Psi] e^{-\int_0^{1/T} d\tau \mathcal{L}'_b}$$

$$\text{with} \quad \mathcal{L}'_b = \sum_i \left(b_i^\dagger \partial_\tau b_i - \mu b_i^\dagger b_i + U b_i^\dagger b_i^\dagger b_i b_i - \Psi b_i^\dagger - \Psi^\star b_i \right) + \sum_{ij} \Psi_i w_{ij}^{-1} \Psi_j .$$

(Warning: if w has negative eigenvalues, so that the gaussian integral over Ψ is well-defined, we need to add a big constant to it, and subtract it from the single-particle terms.) Now integrate out the b fields. It's not gaussian, but notice that the resulting action for Ψ is the connected generating function $W[J]$: $\int [d^2 b] e^{-S[b] + \int \Psi b + h.c.} =$

$e^{-W[\Psi,\Psi^\star]}$. More specifically,

$$Z = \int [d^2\Psi] e^{-\frac{V}{T}\mathcal{F}_0 - \int_0^{1/T} d\tau \mathcal{L}_B}$$

$$\text{with} \quad \mathcal{L}_B = K_1 \Psi^\star \partial_\tau \Psi + K_2 |\partial_\tau \Psi|^2 + K_3 |\vec{\nabla}\Psi|^2 + \tilde{r}|\Psi|^2 + u|\Psi|^4 + \cdots$$

Here $V = Ma^d$ is the volume of space, and $\mathcal{F}_0$ is the mean-field free energy. The coefficients K etc are connected Green's functions of the bs. The choice of which terms I wrote was dictated by Landau, and the order in which I wrote them should have been determined by Wilson. The Mott-SF transition occurs when $\tilde{r}$ changes sign, that is, the condition $\tilde{r} = 0$ determines the location of the Mott-SF boundaries. You can see that generically we have $z = 2$ kinetic terms. Less obvious is that $\tilde{r}$ is proportional to the mean field coefficient r.

Here's the payoff. I claim that the coefficients in the action for Ψ are related by

$$K_1 = -\partial_\mu \tilde{r}. \tag{7.2}$$

This means that $K_1 = 0$ precisely when the boundary of the lobe has a vertical tangent. This means that right at those points (the ends of the dashed lines in the figure) the second-order kinetic term is the leading one, and we have $z = 1$.

Here's the proof of (7.2). $\mathcal{L}_B$ must have the same symmetries as $\mathcal{L}_b$. One such invariance is

$$b_i \to b_i e^{\mathbf{i}\phi(\tau)}, \ \Psi_i \to \Psi_i e^{\mathbf{i}\phi(\tau)}, \mu \to \mu + \mathbf{i}\partial_\tau \phi.$$

This is a funny transformation which acts on the couplings, so doesn't produce Noether currents. It is still useful though, because it implies

$$0 = \delta_\phi \left(K_1 \Psi^\star \partial_\tau \Psi + \tilde{r}|\Psi|^2 + ... \right) = K_1 |\Psi|^2 \mathbf{i}\partial_\tau \phi + \partial_\mu \tilde{r} \mathbf{i} \partial\phi |\Psi|^2 + ...$$

7.1.3 Duality

We have seen above (in §5.1) that the prevention of vortices is essential to superfluidity, which is the condensation of bosons. In $D = 1 + 1$, vortices are events in spacetime. In $D = 2 + 1$, vortices are actual particles, *i.e.* localizable objects, around which the superfluid phase variable winds by 2π (times an integer).

More explicitly, if the boson field which condenses is $b(x) = ve^{\mathbf{i}\phi}$, and we choose polar coordinates in space $x + \mathbf{i}y \equiv Re^{\mathbf{i}\varphi}$, then a vortex is a configuration of the order parameter field of the form $b(x) = f(R)e^{\mathbf{i}\varphi}$, where $f(R) \overset{R\to\infty}{\to} v$ far away: the phase of the order parameter winds around. Notice that the phase is ill-defined in the core of the vortex where $f(R) \overset{R\to 0}{\to} 0$. (This is familiar from our discussion of the Abelian Higgs model.)

To see the role of vortices in destroying superfluidity more clearly, consider superfluid flow in a 2d annulus geometry, with the same polar coordinates $x + \mathbf{i}y = Re^{\mathbf{i}\varphi}$. If the superfluid phase variable is in the configuration $\phi(R, \varphi) = n\varphi$, then the current is

$$\vec{J}(R, \varphi) = \rho_s \vec{\nabla}\phi = \check{\varphi}\rho_s \frac{n}{2\pi R}.$$

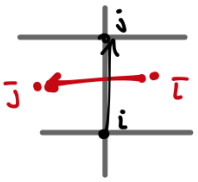
The current only changes if the integer n changes. This happens if vortices enter from the outside; removing the current (changing n to zero) requires n vortices to tunnel all the way through the sample, which if they are gapped and the sample is macroscopic can take a cosmologically long time.

There is a dual statement to the preceding three paragraphs: a state where the bosons themselves are gapped and localized – that is, a Mott insulator – can be described starting from the SF phase by the condensation of vortices. To see this, let us consider again the (simpler-than-Bose-Hubbard) $2 + 1$d rotor model

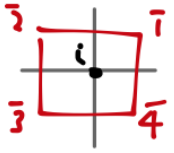
$$\mathbf{H}_{\text{rotors}} = U \sum_i \mathbf{n}_i^2 - J \sum_{\langle ij \rangle} \cos\left(\phi_i - \phi_j\right)$$

and introduce dual variables. Introduce a dual lattice whose sites are (centered in) the faces of the original (direct) lattice; each link of the dual lattice crosses one link of the direct lattice.

• First let $e_{\bar{i}\bar{j}} \equiv \frac{\phi_i - \phi_j}{2\pi}$. Here we define $\bar{i}\bar{j}$ by the right hand rule: $ij \times \bar{i}\bar{j} = +\check{z}$ (ij denotes the unit vector pointing from i to j). This is a lattice version of $\vec{e} = \check{z} \times \vec{\nabla}\phi \frac{1}{2\pi}$. Defining lattice derivatives $\Delta_x \phi_i \equiv \phi_i - \phi_{i+\check{x}}$, the definition is $e_x = -\frac{\Delta_y \phi}{2\pi}, e_y = \frac{\Delta_x \phi}{2\pi}$. It is like an electric field vector.



• The conjugate variable to the electric field is $a_{\bar{i}\bar{j}}$, which must therefore be made from the conjugate variable of ϕ_i, namely $\mathbf{n}_i$: $[\mathbf{n}_i, \phi_j] = -\mathbf{i}\delta_{ij}$. Acting with $\mathbf{n}_i$ translates ϕ_i, which means that it shifts all the $e_{\bar{i}\bar{j}}$ from the surrounding plaquettes. More precisely:



$$2\pi \mathbf{n}_i = a_{\bar{1}\bar{2}} + a_{\bar{2}\bar{3}} + a_{\bar{3}\bar{4}} + a_{\bar{4}\bar{1}}.$$

This is a lattice, integer version of $n \sim \frac{1}{2\pi}\vec{\nabla} \times a \cdot \check{z}$. In terms of these variables,

$$\mathbf{H}_{\text{rotors}} = \frac{U}{2} \sum_i \left(\frac{\Delta \times a}{2\pi}\right)^2 - J \sum_{\langle \bar{i}\bar{j} \rangle} \cos\left(2\pi e_{\bar{i}\bar{j}}\right)$$

with the following *constraint*. If it were really true that $\vec{e} = \frac{1}{2\pi}\check{z} \times \vec{\nabla}\phi$, with single-valued ϕ, then $\vec{\nabla} \cdot \vec{e} = \vec{\nabla} \cdot \left(\check{z} \times \vec{\nabla}\phi\right) = 0$. But there are vortices in the world, where ϕ

is not single valued. The number of vortices $n_v(R)$ in some region R with $\partial R = C$ is determined by the winding number of the phase around C:

$$2\pi n_v(R) = \oint_C d\vec{\ell} \cdot \vec{\nabla}\phi \overset{\text{Stokes}}{=} 2\pi \int_R d^2x \vec{\nabla} \cdot \vec{e}$$

(More explicitly, $2\pi\vec{\nabla} \cdot \vec{e} = \epsilon_{zij}\partial_i\partial_j\phi = [\partial_x, \partial_y]\phi$ clearly vanishes if ϕ is single-valued.) Since this is true for any region R, we have

$$\vec{\nabla} \cdot \vec{e} = 2\pi\delta^2(\text{vortices}).$$

Actually, the lattice version of the equation has more information (and is true) because it keeps track of the fact that the number of vortices is an integer:

$$\Delta_x e_x + \Delta_y e_y \equiv \vec{\Delta} \cdot \vec{e}(\bar{i}) = 2\pi n_v(\bar{i}), \qquad n_v(\bar{i}) \in \mathbb{Z}.$$

It will not escape your notice that this is Gauss' law, with the density of *vortices* playing the role of the charge density.

Phases of the 2d rotors. Since $\vec{e} \sim \vec{\nabla}\phi$ varies continuously, *i.e.* electric flux is not quantized, this is called *noncompact* electrodynamics. Again we will impose the integer constraint $a \in 2\pi\mathbb{Z}$ energetically, *i.e.* let $a \in \mathbb{R}$ and add (something like) $\mathbf{\Delta H} \overset{?}{=} -t\cos a$ and see what happens when we make t finite. The expression in the previous sentence is not quite right, yet, however: This operator does not commute with our constraint $\vec{\Delta} \cdot \vec{e} - 2\pi n_v = 0$ – it jumps $\vec{e}$ but not n_v[44].

We can fix this by introducing explicitly the variable which creates vortices, $e^{-\mathbf{i}\chi}$, with:

$$[n_v(\bar{i}), \chi(\bar{j})] = -\mathbf{i}\delta_{\bar{i}\bar{j}} \ .$$

Certainly our Hilbert space contains states with different number of vortices, so we can introduce an operator which maps these sectors. Its locality might be an issue: certainly it is nonlocal with respect to the original variables, but we will see that we can treat it as a local operator (except for the fact that it carries gauge charge) in the dual description. Since $n_v \in \mathbb{Z}$, $\chi \simeq \chi + 2\pi$ lives on a circle. So:

$$\mathbf{H} \sim \sum_{\bar{i}} \left(\frac{U}{2} \left(\frac{\Delta \times a}{2\pi} \right)^2 + \frac{J}{2} (2\pi e)^2 - t\cos\left(\Delta\chi - a\right) \right)$$

[44] A set of words which has the same meaning as the above: $\cos a$ is not gauge invariant. Understanding these words requires us to think of the operator $G(\bar{i}) \equiv \vec{\Delta} \cdot \vec{e} - 2\pi n_v$ as the generator of a transformation,

$$\delta\mathcal{O} = \sum_{\bar{i}} s(\bar{i})[G(\bar{i}), \mathcal{O}].$$

It can be a useful picture.

still subject to the constraint $\vec{\Delta} \cdot \vec{e} = 2\pi n_v$.

Two regimes:
$\boxed{J \gg U, t}$: This suppresses e and its fluctuations, which means a fluctuates. The fluctuating a is governed by the gaussian hamiltonian

$$\mathbf{H} \sim \sum \left(\vec{e}^2 + \vec{b}^2 \right)$$

with $b \equiv \frac{\Delta \times a}{2\pi}$, which should look familiar. This *deconfined* phase has a gapless photon; a $2+1$d photon has a single polarization state. This is the goldstone mode, and this regime describes the superfluid phase (note that the parameters work out right in the original variables). The relation between the photon a and the original phase variable, in the continuum is

$$\epsilon_{\mu\nu\rho} \partial_\nu a_\rho = \partial_\mu \phi.$$

$\boxed{t \gg U, J}$: In this regime we must satisfy the cosine first. Like in $D = 1+1$, this can be described as the statement that vortices condense. Expanding around its minimum, the cosine term is

$$\mathbf{h} \ni t \left(a - \partial \chi \right)^2$$

– the photon gets a mass by eating the phase variable χ. There is an energy gap. This is the Mott phase.

If the vortices carry other quantum numbers, the (analog of the) Mott phase can be more interesting, as we'll see in section 7.3.

Compact electrodynamics in $D = 2+1$. Note that this free photon phase of $D = 2+1$ electrodynamics is not accessible if e is quantized (so-called compact electrodynamics) where monopole instantons proliferate and gap out the photon. This is the subject of §7.2.

7.1.4 Particle-vortex duality in the continuum

The above is easier to understand (but a bit less precise) in the continuum. Consider a quantum system of bosons in $D = 2+1$ with a $\mathsf{U}(1)$ particle-number symmetry (a real symmetry, not a gauge redundancy). Let's focus on a complex, non-relativistic bose field b with action

$$S[b] = \int \mathrm{d}t \mathrm{d}^2 x \left(b^\dagger \left(\mathbf{i}\partial_t - \vec{\nabla}^2 - \mu \right) b - U (b^\dagger b)^2 \right). \tag{7.3}$$

By Noether's theorem, the symmetry $b \to e^{\mathbf{i}\theta} b$ implies that the current

$$j_\mu = (j_t, \vec{j})_\mu = (b^\dagger b, \mathbf{i} b^\dagger \vec{\nabla} b + h.c.)_\mu$$

satisfies the continuity equation $\partial^\mu j_\mu = 0$.

This system has two phases of interest here. In the ordered/broken/superfluid phase, where the groundstate expectation value $\langle b \rangle = \sqrt{\rho_0}$ spontaneously breaks the U(1) symmetry, the goldstone boson θ in $b \equiv \sqrt{\rho_0} e^{\mathbf{i}\theta}$ is massless

$$S_{\text{eff}}[\theta] = \frac{\rho_0}{2} \int \left(\dot{\theta}^2 - \left(\vec{\nabla} \theta \right)^2 \right) \mathrm{d}^2 x \mathrm{d}t, \quad j_\mu = \rho_0 \partial \theta \ .$$

In the disordered/unbroken/Mott insulator phase, $\langle b \rangle = 0$, and there is a mass gap. A dimensionless parameter which interpolates between these phases is $g = \mu/U$; large g encourages condensation of b.

We can 'solve' the continuity equation by writing

$$j^\mu = \epsilon^{\mu \cdots} \partial_\cdot a_\cdot \tag{7.4}$$

where $a_\cdot$ is a gauge potential. The time component of this equation says that the boson density is represented by the magnetic flux of a. The spatial components relate the boson charge current to the electric flux of a. The continuity equation for j is automatic – it is the Bianchi identity for a – as long as a is single-valued. That is: as long as there is no magnetic charge present. A term for this condition which is commonly used in the cond-mat literature is: "a is non-compact." (More on the other case below.)

The relation (7.4) is the basic ingredient of the duality, but it is not a complete description: in particular, how do we describe the boson itself in the dual variables? In the disordered phase, adding a boson is a well-defined thing which costs a definite energy. The boson is described by a localized clump of magnetic flux of a. Such a configuration is energetically favored if a participates in a superconductor – i.e. if a is coupled to a condensate of a charged field. The Meissner effect will then ensure that its magnetic flux is bunched together. So this suggests that we should introduce into the dual description a scalar field, call it Φ, minimally coupled to the gauge field a:

$$S[b] \leftrightsquigarrow S_{\text{dual}}[a, \Phi] \ .$$

And the disordered phase should be dual to a phase where $\langle \Phi \rangle \neq 0$, which gives a mass to the gauge field by the Anderson-Higgs mechanism.

Who is Φ? More precisely, what is the identity in terms of the original bosons of the particles it creates? When Φ is not condensed and its excitations are massive, the gauge field is massless. This the Coulomb phase of the Abelian Higgs model $S[a, \Phi]$; at low energies, it is just free electromagnetism in $D = 2 + 1$. These are the properties of the ordered phase of b. (This aspect of the duality is explained in Wen, §6.3.) The

photon has one polarization state in $D = 2 + 1$ and is dual to the goldstone boson. This is the content of (7.4) in the ordered phase: $\epsilon^{\mu\cdots}\partial.a. = \rho_0\partial_\mu\theta$ or $\star \mathrm{d}a = \rho_0 \mathrm{d}\theta$.

Condensing Φ gives a mass to the Goldstone boson whose masslessness is guaranteed by the broken $\mathsf{U}(1)$ symmetry. Therefore Φ is a disorder operator: its excitations are vortices in the bose condensate, which are gapped in the superfluid phase. The transition to the insulating phase can be described as a condensation of these vortices.

The vortices have relativistic kinetic terms, *i.e.* particle-hole symmetry. This is the statement that in the ordered phase of the time-reversal invariant bose system, a vortex and an antivortex have the same energy. An argument for this claim is the following. We may create vortices by rotating the sample, as was done in the figure at right. With time-reversal symmetry, rotating the sample one way will cost the same energy as rotating it the other way.

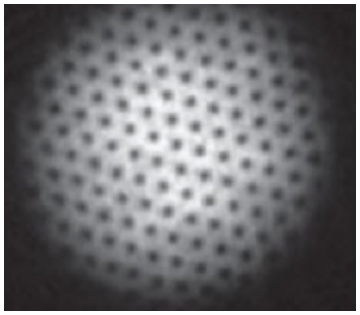


Fig: M. Zwierlein.

This means that the mass of the vortices $m_V^2\Phi^\dagger\Phi$ is distinct from the vortex chemical potential $\mu_V\rho_V = \mu_V \mathbf{i}\Phi^\dagger\partial_t\Phi + h.c.$. The vortex mass^2 maps under the duality to the boson chemical potential. Taking it from positive to negative causes the vortices to condense and disorder (restore) the $\mathsf{U}(1)$ symmetry.

To what does the vortex chemical potential map? It is a term which breaks time-reversal, and which encourages the presence of vortices in the superfluid order. It's an external magnetic field for the bosons. (This also the same as putting the bosons into a rotating frame.)

To summarize, a useful dual description is the Abelian Higgs model

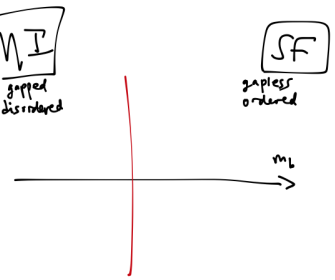
$$S[a,\Phi] = \int \mathrm{d}^2x\mathrm{d}t \left(\Phi^\dagger \left((\mathbf{i}\partial_t - \mathbf{i}A_t - \mu)^2 + \left(\vec{\nabla} + \vec{A}\right)^2 \right) \Phi - \frac{1}{e^2} f_{\mu\nu}f^{\mu\nu} - V(\Phi^\dagger\Phi) \right).$$

We can parametrize V as

$$V = \lambda \left(\Phi^\dagger\Phi - v\right)^2$$

– when $v < 0$, $\langle\Phi\rangle = 0$, Φ is massive and we are in the Coulomb phase. When $v > 0$ Φ condenses and we are in the Anderson-Higgs phase.

The description above is valid near the boundary of one of the MI phases. At the tips of the lobes are special points where the bosons b themselves have particle-hole symmetry (*i.e.* relativistic kinetic terms). For more on this diagram, see *e.g.* chapter 9 of Sachdev.



In the previous discussion I have been assuming that the vortices of b have unit charge under a and are featureless bosons, *i.e.* do not carry any non-trivial quantum numbers under any other symmetry. If *e.g.* the vortices have more-than-minimal charge under a, say charge q, then condensing them leaves behind a $\mathbb{Z}_q$ gauge theory and produces a state with topological order. If the vortices carry some charge under some other symmetry (like lattice translations or rotations) then condensing them breaks that symmetry. If the vortices are minimal-charge fermions, then they can only condense in pairs, again leaving behind an unbroken $\mathbb{Z}_2$ gauge theory.

7.2 Compact electrodynamics in $D = 2 + 1$

Since I emphasized above that we were speaking about non-compact electrodynamics, I should explain what is compact electrodynamics, why this makes a difference, and where it arises. First I'll introduce it in a lattice model of bosons. It can also emerge from spin systems; in this context, when a deconfined phase is realized, it is called a $\mathsf{U}(1)$ spin liquid.

Consider a quantum system on a two-dimensional lattice (say, square) with rotors $\Theta_l \equiv \Theta_l + 2\pi m$ on the links l. (Think of this as the phase of a boson or the direction of an easy-plane spin.) The conjugate variable $\mathbf{n}_l$ is an integer

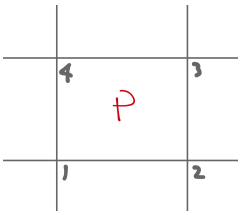
$$[\mathbf{n}_l, \Theta_{l'}] = -\mathbf{i}\delta_{l,l'}.$$

Here $\mathbf{n}_{ij} = \mathbf{n}_{ji}, \Theta_{ij} = \Theta_{ji}$ – we have not oriented our links (yet). We also impose the *Gauss' law constraint*

$$\mathbf{G}_s \equiv \sum_{l \in v(s)} \mathbf{n}_l = 0 \quad \forall \text{ sites } s,$$

where the notation $v(s)$ means the set of links incident upon the site s ('v' is for 'vicinity').

We'll demand that the Hamiltonian is 'gauge invariant', that is, that $[\mathbf{H}, \mathbf{G}_s] = 0 \forall s$. Any terms which depend only on $\mathbf{n}$ are OK. The natural single-valued object made from Θ is $e^{\mathbf{i}\Theta_l}$, but this is not gauge invariant. A combination which is gauge invariant is the *plaquette operator*, associated to a face p of the lattice:



$$\prod_{l \in \partial p} e^{(-1)^y \mathbf{i}\Theta_l} \equiv e^{\mathbf{i}(\Theta_{12} - \Theta_{23} + \Theta_{34} - \Theta_{41})}$$

– we put a minus sign on the horizontal links. ∂p denotes the links running around the boundary of p. So a good hamiltonian is

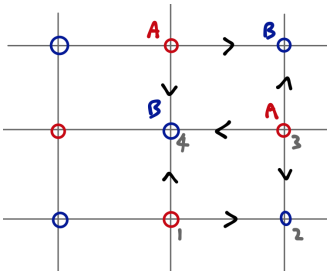
$$\mathbf{H} = \frac{U}{2} \sum_l \mathbf{n}_l^2 - K \sum_\square \cos\left(\sum_{l \in \partial \square} (-1)^y \Theta_l \right).$$

Local Hilbert space. The space of gauge-invariant states is not a tensor product over local Hilbert spaces. This sometimes causes some confusion, and seems like an obstruction to such a system emerging from condensed matter. Notice, however, that we can arrive at the gauge-theory hilbert space by imposing the Gauss' law constraint energetically (as in the toric code): Start with the following Hamiltonian acting on the full unconstrained rotor Hilbert space:

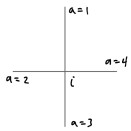
$$H_{\text{big}} = +\Gamma_\infty \sum_i \mathbf{G}_i + \mathbf{H}.$$

True to its name, the coefficient Γ_∞ is some huge energy scale which penalizes configurations which violate Gauss' law (if you like, such configurations describe some matter with rest mass Γ_∞). So, states with energy $\ll \Gamma_\infty$ all satisfy Gauss' law. Then further, we want $\mathbf{H}$ to act *within* this subspace, and not create excitations of enormous energies like Γ_∞. This requires $[\mathbf{G}_i, \mathbf{H}] = 0, \forall i$, which is exactly the condition that $\mathbf{H}$ is gauge invariant.

A useful change of variables gets rid of these annoying signs. Assume the lattice is bipartite: made of two sublattices A, B each of which only touches the other. Then draw arrows from A sites to B sites, and let



$$\begin{aligned} \mathbf{e}_{ij} &\equiv \eta_i \mathbf{n}_{ij} \\ \mathbf{a}_{ij} &\equiv \eta_i \Theta_{ij} \end{aligned}, \quad \eta_i \equiv \begin{cases} +1, & i \in A \\ -1, & i \in B \end{cases}.$$



Then the Gauss constraint now reads

$$0 = \mathbf{e}_{\bar{i}\bar{1}} + \mathbf{e}_{\bar{i}\bar{2}} + \mathbf{e}_{\bar{i}\bar{3}} + \mathbf{e}_{\bar{i}\bar{4}} \equiv \Delta \cdot \mathbf{e}(\bar{i}).$$

This is the lattice divergence operation. The plaquette term reads

$$\cos\left(\Theta_{12} - \Theta_{23} + \Theta_{34} - \Theta_{41}\right) = \cos\left(\mathbf{a}_{12} + \mathbf{a}_{23} + \mathbf{a}_{34} + \mathbf{a}_{41}\right) \equiv \cos\left(\Delta \times \mathbf{a}\right)$$

– the lattice curl (more precisely, it is $(\Delta \times \mathbf{a}) \cdot \check{z}$). In these variables,

$$\mathbf{H} = \frac{U}{2} \sum_l \mathbf{e}_l^2 - K \sum_\square \cos\left((\Delta \times \mathbf{a}) \cdot \check{n}_\square\right)$$

(in the last term we emphasize that this works in $D \geq 2+1$ if we remember to take the component of the curl normal to the face in question). This is (compact) lattice $\mathsf{U}(1)$ gauge theory, with no charges. The word 'compact' refers to the fact that the charge is quantized; the way we would add charge is by modifying the Gauss' law to

$$\underbrace{\Delta \cdot \mathbf{e}(\bar{i})}_{\in \mathbb{Z}} = \underbrace{\text{charge at } \bar{i}}_{\Longrightarrow \in \mathbb{Z}}$$

where the charge must be quantized because the LHS is an integer. (In the noncompact electrodynamics we found dual to the superfluid, it was the continuous angle variable which participated in the Gauss' law, and the discrete variable which was gauge variant.)

What is it that's compact in compact QED?

The operator appearing in Gauss' law

$$\mathbf{G}(x) \equiv \left(\vec{\nabla} \cdot \vec{\mathbf{e}}(x) - 4\pi \mathbf{n}(x)\right)$$

(here $\mathbf{n}(x)$ is the density of charge) is the generator of gauge transformations, in the sense that a gauge transformation acts on any operator $\mathcal{O}$ by

$$\mathcal{O} \mapsto e^{-\mathbf{i}\sum_x \alpha(x)\mathbf{G}(x)} \mathcal{O} e^{\mathbf{i}\sum_x \alpha(x)\mathbf{G}(x)} \tag{7.5}$$

This is a fact we've seen repeatedly above, and it is familiar from ordinary QED, where using the canonical commutation relations

$$[\mathbf{a}^i(x), \mathbf{e}^j(y)] = -\mathbf{i}\delta^{ij}\delta(x-y), \quad [\phi(x), \mathbf{n}(y)] = -\mathbf{i}\delta(x-y)$$

(ϕ is the phase of a charged field, $\Phi = \rho e^{\mathbf{i}\phi}$) in (7.5) reproduce the familiar gauge transformations

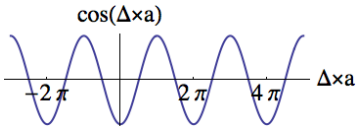
$$\vec{\mathbf{a}} \to \vec{\mathbf{a}} + \vec{\nabla}\alpha, \qquad \phi \to \phi + \alpha \quad .$$

SO: if all the objects appearing in Gauss' law are integers (which is the case if charge is quantized and electric flux is quantized), it means that the gauge parameter α itself only enters mod 2π, which means the gauge transformations live in $\mathsf{U}(1)$, as opposed to $\mathbb{R}$. So it's the gauge group that's compact.

This distinction is very important, because (in the absence of matter) this model does not have a deconfined phase! To see this result (due to Polyakov), first consider strong coupling:

$\boxed{U \gg K}$: The groundstate has $\mathbf{e}_{\bar{l}} = 0$, $\forall \bar{l}$. (Notice that this configuration satisfies the constraint.) There is a gap to excitations where some link has an *integer* $\mathbf{e} \neq 0$, of order U. (If $\mathbf{e}$ were continuous, there would not be a gap!) In this phase, electric flux is confined, *i.e.* costs energy and is generally unwanted.

$\boxed{U \ll K}$: The surprising thing is what happens when we make the gauge coupling weak.



Then we should first minimize the magnetic flux term: minimizing $-\cos(\Delta \times \mathbf{a})$ means $\Delta \times \mathbf{a} \in 2\pi\mathbb{Z}$. Near each minimum, the physics looks like Maxwell, $\mathbf{h} \sim \mathbf{e}^2 + \mathbf{b}^2 + \cdots$. BUT: it turns out to be a colossally bad idea to ignore the tunnelling *between* the minima. To see this, begin by solving the Gauss law constraint $\Delta \cdot \mathbf{e} = 0$ by introducing

$$\mathbf{e}_{\bar{1}\bar{2}} \equiv \frac{1}{2\pi}\left(\chi_2 - \chi_1\right) \tag{7.6}$$

(*i.e.* $\vec{\mathbf{e}} = \check{z} \cdot \Delta\chi \frac{1}{2\pi}$.) χ is a (discrete!) 'height variable'. Then the operator

$$e^{\mathbf{i}(\Delta\times\mathbf{a})(\bar{i})}$$

increases the value of $\mathbf{e}_{\bar{i}\bar{a}}$ for all neighboring sites $\bar{a}$, which means it jumps $\chi_{\bar{i}} \to \chi_{\bar{i}} + 2\pi$. So we should regard

$$(\Delta \times \mathbf{a})\,(\bar{i}) \equiv \Pi_\chi(\bar{i})$$

as the conjugate variable to χ, in the sense that

$$[\Pi_\chi(r), \chi(r')] = -\mathbf{i}\delta_{rr'}.$$

Notice that this is consistent with thinking of χ as the dual scalar related to the gauge field by our friend the (Hodge) duality relation

$$\partial_\mu \chi = \epsilon_{\mu\nu\rho}\partial_\nu a_\rho.$$

The spatial components i say $\partial_i \chi = \epsilon_{ij} f_{0j}$, which is the continuum version of (7.6). The time component says $\dot{\chi} = \epsilon_{ij} f_{ij} = \nabla \times a$, which indeed says that (if χ has quadratic kinetic terms), the field momentum of χ is the magnetic flux. So χ is the would-be transverse photon mode.

The hamiltonian is now

$$\mathbf{H} = \frac{U}{2}\sum_l (\Delta\chi)^2 - K\sum_r \cos\Pi_\chi(r)$$

with no constraint, but $\chi \in 2\pi\mathbb{Z}$. In the limit $U \gg K$, the spatial gradients of χ are forbidden – χ wants to be uniform. From the definition (7.6), uniform χ means there are no electric field lines, this is the confined phase. Deconfinement limit should be $K \gg U$, in which case it looks like we can Taylor expand the cosine $\cos\Pi_\chi \sim 1 - \frac{1}{2}\Pi_\chi^2$ about one of its minima, and get harmonic oscillators. But: tunneling between the neighboring vacua of $\Delta \times \mathbf{a}$ is accomplished by the *flux-insertion operator* (or *monopole operator*)

$$e^{\mathbf{i}\chi}, \quad \text{which satisfies } [e^{\mathbf{i}\chi(r)}, (\Delta \times \mathbf{a})\,(r')] = e^{\mathbf{i}\chi(r)}\delta_{rr'}$$

– that is, $e^{\mathbf{i}\chi}$ is a raising operator for $\Delta \times \mathbf{a}$. To analyze whether the Maxwell limit survives this, let's go to the continuum and study perturbations of the free hamiltonian

$$\mathbf{H}_0 = \int \left(\frac{U}{2} \left(\vec{\nabla} \chi \right)^2 + \frac{K}{2} \Pi_\chi^2 \right)$$

by

$$\mathbf{H}_1 = - \int V_0 \cos \chi \ .$$

This operator introduces tunneling events by $\Pi_\chi \to \Phi_\chi \pm 2\pi$ with rate V_0. Alternatively, notice that again we can think of the addition of this term as energetically imposing the condition that $\chi \in 2\pi\mathbb{Z}$.

So: is V_0 irrelevant? Very much no. In fact

$$\langle \cos \chi(r) \cos \chi(0) \rangle_0 \sim \text{const} \tag{7.7}$$

has constant amplitude at large r! That means that the operator has dimension zero, and the perturbation in the action has $[S_1 = - \int V_0 \cos \chi d^2x d\tau] \sim L^3$, very relevant. The result is that it pins the χ field (the would-be photon mode) to an integer, from which it can't escape. This result is due to Polyakov.

To see (7.7) begin with the gaussian identity

$$\left\langle e^{\mathbf{i}s\chi(x)} e^{\mathbf{i}s'\chi(0)} \right\rangle = e^{-\frac{ss'}{2}\langle \chi(x)\chi(0) \rangle} \ ,$$

with $s, s' = \pm$. The required object is

$$\begin{aligned} \langle \chi(x)\chi(0) \rangle &= \frac{\mathbf{i}}{T} \int \text{đ}^3 p \frac{e^{\mathbf{i}\vec{p}\cdot\vec{x}}}{p^2} = \mathbf{i} \frac{2\pi}{(2\pi)^3 T} \int_0^\infty dp \underbrace{\int_{-1}^{1} d\cos\theta e^{\mathbf{i}px\cos\theta}}_{=\frac{2\sin px}{px}} \\ &= \mathbf{i} \frac{2}{(2\pi)^2 T} \int_0^\infty dp \frac{\sin px}{px} \\ &= \mathbf{i} \frac{2}{2\pi T} \frac{1}{x} \frac{1}{2} \underbrace{\int_{-\infty}^{\infty} d\bar{p} \frac{\sin \bar{p}}{\bar{p}}}_{=\pi} \\ &= \frac{\mathbf{i}}{2Tx}. \end{aligned} \tag{7.8}$$

(I have set the velocity of propagation to 1, and $T \equiv U/K$ is the coefficient in front of the Lagrangian, $S = T \int d^3x \partial_\mu \chi \partial^\mu \chi$.) So

$$\left\langle e^{\mathbf{i}s\chi(x)} e^{\mathbf{i}s'\chi(0)} \right\rangle = e^{-\mathbf{i}\frac{ss'}{4xT}} \ .$$

And

$$\langle \cos\chi(x)\cos\chi(0)\rangle = \cos\frac{1}{4Tx}$$

which does not decay at long distance, and in fact approaches a constant.

- The fact that the would-be-transverse-photon χ is massive means confinement of the gauge theory. To see that external charge is confined, think as usual about the big rectangular Wilson loop $\langle W(\square)\rangle = \left\langle e^{\mathbf{i}\oint_\square A}\right\rangle \overset{\text{euclidean}}{\sim} e^{-E(R)T}$ as an order parameter for confinement. In term of χ,

$$\oint_\square A = \int_\blacksquare F_{12} = \int_\blacksquare g\dot{\chi}$$

  (I've absorbed a factor of the gauge coupling into χ to make the dimensions work nicely, $\epsilon_{\mu\nu\rho}\partial_\nu A_\rho = g\partial_\mu\chi$) and the expectation is

$$\langle W(\square)\rangle = Z^{-1}\int [d\chi] e^{-S_\chi + g\mathbf{i}\int_\blacksquare \dot{\chi}} \sim e^{-cg^2 m_\chi \cdot \text{area}(\blacksquare)}.$$

  In the last step we did the gaussian integral from small χ fluctuations. This area-law behavior proportional to m_χ means that the mass for χ confines the gauge theory. This is the same (Polyakov) effect we saw in the previous section, where the monopole tunneling events produced the mass.

- Adding matter helps to produce a deconfined phase! In particular, the presence of enough massless charged fermions can render the monopole operator irrelevant. I recommend this paper by Tarun Grover for more on this.

- Think about the action of $e^{\mathbf{i}\chi(x,t)}$ from the point of view of $2+1$d spacetime: it inserts 2π magnetic flux at the spacetime point x,t. From that path integral viewpoint, this is an event localized in three dimensions which is a source of magnetic flux – a magnetic monopole. In Polyakov's paper, he uses a UV completion of the abelian gauge theory (not the lattice) in which the magnetic monopole is a smooth solution of field equations (the 't Hooft-Polyakov monopole), and these solutions are instanton events. The $\cos\chi$ potential we have found above arises from, that point of view, by the same kind of dilute instanton gas sum that we did in the $D = 1+1$ Abelian Higgs model.

7.3 Deconfined Quantum Criticality

[The original papers are this and this; this treatment follows Ami Katz' BU Physics 811 notes.] Consider a square lattice with quantum spins (spin half) at the sites, governed

by the Hamiltonian

$$H_{JQ} \equiv J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j + Q \sum_{[ijkl]} \left(\vec{S}_i \cdot \vec{S}_j - \frac{1}{4} \right) \left(\vec{S}_k \cdot \vec{S}_l - \frac{1}{4} \right).$$

Here $\langle ij \rangle$ denotes pairs of sites which share a link, and $[ijkl]$ denotes groups of four sites at the corners of a plaquette. This JQ-model is a somewhat artificial model designed to bring out the following competition which also exists in more realistic models:

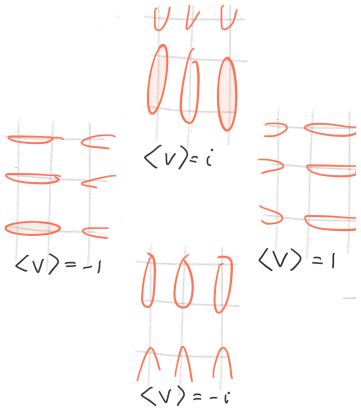
$\boxed{J \gg Q}$: the groundstate is a Neel antiferromagnet (AFM), with local order parameter $\vec{n} = \sum_i (-1)^{x_i + y_i} \vec{S}_i$, whose expectation value breaks the spin symmetry $\mathsf{SU}(2) \to \mathsf{U}(1)$. Hence, the low-energy physics is controlled by the (two) Nambu-Goldstone modes. This is well-described by the field theory we studied in §3.3.

$\boxed{Q \gg J}$: The Q-term is designed to favor configurations where the four spins around each square form a pair of singlets. A single Q-term has a two-fold degenerate groundstate, which look like $|=\rangle$ and $|||\rangle$. The sum of all of them has four groundstates, which look like ... These are called *valence-bond solid* (VBS) states. The VBS order parameter on the square lattice is

$$V = \sum_i \left((-1)^{x_i} \vec{S}_i \cdot \vec{S}_{i+x} + \mathbf{i} (-1)^{y_i} \vec{S}_i \cdot \vec{S}_{i+y} \right) \in \mathbb{Z}_4.$$

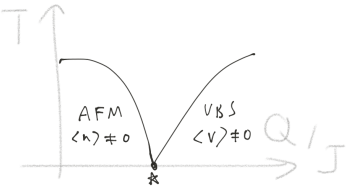
In the four solid states, it takes the values $1, \mathbf{i}, -1, -\mathbf{i}$. Notice that they are related by multiplication by $\mathbf{i} = e^{\mathbf{i}\pi/2}$. V is a singlet of the spin $\mathsf{SU}(2)$, but the VBS states do break spacetime symmetries: a lattice rotation acts by $R_{\pi/2} : V \to -\mathbf{i}V$ (the Neel order $\vec{n}$ is invariant), while a translation by a single lattice site acts by

$$T_{x,y} : \vec{n} \to -\vec{n}, \; T_x : V \to -V^\dagger, T_y : V \to V^\dagger. \tag{7.9}$$

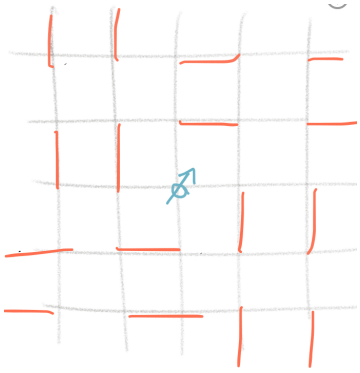


The VBS phase is gapped (it only breaks discrete symmetries, so no goldstones).

Claim: There seems to be a continuous transition between these two phases as a function of Q/J. (If it is first order, the latent heat is very small.) Here's why this is weird and fascinating: naively, the order parameters break totally different symmetries, and so need have nothing to do with each other. Landau then predicts that generically there should be a region where both are nonzero or where both are zero. Why should the transitions coincide? What are the degrees of freedom at $\star$?



To get a big hint, notice that the VBS order parameter is like a discrete rotor: if we had a triangular lattice it would be in $\mathbb{Z}_6$ and would come closer to approximating a circle-valued field. In any case, we can consider vortex configurations, where the phase of V rotates (discretely, between the four quadrants) as we go around a point in space. Such a vortex looks like the picture at right.



Notice that inside the core of the vortex, there is necessarily a spin which is not paired with another spin: The vortex carries spin: it transforms as a doublet under the spin $\mathsf{SU}(2)$. Why do we care about such vortices? I've been trying to persuade you for the past two sections that the way to think about destruction of (especially $\mathsf{U}(1)$) ordered phases is by proliferating vortex defects. Now think about proliferating this kind of VBS vortex. Since it carries spin, it necessarily must break the $\mathsf{SU}(2)$ symmetry, as the Neel phase does. This is why the transitions happen at the same point.

To make this more quantitative, let's think about it from the AFM side: how do we make V from the degrees of freedom of the low energy theory? It's not made from n since it's a spin singlet which isn't 1 (spin singlets made from n are even under a lattice translation). What about the $\mathbb{CP}^1$ version, aka the Abelian Higgs model, aka scalar QED (now in $D = 2+1$)?

$$L = -\frac{1}{4g^2}F^2 + |Dz|^2 - m^2|z|^2 - \frac{\lambda}{4}|z|^4$$

where $z = \begin{pmatrix} z_\uparrow \\ z_\downarrow \end{pmatrix}$, and $D_\mu z = (\partial_\mu - \mathbf{i}A_\mu)z$ as usual. Let's think about the phases of this model.

$\boxed{m^2 < 0}$: Here z condenses and breaks $\mathsf{SU}(2) \to \mathsf{U}(1)$, and A_μ is higgsed. A gauge invariant order parameter is $\vec{n} = z^\dagger \vec{\sigma} z$, and there are two goldstones associated with its rotations. This is the AFM. The cautionary tale I told you about this phase in $D = 1+1$ doesn't happen because now the vortices are *particles* rather than instanton events. More on these particles below.

$\boxed{m^2 > 0}$: Naively, in this phase, z are uncondensed and massive, leaving at low energies only $L_{\text{low-E}} \stackrel{?}{=} -\frac{1}{4g^2}F^2$, Maxwell theory in $D = 2+1$. This looks innocent but it will occupy us for quite a few pages starting now. This model has a conserved current (conserved by the Bianchi identity)

$$J_F^\mu \equiv \epsilon^{\mu\nu\rho}F_{\nu\rho} = \frac{1}{8\pi}n^a\partial_\mu n^b \partial_\nu n^c \epsilon_{abc}.$$

In terms of the gauge variables, the thing that's conserved is the magnetic flux; in terms of the spins, it's the skyrmion number (as you showed on the homework) We can follow these around more effectively by introducing the dual scalar field by a by-now-familiar duality relation:

$$J_F^\mu \equiv \epsilon^{\mu\nu\rho} F_{\nu\rho} \equiv g\partial^\mu \chi. \tag{7.10}$$

You can think of the last equation here as a *solution* of the conservation law $\partial_\mu J_F^\mu = 0$. The symmetry acts on χ by shifts: $\chi \to \chi + \text{constant}$. In terms of χ, the Maxwell action is

$$L_{\text{low-E}} \stackrel{?}{=} -\frac{1}{4g^2} F^2 = \frac{1}{2}\partial_\mu \chi \partial^\mu \chi.$$

But this is a massless scalar, a gapless theory. And what is the $\chi \to \chi + c$ symmetry in terms of the spin system? I claim that it's the rotation of the phase of the VBS order parameter, which is explicitly broken by the squareness of the square lattice. An improvement would then be

$$L_{\text{low-E}} = \frac{1}{2}\partial_\mu \chi \partial^\mu \chi - V_k \cos\left(k\chi/g\right)$$

where $V_k \sim \frac{1}{a^2}$ (a is the lattice spacing) comes from the breaking the rotation invariance by the lattice down to $\mathbb{Z}_k$ ($k = 4$ for the square lattice).

To see that shifts of χ are VBS rotations, let's reproduce the lattice symmetries in the Abelian Higgs model. Here's the action of lattice translations $T \equiv T_x$ or T_y (take a deep breath.): $T : n^a \to -n^a$ but $n^a = z^\dagger \sigma^a z$, so on z we must have $T : z \to \mathbf{i}\sigma^2 z^\star$. The gauge current is $j_\mu = \mathbf{i} z^\dagger \partial_\mu z + h.c. \to -j_\mu$ which means we must have $A_\mu \to -A_\mu$ and $F_{\mu\nu} \to -F_{\mu\nu}$. Therefore by (7.10) we must have $T : \partial\chi \to -\partial\chi$ which means that

$$T_{x,y} : \chi \to -\chi + g\alpha_{x,y}$$

where $\alpha_{x,y}$ are some so-far-undetermined numbers, and g is there on dimensional grounds. Therefore, by choosing $T_{x,y}\chi \to -\chi \pm g\pi/2, R_{\pi/2} : \chi \to \chi - g\pi/2$ we can reproduce the transformation (7.9) by identifying

$$V = ce^{\mathbf{i}\chi/g}$$

(up to an undetermined overall complex number). Notice for future reference the canonical commutation relation between the flux current density ($J_F^0 = g\dot{\chi} = \frac{g}{\mathbf{i}}\frac{\delta}{\delta\chi}$) and V:

$$[J_F^0(x), V(0)] = V(0)\delta^2(x). \tag{7.11}$$

It creates flux.

So χ is like the phase of the bosonic operator V which is condensed in the VBS phase; lattice effects break the $\mathsf{U}(1)$ symmetry down to some discrete subgroup ($\mathbb{Z}_4$ for

the square lattice, $\mathbb{Z}_6$ for triangular, $\mathbb{Z}_3$ for honeycomb), with a potential of the form $\mathcal{V}(V^k) = m_\chi^3 \cos(4\chi/g) + \cdots$, where $k = 4, 6, 3...$ depends on the lattice, which has k minima, corresponding to the k possible VBS states. By (7.11), such a potential has charge k under J_F.

Consider this phase from the point of view of the gauge theory now. Notice that χ is the same (up to a factor) dual variable we introduced in our discussion of compact QED, and the Wilson loop will again produce an area law if χ is massive, as with the Polyakov effect.

In order for this story to make sense, we need that $M, g^2 \ll \frac{1}{a^2}$, so that χ is actually a low-energy degree of freedom. The idea is that the critical point from tuning J/Q to the critical value is reached by taking $m_\chi \to 0$. What is the nature of this critical theory? It has emergent deconfined gauge fields, even though the phases on either side of the critical point do not (they are confined $m > 0$ and Higgsed $m < 0$ respectively). Hence the name deconfined quantum criticality.

The conjecture (which would explain the phase diagram above) is that this gauge theory is a critical theory (in fact a conformal field theory) with only one relevant operator (the one which tunes us through the phase transition, the mass for χ) which is a singlet under all the symmetries. Recall that $e^{\mathbf{i}k\chi}$ has charge k under the J_F symmetry, and the square lattice preserves a $\mathbb{Z}_4 \subset \mathsf{U}(1)$ subgroup, so only allows the 4-vortex-insertion operator $e^{\mathbf{i}4\chi}$. What is the dimension of this operator at the critical point? The conjecture is that it has dimension larger than 3.

Insanely brief sketch of a check at large N. Actually, this can be checked very explicitly in a large-N version of the model, with N component z fields, so that the spin is $\phi^A = z^\dagger T^A z, A = 1..N^2 - 1$. This has $\mathsf{SU}(N)$ symmetry. When $m^2 < 0$, it is broken to $\mathsf{SU}(N-1)$, with $2(N-1)$ goldstone bosons. (Actually there is a generalization of the lattice model which realizes this – just make the spins into $N \times N$ matrices.)

Introducing an H-S field σ to decouple the $|z|^4$ interaction, we can make the z integrals gaussian, and find (this calculation is just like our earlier analysis in §4.4)

$$S[A,\sigma] = \int đp \left(\frac{1}{4} F_{\mu\nu}(p) \left(\frac{1}{g_{UV}^2} + \frac{c_1 N}{\mathbf{i}p} \log \frac{2m + \mathbf{i}p}{2m - \mathbf{i}p} \right) F^{\mu\nu}(-p) + \sigma(p) \left(-\frac{1}{\lambda} + \frac{c_2 N}{\mathbf{i}p} \log \frac{2m + \mathbf{i}p}{2m - \mathbf{i}p} \right) \sigma(-p) \right)$$

In the IR limit, $m \ll p \ll g_{UV}^2 N, \lambda N$, this is a scale-invariant theory with $\langle FF \rangle \sim p, \langle \sigma\sigma \rangle \sim p$ so that both F and σ have dimension near 2. (Actually the dimension of F is fixed at 2 by flux conservation.) z doesn't get any anomalous dimension at leading order in N.

This is all consistent with the claim so far. What is the dimension of $V_4 = e^{\mathbf{i}4\chi}$? To answer this question, we use a powerful tool of conformal field theory called *radial*

quantization. Consider the theory on a cylinder, $S^2 \times \mathbb{R}$, where the last factor we can interpret as time. In a conformal field theory there is a one-to-one map between local operators and states of the theory on $S^d \times \mathbb{R}$. The state corresponding to an operator $\mathcal{O}$ is just $\mathcal{O}(0)\,|0\rangle$. The energy of the state on the sphere is the scaling dimension of the operator. (For an explanation of this, I refer to §4 of these notes.)

The state created by acting with $V_k(0)$ on the vacuum maps by this transformation to an initial state with flux k spread over the sphere (think of it as the 2-sphere surrounding the origin in spacetime): this state has charge k under $Q_F = \int_{S^2} J_F^0 = \int_{S^2} F_{12}$. The dimension of V_k is the energy of the lowest-energy state with $Q_F = k$. We can compute this by euclidean-time path integral:

$$Z_k = \mathrm{tr}_{Q_F=k} e^{-TH_{\mathrm{cyl}}} \overset{T\to 0}{\to} e^{-T\Delta_k}.$$

This is

$$Z_k = \int [dA]\delta\left(\int F - k\right) \int [dz dz^\dagger] e^{-S[z,A]} \equiv e^{-F_k}$$

which at large-N we can do by saddle point. The dominant configuration of the gauge field is the charge-k magnetic monopole $\underline{A}_\varphi = \frac{k}{2}(1-\cos\varphi)$, and we must compute

$$\int [d^2 z] e^{z^\dagger\left(-D_{\underline{A}}^\dagger D_{\underline{A}} + m^2\right)z} = \det\left(-D_{\underline{A}}^\dagger D_{\underline{A}} + m^2\right)^{-N/2} = e^{-\frac{N}{2}\mathrm{tr}\log\left(-D_{\underline{A}}^\dagger D_{\underline{A}} + m^2\right)}$$

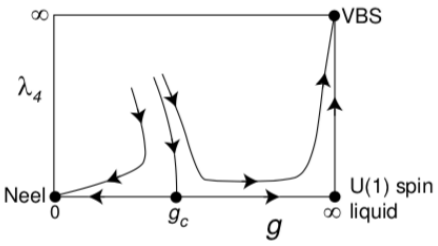
The free energy is then a sum over eigenstates of this operator

$$\left(-\partial_\tau^2 - \vec{D}_{\underline{A}}^2\right) f_\ell e^{\mathrm{i}\omega\tau} = \left(\omega^2 + \lambda_\ell(k)\right) f_\ell e^{\mathrm{i}\omega\tau}$$

$$F_k = NT \int \mathrm{đ}\omega \sum_\ell (2\ell+1)\log(\omega^2 + \lambda_\ell(k) + m^2).$$

The difference $F_k - F_0$ is UV finite and gives $\Delta_k = N c_k,\ c_1 \sim .12, c_4 \sim .82$. Unitarity requires $\Delta_1 \geq \frac{1}{2}$ (= the free scalar dimension), so don't trust this for $N < 4$.

So the idea can be summarized by the flow diagram at right, where the horizontal axis is $m_z^2 \sim g - g_c$. For $m_z^2 < 0$, we are in the Higgs phase of the gauge theory, where the spin symmetry is spontaneously broken – the Neel phase. For $m_z^2 > 0$, if we managed to tune $V_k = 0$, we would end up at a state with a massless photon. But this state is unstable to the proliferation of monopoles (created by the operator whose coefficient in the action is V_k). And on the lattice, the coefficient V_k of the monopole fugacity is inevitably nonzero, so we end up instead in the gapped VBS phase. But right at the critical point, there's a deconfined $\mathsf{U}(1)$ gauge theory!



[Fig is from this nice summary].

Pure field theory description. We've been discussing a theory with $\mathsf{U}(1)_{\rm VBS} \times \mathsf{SU}(2)_{\rm spin}$ symmetry. Lattice details aside, how can we encode the way these two symmetries are mixed up which forces the order parameter of one to be the disorder operator for the other? To answer this, briefly consider enlarging the symmetry to $\mathsf{SO}(5) \supset \mathsf{U}(1)_{\rm VBS} \times \mathsf{SU}(2)_{\rm spin}$, and organize $(\mathrm{Re}\, V, \mathrm{Im}\, V, n^1, n^2, n^3) \equiv n^a$ into a 5-component mega-voltron-spin vector. We saw that in $D = 0+1$, we could make a WZW term with a 3-component spin

$$\mathcal{W}_0[(n^1, n^2, n^3)] = \int_{B_2} \epsilon^{abc} n^a dn^b \wedge dn^c.$$

Its point in life was to impose the spin commutation relations at spin s when the coefficient is $2s$. In $D = 1+1$, we can make a WZW term with a 4-component spin, which can have $\mathsf{SO}(4)$ symmetry

$$\mathcal{W}_1[(n^1, n^2, n^3, n^4)] = \int_{B_3} \epsilon^{abcd} n^a dn^b \wedge dn^c \wedge dn^d.$$

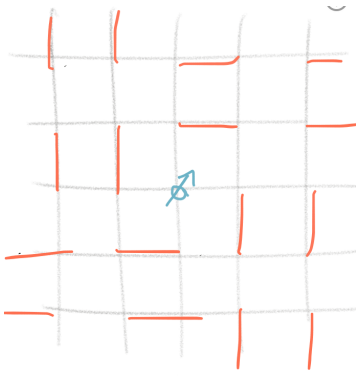
[45] Once we've got this far, how can you resist considering

$$\mathcal{W}_2[(n^1, n^2, n^3, n^4, n^5)] = \int_{B_4} \epsilon^{abcde} n^a dn^b \wedge dn^c \wedge dn^d \wedge dn^e.$$

What does this do? Break the $\mathsf{SO}(5) \to \mathsf{U}(1) \times \mathsf{SU}(2)$ and consider a vortex configuration of V at $x^2 = x^3 = 0$. Suppose our action contains the term $k\mathcal{W}_2[n]$ with $k = 1$. Evaluate this in the presence of the vortex:

$$k\mathcal{W}_2[(n^1, n^2, n^3, n^4, n^5)|_{\text{vortex of } n^1 + \mathbf{i} n^2 \text{ at } x^2 = x^3 = 0}] = \frac{k}{16\pi^2} \int_{B_2 | x^2 = x^3 = 0} \epsilon^{abc} n^a dn^b \wedge dn^c = k\mathcal{W}_0[(n^1, n^2, n^3)].$$

This says the remaining three components satisfy the spin-half commutation relations: there is a spin in the core of the vortex, just as in the lattice picture at right.



[End of Lecture 20]

[45] In fact the $D = 1+1$ version of this is extremely interesting. A few brief comments: (1) involves a *real* VBS order parameter n^4.) (2) The $D = 1+1$ term has the same number of derivatives (in the EOM) as the kinetic term $\partial n^a \partial n^a$. This means they can compete at a fixed point. The resulting CFTs are called WZW models. (3) The above is in fact a description of the spin-half chain, which previously we've described by an $\mathsf{O}(3)$ sigma model at $\theta = \pi$.

7.4 Bosonization

7.4.1 Bosonization dictionary

[Fradkin 2d ed, page 133; Sachdev 2d ed, page 420] Let's set $T = \frac{1}{\pi}$ for a bit. The left and right $\mathsf{U}(1)$ currents in terms of bosons and fermions are

$$\Psi_L^\dagger \Psi_L = \frac{1}{2\pi}\partial_z \phi, \quad \Psi_R^\dagger \Psi_R = \frac{1}{2\pi}\bar{\partial}_z \phi.$$

The fermions themselves are:

$$\Psi_L(z)^\dagger =: e^{\mathbf{i}\phi_L(z)} :, \quad \Psi_R(\bar{z})^\dagger =: e^{\mathbf{i}\phi_R(\bar{z})} : \ .$$

How might a human have figured this out? (One human who did was Mandelstam.)

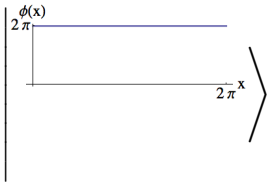
The relation for the charges says that the total fermion charge is the winding number:

$$\Psi_L^\dagger \Psi_L + \Psi_R^\dagger \Psi_R = \frac{1}{2\pi}\partial_x \phi. \tag{7.12}$$

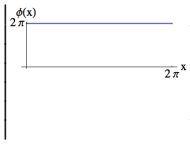
Let's consider the refined statement:

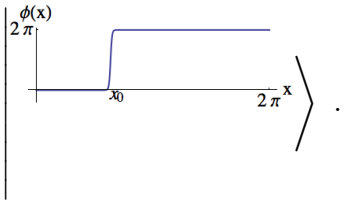
$$\Psi_L^\dagger \Psi_L = \frac{1}{2\pi}\partial_x \phi_L = j_L. \tag{7.13}$$

This means that if we create a left-moving fermion at $x = x_0$, the value of $\phi_L(x)$ should *jump* for all $x < x_0$, by a fixed amount so that if we integrate the BHS of (7.12) we get 1. (That amount is 2π.) For example, according to (7.13) a state with no fermions looks like:

$|\text{no fermions}\rangle =$ 

Acting on this state with one fermion at $x = x_0$ should do the following:

$\Psi_L^\dagger(x_0)$ 

$=$ 

.

Such a jump is produced by the conjugate field momentum (recall the translation operator in QM is $e^{-\mathbf{i}\mathbf{p}a}\psi(x) = \psi(x+a)$):

$$\Psi_L(x_0) \sim e^{\mathbf{i}2\pi \int_{-\infty}^{x_0} dx \boldsymbol{\pi}_L(x)} \ .$$

Now who is the canonical momentum $\boldsymbol{\pi}_L(x)$ in terms of ϕ_L, ϕ_R? It's the thing with a canonical commutator with ϕ_L. One way to figure this out is to go back to the mode expansions and show that

$$[\partial_x \boldsymbol{\phi}_L(x), \boldsymbol{\phi}_L(y)] = \frac{2\mathbf{i}}{T}\delta(x-y) \tag{7.14}$$

which shows that $\boldsymbol{\pi}_L(x) = \frac{T}{2}\partial_x \boldsymbol{\phi}_L(x)$ (and similarly $\boldsymbol{\pi}_R = -\frac{T}{2}\partial_x \boldsymbol{\phi}_R(x)$). (Notice that this equation is roughly the imaginary part of the x derivative of (5.10).) So (when $T = \frac{1}{\pi}$, which is when this operator is single-valued!) the fermion operator is

$$\Psi_L(x_0) \sim e^{\mathbf{i}\int_{-\infty}^{x_0} dx\ \partial_x \boldsymbol{\phi}_L} = e^{\mathbf{i}\boldsymbol{\phi}_L}\ .$$

We already saw that this kind of object creates winding in (5.11). So again the fermion is a domain-wall creation operator; notice that because the field ϕ is periodic, it can be possible to have a single such domain wall on a circle. We'll say a bit more about the periodicity of ϕ_L below.

In order for this to be a fermion operator, we require $\{\Psi_L(x), \Psi_L(x')\} = 0, x \neq x'$. Is it true? Yes in fact, as we can see by BCH, in the form $e^{\mathbf{A}}e^{\mathbf{B}} = e^{\mathbf{B}}e^{\mathbf{A}}e^{-[\mathbf{A},\mathbf{B}]}$ if $[\mathbf{A}, \mathbf{B}]$ is a c-number. A useful intermediate fact comes from the x-integral of (7.14) (from $-\infty$ to x) which says

$$[\boldsymbol{\phi}_L(x), \boldsymbol{\phi}_L(y)] = \mathbf{i}\pi \mathsf{sign}(x-y)$$

(the additive constant of integration on the RHS is fixed by demanding antisymmetry in $x \to y$ which is manifest on the LHS). Therefore:

$$:e^{\mathbf{i}\phi_L(x)}::e^{\mathbf{i}\phi_L(0)}:= e^{-\mathbf{i}\pi\mathsf{sign}(x)}:e^{\mathbf{i}\phi_L(0)}::e^{\mathbf{i}\phi_L(x)}:= -:e^{\mathbf{i}\phi_L(0)}::e^{\mathbf{i}\phi_L(x)}:.$$

More generally, the same calculation shows that

$$:e^{\mathbf{i}n\phi_L(x)}::e^{\mathbf{i}m\phi_L(0)}:= (-1)^{nm}:e^{\mathbf{i}m\phi_L(0)}::e^{\mathbf{i}n\phi_L(x)}:$$

So $e^{\mathbf{i}n\phi_L}$ is bosonic for even n and fermionic for odd n.

You can also see the fermi statistics from the OPEs:

$$:e^{\mathbf{i}\phi_L(z_1)}::e^{-\mathbf{i}\phi_L(z_2)}:\sim \frac{1}{z_1-z_2} = -\frac{1}{z_2-z_1}.$$

$$:e^{\mathbf{i}\phi_L(z_1)}::e^{\mathbf{i}\phi_L(z_2)}:\sim (z_1-z_2):e^{\mathbf{i}\phi_L(z_1)+\mathbf{i}\phi_L(z_2)}:$$

Notice that this object

$$e^{\mathbf{i}\frac{1}{\pi}\int_{-\infty}^{x_0} dx \partial_x \boldsymbol{\phi}_L} = e^{\mathbf{i}\int_{-\infty}^{x_0} dx j_L(x)}$$

has a lot in common with the Jordan-Wigner string, which counts the number of particles to the left.

Periodicity of chiral bosons. Since $\phi \simeq \phi + 2\pi m, m \in \mathbb{Z}$ and $\phi = \phi_L + \phi_R$ it is tempting to suggest that ϕ_L has period π. Such a period would mean that $e^{-\mathbf{i}\phi_L} \sim \Psi_L$ is not single-valued. But it is not quite correct. To see the correct statement, notice that the T-dual field is $\tilde{\phi} = \phi_L - \phi_R$. With our normalization, this field also has periodicity 2π. The winding modes of the dual fields are momentum modes of the original one, so

$$\begin{matrix} \phi_L + \phi_R \simeq \phi_L + \phi_R + 2\pi m \\ \phi_L - \phi_R \simeq \phi_L + \phi_R + 2\pi j \end{matrix} \implies \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix} \simeq \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix} + \pi \begin{pmatrix} m+j \\ m-j \end{pmatrix} ,$$

where m and j are independent integers. But then $m \pm j$ are *not* independent integers: when $m+j$ is odd or even, so is $m-j$. This is the same fact as my earlier statement about the labels α, β on $\mathcal{V}_{\alpha,\beta}$ at Eq. (5.12).

Note that the T-dual field is often called $\vartheta \equiv \phi_L - \phi_R$ in discussions of Luttinger liquids (*e.g.* in Fradkin's and Sachdev's books).

Possibly Depressing comment. So now you are starting to see that this duality business is actually often a sad story: we thought we could solve *two* systems (free bosons and free fermions) but since they are really the same system in disguise, it turns out we can only solve one!

Multiple fermions. Notice that if we had N complex fermions ψ^a (for example a could be a spin index), we could bosonize them by N scalar fields at the free fermion radius:

$$\psi^a(z) \overset{?}{\sim} e^{\mathbf{i}\phi^a(z)}, \quad a = 1..N \ . \tag{7.15}$$

This formula *almost* works, but it has the problem that the operators on the RHS with different values of a will still commute (since the associated scalars don't have any singularities in their OPE, *i.e.* they commute). This must be fixed by so-called *Klein factors* (or 2-cocycles) which introduce the extra necessary signs. The necessary object is constructed by ordering the species of fermions $a = 1..N$ and defining:

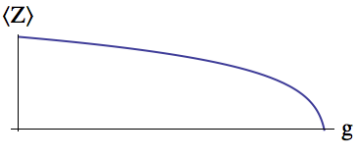
$$\mathfrak{c}_a \equiv (-1)^{\sum_{b<a} \mathbf{N}_b}$$

where $\mathbf{N}_b$ is the total fermion number of species b. The correct version of (7.15)

$$\psi^a(z) = \mathfrak{c}_a e^{\mathbf{i}\phi^a(z)}, \quad a = 1..N \quad \text{(no sum on } a\text{)}. \tag{7.16}$$

Spin fields in (two copies of) the Ising model.

Now I will partially fulfil my promise to use bosonization to determine the scaling dimension of the spin operator at the Ising critical point. Recall that I claimed without any justification that in the TFIM, the behavior of the magnetization $\langle Z \rangle$ for $g \lesssim g_c$ was



$$\langle Z \rangle \sim (g_c - g)^{1/8}$$

(as in the figure). This 1/8 is the scaling dimension of the order parameter operator in the critical Ising CFT, the operator onto which the spin Z_j matches.

To begin, let's consider just the holomorphic part of a complex fermion $\psi(z) = \frac{1}{\sqrt{2}}(\chi_1(z) + \mathbf{i}\chi_2(z))$. This is not exactly the Ising critical theory, which is instead a non-chiral majorana mode, $\chi_L(z), \chi_R(\bar{z})$, but this is what we know how to bosonize. The bosonization map is

$$\psi(z) \sim e^{\mathbf{i}\phi(z)}.$$

Recall that the fermions in our spin chain arose as domain wall operators. They satisfy

$$\mathbf{Z}_l \mathbf{c}_j = -\mathbf{c}_j \mathbf{Z}_l, \quad l > j.$$

We can interpret this formula to say that the spin $\mathbf{Z}_j$ creates a branch cut for the fermion field. The CFT version of this statement is:

$$\psi(z)\sigma(0) \sim z^{-\frac{1}{2}}\mu(0) \quad . \tag{7.17}$$

Here σ, μ are *spin fields* (or more generally, *twist fields*) whose presence at $z = 0$ creates a branch cut. The actual location of the branch cut is not physical, but the location of the branch *points* (here $0, \infty$) is meaningful. Can we find operators which accomplish this?

Yes: they are

$$\sigma(w) = e^{-\frac{\mathbf{i}}{2}\phi(w)}, \quad \mu(w) = e^{+\frac{\mathbf{i}}{2}\phi(w)}. \tag{7.18}$$

To check this claim, their OPEs with the fermion operator are:

$$\begin{aligned} \psi(z)\sigma(0) &=: e^{\mathbf{i}\phi(z)} :: e^{-\frac{\mathbf{i}}{2}\phi(0)} :\sim e^{+\frac{\mathbf{i}}{2}\phi(0)} z^{-\frac{1}{2}} + \text{regular} \\ \psi(z)\mu(0) &=: e^{\mathbf{i}\phi(z)} :: e^{+\frac{\mathbf{i}}{2}\phi(0)} :\sim e^{+3\frac{\mathbf{i}}{2}\phi(0)} z^{+\frac{1}{2}} + \text{regular} \end{aligned} \tag{7.19}$$

As a check on my ability to do wick contractions, consider the OPE between (and hence the two-point function of) two of the spin fields, which is a special case of (5.13):

$$e^{\alpha \mathbf{i}\phi(z)} e^{-\alpha \mathbf{i}\phi(0)} \sim \frac{1}{z^{\alpha^2}} \ .$$

For $\alpha = \pm\frac{1}{2}$, this gives

$$\Delta_\sigma = \Delta_\mu = \frac{1}{2}\alpha^2|_{\alpha=\frac{1}{2}} = \frac{1}{8}.$$

The dimension of $e^{\mathbf{i}\frac{3}{2}\phi}$ is $\frac{1}{2}\left(\frac{3}{2}\right)^2 = \frac{9}{8}$ so the dimensional analysis in (7.19) works.

The answer $\Delta_\sigma = \frac{1}{8}$ is the dimension of the spin field in the theory of a *complex* chiral fermion. This is two copies of the left-moving part of the Ising CFT. The spin field in the Ising model which makes a branch cut for *both* χ_L and χ_R has the same dimension (the one that makes a branch cut in just χ_L has dimension $\frac{1}{16}$). The basic fact I am using here is that the spin-field dimension is additive in the number of fermion fields which feel the branch cut, that is, in the number of fermion fields whose boundary conditions are twisted.

To see this, let's make spin fields for N complex fermions ψ^a, each bosonized as we just did by N scalar fields

$$\psi^a(z) \sim e^{\mathbf{i}\phi^a(z)}, \quad a = 1..N$$

The operators

$$\sigma_s(z) \equiv e^{\mathbf{i}s^a\phi^a(z)}, \quad \{s^a = \pm\frac{1}{2}\}$$

create branch cuts for all N fermions at once:

$$\psi^a(z)\sigma_s(0) \sim z^{s^a}\sigma_{s'}(0) + \text{regular}$$

where $(s')^a = s^a + 1, (s')^{b\neq a} = s^b$. The dimension of these operators is the sum of the dimensions:

$$\Delta_s = \frac{1}{2}\left(s_1^2 + s_2^2 + \cdots + s_N^2\right) = \frac{N}{8}.$$

Now take $N = \frac{1}{2}$ to get the answer for a single majorana fermion. If you are not satisfied by this argument, there is a more direct way to compute the dimension of the Ising model spin field, using the defining equation (7.17) which requires a bit more CFT technology (it's explained on p. 74 of Ginsparg's CFT notes). We'll come to that soon.

Derivation of spin field.

Notice that the object (7.18) would not be considered single-valued on the original boson hilbert space! However, the operator which creates a branch cut for *both* ψ_L and ψ_R at the same time is $e^{\frac{\mathbf{i}}{2}(\phi_L(z)\pm\phi_R(\bar{z}))}$ and this *is* an allowed operator. And in fact, here is a *derivation* of its relation to the spin operator. Recall the JW formula

$$\boldsymbol{\sigma}_j^+ = (-1)^{\sum_{i<j} n_i}\mathbf{c}_j^\dagger.$$

(Previously we put the JW string to the right; here I defer to the bosonization convention we used above.) Here (in the basis we used earlier, where $\mathbf{Z}$ is the Ising order parameter), $\boldsymbol{\sigma}^+ \equiv \mathbf{Z} + \mathbf{i}\mathbf{Y}$, so $\boldsymbol{\sigma}^+ + \boldsymbol{\sigma}^- = 2\mathbf{Z}$. In the continuum, in terms of the fermions, this is

$$\boldsymbol{\sigma}^+(x) = e^{\mathbf{i}\pi \int_{-\infty}^{x} dx' j_0(x')} \Psi(x)^\dagger \ .$$

Now, we expand our fermion operator near the fermi surface:

$$\boldsymbol{\Psi}(x,\tau) = e^{\mathbf{i}k_F x}\Psi_R(x,\tau) + e^{-\mathbf{i}k_F x}\Psi_L(x,\tau) \tag{7.20}$$

Let's do just the R piece first (then we'll put it all together):

$$\begin{aligned} & e^{\mathbf{i}\pi \int_{-\infty}^{x} dx' j_0(x')} \Psi_R(x)^\dagger \\ \overset{(7.12)}{=} \ & e^{-\mathbf{i}\pi \int_{-\infty}^{x} dy \ \frac{\partial_y \phi_L + \partial_y \phi_R}{2\pi}} e^{-\mathbf{i}\phi_R(x)} \\ = \ & e^{\mathbf{i}\frac{\phi_L(x)+\phi_R(x)}{2}} e^{-\mathbf{i}\phi_R(x)} \\ \sim \ & e^{\mathbf{i}\frac{\phi_L(x)-\phi_R(x)}{2}} \end{aligned} \tag{7.21}$$

OK, that was practice; the full expression is:

$$\begin{aligned} \boldsymbol{\sigma}^+(x) \quad = \quad & e^{\mathbf{i}\pi \int_{-\infty}^{x} dx' j_0(x')} \Psi(x)^\dagger \\ \overset{(7.12),(7.20)}{=} \quad & e^{-\mathbf{i}\pi \int_{-\infty}^{x} dy \ \frac{\partial_y \phi_L + \partial_y \phi_R}{2\pi}} \left(e^{-\mathbf{i}k_F x} e^{-\mathbf{i}\phi_R(x)} + e^{+\mathbf{i}k_F x} e^{-\mathbf{i}\phi_L(x)} \right) \\ \sim \quad & e^{\mathbf{i}\frac{\phi_L(x)-\phi_R(x)}{2}} e^{-\mathbf{i}k_F x} + e^{-\mathbf{i}\frac{\phi_L(x)-\phi_R(x)}{2}} e^{+\mathbf{i}k_F x} \end{aligned} \tag{7.22}$$

The role of the anomaly in bosonization. Recall that I promised that the chiral anomaly would play a role in bosonization. Here's one way in which it does. Consider the canonical commutator of the bose mode operators ($\partial_z \phi \propto \sum_n \boldsymbol{\rho}_n z^{-n-1}$):

$$[\boldsymbol{\rho}_n, \boldsymbol{\rho}_{-n'}] = \delta_{n,n'}$$

in terms of the fermions, $\boldsymbol{\rho}_n = \sum_l \mathbf{L}_l^\dagger \mathbf{L}_{l+n}$, we have:

$$\begin{aligned} [\boldsymbol{\rho}_n, \boldsymbol{\rho}_{-n'}] &= \sum_{l,'l} [\mathbf{L}_l^\dagger \mathbf{L}_{l+n}, \mathbf{L}_{l'}^\dagger \mathbf{L}_{l'-n'}] \\ &= \sum_{l'} \left(\mathbf{L}_{l'-n}^\dagger \mathbf{L}_{l'-n'} - \mathbf{L}_{l'}^\dagger \mathbf{L}_{l'+n-n'} \right) \overset{?}{=} 0 \end{aligned} \tag{7.23}$$

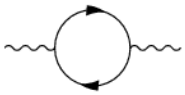
This expression has the classic form of an anomaly: it looks like this vanishes since we can redefine the dummy index in the second term by $n' \to n' + n$ and make the two terms look the same with opposite sign. *However*, in the presence of a UV cutoff, *i.e.* in the regulated theory, that shift will shift the cutoff and will not be innocuous since the

summand is not zero for arbitrarily high energies.[46] If, for example, we regulate the short distance behavior using normal-ordered operators, we have

$$\begin{aligned}[\boldsymbol{\rho}_n, \boldsymbol{\rho}_{-n'}] &= \sum_{l,'l} [: \mathbf{L}_l^\dagger \mathbf{L}_{l+n} :, : \mathbf{L}_{l'}^\dagger \mathbf{L}_{l'-n'} :] \\ &= \delta_{nn'} \sum_l \left(\left\langle \mathbf{L}_{l-n}^\dagger \mathbf{L}_{l-n} \right\rangle - \left\langle \mathbf{L}_n^\dagger \mathbf{L}_n \right\rangle \right) = \delta_{nn'} n. \end{aligned} \tag{7.24}$$

This calculation is related to the anomaly because it says that the OPE of the chiral current j_R with itself has a singularity with a definite coefficient. Specifically (as you can check on HW 6),

$$j(z)j(0) \sim \frac{k}{z^2}$$

with $k = 1$ is called the *level* of the (here $\mathsf{U}(1)$) current algebra. This singularity implies the anomalous nonconservation when coupling to a background gauge field. In terms of the fermions, recall that both of these effects come from the 'diangle' diagram: 

Twist fields and boundary conditions. Above I used the phrase "fermion fields whose boundary conditions are twisted" by the spin fields. With this phrase I betrayed that I am secretly using the following picture, which is called *radial quantization* (more on this at §6.2). Imagine that the radial coordinate of the plane is the time coordinate (it is euclidean so this interpretation is not problematic). Then equal-time surfaces are circles centered at the origin. (We'll see below that in a CFT this is equivalent to the ordinary notion of time evolution by a conformal transformation.) In this case, the boundary conditions on the spatial coordinate are determined by going around the origin; if there's a branch point at the origin, we must cross the branch cut in doing so, and the field comes back to itself up to the discontinuity across the branch cut, which here is a minus sign. So the spin fields can be regarded as "boundary-condition-changing operators": they take the fermions from the sector with PBC (and hence a fourier expansion with integer powers of z: $\psi_{PBC}(z) = \sum_{n\in\mathbb{Z}} z^n \psi_n$; this is called the *Ramond sector*) to APBC (half-integer mode expansion $\psi_{APBC}(z) = \sum_{n\in\mathbb{Z}} z^{n+\frac{1}{2}} \psi_{n+\frac{1}{2}}$; this is called the *Neveu-Schwarz sector*) and vice versa.

Understanding in terms of group theory. I must comment on the very essential group theory underlying the previous discussion of spin fields for multiple fermions. The theory of N complex free chiral fermions has a manifest $\mathsf{U}(N)$ symmetry rotating the

[46] To see how the high-energy stuff comes in consider the analogous infinite-volume expression

$$0 = \int_{-\infty}^{\infty} dp (f(p) - f(p)) \stackrel{?}{=} \int_{-\infty}^{\infty} dp (f(p+s) - f(p)) = \int dp (sf'(p) + \mathcal{O}(s^2)) = s(f(\infty) - f(-\infty)).$$

complex fermions into each other: $\psi^a \to \mathsf{U}^a_b \psi^b$, $\mathsf{U}\mathsf{U}^\dagger = 1$. The free fermion action actually enjoys a larger symmetry which ignores the grouping into complex fermions: $\psi^a = \frac{1}{\sqrt{2}}(\chi^{2a-1} + \mathbf{i}\chi^{2a})$; since $S \propto \int \delta_{AB}\chi^A \bar{\partial}\chi^B$, these $2N$ chiral majorana fermions transform in the fundamental of an $\mathsf{SO}(2N)$ symmetry, $\chi^A \to \mathsf{O}^A_B \chi^B, \mathsf{O}^\mathsf{T}\mathsf{O} = 1$.

This $\mathsf{SO}(2N)$ acts on the hilbert space. In terms of the fermions, the generators are just fermion bilinears $j^{AB} = \chi^A\chi^B$, the currents (recall that the adjoint (the rep that the generators transform in) of $\mathsf{SO}(2N)$ is made from antisymmetrizing two fundamentals). There are $2N(2N-1)/2$ of these.

In terms of the bosons, the generators come in two types (in the theory of Lie groups this is called Cartan-Weyl basis). N of them are just the ordinary boson currents $\partial\phi^a$. This is called the *Cartan subalgebra* (the number of them is $N = \text{rank}(\mathsf{SO}(2N))$ – they all commute with each other. The non-commuting (charged) generators are vertex operators $e^{\mathbf{i}\phi^a w_a}$ where in order for this to be a conserved current, it must have dimension 1, which requires $\sum_{a=1}^N w_a w_a = 2$. (in Lie group terminology, w_a is a *root vector*). Recall that the entries of w must be integers. This condition is solved by vectors of the form $\{w_a\} = \{(0...0, \pm 1, 0\cdots, 0, \pm 1, ...)\}$, of which there are $N(N-1)/2 \times 4$. Altogether we have recovered the same number of generators $N + 2N(N-1) = N(2N-1), \checkmark$.

The states in the Neveu-Schwarz (APBC) sector are of the form

$$\psi^A_{-\frac{1}{2}}\psi^B_{-\frac{3}{2}}\cdots|0\rangle\,, \qquad \psi^A_{r>0}|0\rangle = 0, \forall r$$

and therefore transform in various antisymmetric tensor representations of this $\mathsf{SO}(2N)$.

What is going on with these spin field objects $e^{\mathbf{i}s^a\phi^a} = e^{\mathbf{i}\left(\pm\frac{1}{2}\phi^1 \pm \frac{1}{2}\phi^2 \cdots \pm\frac{1}{2}\phi^N\right)}$? These operators map the Neveu-Schwarz sector to the Ramond sector. For a hint about what they are doing, notice that the fermion zeromodes in the Ramond (PBC) sector satisfy

$$\{\psi^a_0, \psi^b_0\} = 2\delta^{ab} \implies \{\chi^A_0, \chi^B_0\} = \delta^{AB}$$

which is the algebra of Dirac gamma matrices for $\mathsf{SO}(2N)$, *i.e.* Clifford algebra. This algebra must be represented on the Ramond groundstates, which are therefore not unique, *i.e.* they are degenerate. The point in life of Clifford algebra is to construct spinor representations. So the answer is that these vectors $\{s^a\} = \{\pm\frac{1}{2}, \pm\frac{1}{2}, \cdots, \pm\frac{1}{2}\}$ are *weight vectors* for the spinor representation(s) of $\mathsf{SO}(2N)$, in which the Ramond groundstates transform. The spin fields create the Ramond groundstates from the NS groundstate:

$$\sigma_s(0)|0\rangle_{NS} = |s\rangle\,.$$

A very interesting application of this group theory arises in the demonstration (by Fidkowski and Kitaev) that eight free majorana chains in the topological phase (with

the dangling majoranas at the ends) can be adiabatically connected to the trivial phase (only) via interacting Hamiltonians.

7.4.2 Application 1: Briefly, what is a Luttinger liquid?

[Fradkin p. 152] Let us consider a model which is actually made of fermions (which may have spin) at finite chemical potential. The actual density of the microscopic fermions Ψ is

$$\boldsymbol{\rho}_{\rm actual}(x) = \sum_\sigma \Psi^\dagger_\sigma(x)\Psi_\sigma(x) = \underbrace{\rho_0}_{=N_{\rm spin}\frac{k_F}{\pi}} + \underbrace{\boldsymbol{\rho}_L}_{=\psi^\dagger_L\psi_L} + \boldsymbol{\rho}_R + \sum_\sigma \left(e^{2k_F\mathbf{i}x}\psi^\dagger_R\psi_L + h.c.\right) ...$$

where I am reminding you that $\Psi(x) = e^{\mathbf{i}k_Fx}\psi_R(x) + e^{-\mathbf{i}k_Fx}\psi_L(x)$. The .... are even-higher momentum contributions which we will ignore. The quantity $2k_F$ is important in any system with a fermi surface as it is the *diameter* of the fermi surface, and (bosonic) particle-hole excitations near (just below) this momentum can be cheap despite its large value.

Consider the hamiltonian $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_{\rm pot}$ where the free Galilean-invariant hamiltonian is

$$\mathbf{H}_0 = \sum_\sigma \int đp\ pv_F \left(\psi^\dagger_{\sigma R}\psi_{\sigma R} - \psi^\dagger_{\sigma L}\psi_{\sigma L}\right) \ .$$

Subject this system to an (at-the-moment) external potential

$$\mathbf{H}_{\rm pot} = \int dx\ eV(x)\boldsymbol{\rho}(x) = -e\int đp \int đq \tilde{V}(q)\Psi^\dagger(p+q)\Psi(p) \ .$$

Let us consider the special wavenumber $V(x) = V_0\cos(2k_Fx)$, so

$$\mathbf{H}_{\rm pot} = -eV_0\int dp\psi^\dagger_R(p)\psi_L(p) + h.c. = -eV_0\int dx\psi^\dagger_R(x)\psi_L(x) + h.c.$$

In the relativistic language, this is a mass term:

$$\mathbf{H} = \int dx\psi^\dagger\left(\sigma^3v_F\mathbf{i}\partial_x + eV_0\sigma^1\right)\psi = \int dx\left(\bar{\psi}v_F\gamma^x\partial_x\psi + eV_0\bar{\psi}\psi\right)$$

(here I used $\gamma^0 = \sigma^1, \gamma^x = \sigma^1\sigma^3, \bar{\psi} \equiv \psi^\dagger\gamma^0$).

Where does this potential come from? It can be generated by (*e.g.* Coulomb) interactions between the fermions themselves

$$\mathbf{H}_{\rm int} = \int dxeV\boldsymbol{\rho} + \frac{e}{2}\int\int VKV \ .$$

If we treat this interaction in mean field theory, we get back (for some choice of K more on which more below) the previous background potential, $\mathbf{H}_{\rm int} \sim \mathbf{H}_{\rm pot}$. On the other hand, if we integrate out V, we find a density-density interaction

$$\mathbf{H}_{\rm int} = e \int \int \boldsymbol{\rho} K^{-1} \boldsymbol{\rho}$$

which for the right kernel, K, might be the Coulomb interaction. In a metal, the coulomb interaction is screened and therefore short-ranged. Expanding in this range, we find

$$\mathbf{H}_{\rm int} = \int dx \left(\underbrace{2g_2 \boldsymbol{\rho}_R(x)\boldsymbol{\rho}_L(x)}_{\text{back-scattering, 1d}} + \underbrace{g_4 \left(\boldsymbol{\rho}_R^2 + \boldsymbol{\rho}_L^2\right)}_{\text{forward scattering, ubiquitous}} + g_u \lim_{y\to x} \left(\psi_R^\dagger(x)\psi_R^\dagger(y)\psi_L(x)\psi_L(y) + h.c. \right) \right)$$

The gs can be written in terms of various (fourier) components of the interaction potential K^{-1}. In the last term, you should think of $y = x + a$, one lattice spacing. The last term describes a process whereby two L/R fermions turn into two R/L fermions, which clearly violates momentum conservation by $\pm 2k_F$; this can happen in a lattice model if the two sides of the fermi surface differ by a reciprocal lattice vector:

$$k_F \equiv -k_F + \frac{2\pi}{a}.$$

This is called *umklapp* scattering. When does it happen? k_F is determined by the density of electrons. The maximum density is when all states are filled and is $N_s \frac{2\pi}{a}$, and then there is no fermi surface. Umklapp happens at *half-filling*, when $k_F = \frac{\pi}{a}$. Otherwise g_u can be ignored. Notice that this effect explicitly breaks the chiral symmetry $\psi_{L/R} \to e^{\mathbf{i}\alpha_{L/R}}\psi_{L/R}$ which independently rotates the phases of the left-movers and right-movers down to a $\mathbb{Z}_2$ symmetry, which is just fermion number (which is hard to violate).

Bosonize. Now we bosonize. Using $\phi = \phi_L + \phi_R, \Theta = \phi_L - \phi_R$ and our dictionary above, we find $\mathbf{H}_0 \equiv \int dx \mathbf{h}_0(x)$, with

$$\mathbf{h}_0(x) = \frac{v_F}{2}\left(\Pi^2 + (\partial_x\phi)^2\right) = \frac{v_F}{2}\left((\partial_x\Theta)^2 + (\partial_x\phi)^2\right)$$

where in the second step we used the canonical commutators we found previously. Using $\boldsymbol{\rho}_L = \frac{1}{2\pi}\partial_x\phi_L$ etc this is

$$\mathbf{h}_0(x) = \pi v_F \left((\boldsymbol{\rho}_L + \boldsymbol{\rho}_R)^2 + (\boldsymbol{\rho}_L - \boldsymbol{\rho}_R)^2\right) = 2\pi v_F \left(\boldsymbol{\rho}_L^2 + \boldsymbol{\rho}_R^2\right).$$

The forward scattering hamiltonian density is

$$\mathbf{h}_{\rm forward}(x) = g_4 \left(\boldsymbol{\rho}_R^2 + \boldsymbol{\rho}_L^2\right)$$

of exactly the same form as the free term (!) and therefore just renormalizes the velocity, the coefficient of the spatial gradient term. The back-scattering term is less innocuous

$$\mathbf{h}_{\text{back}}(x) = 2g_2 \boldsymbol{\rho}_R \boldsymbol{\rho}_L = \frac{g_2}{\#} \left((\partial_x \phi)^2 - \Pi^2 \right)$$

Umklapp is a little trickier: to define the bosonized form we must absorb the OPE singularity

$$: e^{-\mathbf{i}\phi_R(x)} :: e^{-\mathbf{i}\phi_R(y)} :\sim (x-y)^{\frac{1}{2\pi T}} : e^{-2\mathbf{i}\phi_R(x)} : + ...$$

into the coupling constant:

$$\mathbf{h}_u = \tilde{g}_u \cos 2\phi.$$

(It is useful to set $\tilde{g}_u = g_u a^{-\frac{1}{2\pi T}}$ where a is a short-distance cutoff.) With $\tilde{g}_u \neq 0$, the remaining chiral symmetry acts by $\phi \to \phi + \pi$.

The full hamiltonian density (without umklapp, so away from half-filling) is

$$\mathbf{h} = \frac{v}{2} \left(\frac{1}{K} (\partial_x \Theta)^2 + K (\partial_x \phi)^2 \right)$$

where v and K can be written in terms of the gs and v_F[47]. Notice that T-duality is clear in this expression as $\mathsf{T} : K \leftrightarrow \frac{1}{K}, \phi \leftrightarrow \Theta$.

A few comments:

- The backscattering term g_2 affects both the velocity and the 'stiffness' K (this is the parameter we called the tension T earlier). The signs are such that a repulsive interaction $g_2 > 0$ increases K.
- In the absence of umklapp, the chiral symmetry acts on the boson as $\phi \to \phi + \alpha$, $\alpha \in [0, 2\pi)$. Think about how this acts on the fermions: since $\boldsymbol{\rho} = ... + \frac{1}{2\pi}\partial\phi$, the density transforms as

  $$\boldsymbol{\rho}(x) \to \rho_0 + \boldsymbol{\rho}_R + \boldsymbol{\rho}_L + \left(e^{\mathbf{i}2(k_F x - \alpha)} \psi_R^\dagger(x) \psi_L(x) + h.c. \right) = \boldsymbol{\rho} \left(x - \frac{\alpha}{k_F} \right)$$

  – this is a rigid translation of the density profile! It makes sense then that lattice effects would break this symmetry.

[47] The relation is

$$v = \sqrt{(v_F + g_4/\pi)^2 - (g_2/\pi)^2}, \quad K = \sqrt{\frac{v_F + g_4/\pi + g_2/\pi}{v_F + g_4/\pi - g_2/\pi}}.$$

- We have ignored the fact that the dispersion will not be exactly linear arbitrarily far from the Fermi surface. Such terms can also be bosonized and become things like $(\partial_x \phi)^3$ which change the story dramatically; in particular they will lead to damping of the 'sound mode' described by ϕ even away from half-filling.

- If we study spinful fermions, there are even more 4-fermion terms to consider. A payoff of studying the case with spin is that it provides an example of *spin-charge separation*: by bosonizing like $\phi_c \equiv \phi_\downarrow + \phi_\uparrow, \phi_s \equiv \phi_\downarrow - \phi_\uparrow$ we can see that the charge sector (ϕ_c) and the spin sector (ϕ_s) decouple and can have independent vs and Ks – the spin excitations of the electron travel at a different speed than its charge! There is a lot more to say about this and I recommend Fradkin's treatment in chapter 6.

How is a Luttinger liquid different from a Fermi liquid?

Continuing with the model with actual microscopic fermions, we have encountered a gapless phase in $D = 1 + 1$ described by free fermions, which we also know how to build in $D > 1 + 1$. The excitations are characterized by long-lived fermionic quasiparticles with the same $\mathsf{U}(1)$ charge as the constituent fermions $\mathbf{c}_j$ – they are Landau quasiparticles. This has the consequence that there are sharp peaks in the fermion green's functions,

$$\int e^{\mathbf{i}\omega t - \mathbf{i}kja} \left\langle \mathbf{c}_j^\dagger(t)\mathbf{c}_0(0) \right\rangle = G(\omega, k) \sim \frac{Z}{\omega - v_F(k - k_F)}$$

(the RHS is the contribution from the right-mover $\Psi_R(t - x)$). There is (therefore) a *jump* in the fermion occupation number at $k = k_F$:

$$n(k) \equiv \mathbf{c}^\dagger(k)\mathbf{c}(k) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \underbrace{\mathrm{Im}G(\omega, k)}_{\sim Z\pi\delta(\omega - v_F(k-k_F))} = Z\theta(k - k_F).$$

[Fradkin §6.8] These last two characteristic features of Fermi liquid theory can be destroyed by interactions in 1+1 dimensions, as we can see using the boson description. As a simple first step, notice that as we change T away from $\frac{1}{\pi}$, the dimension of the operator which at $T = \frac{1}{\pi}$ creates the right-moving fermion $e^{\mathbf{i}\phi_R}$ changes away from $\frac{1}{2}$:

$$\left\langle e^{\mathbf{i}\phi_R(x,t)} e^{-\mathbf{i}\phi_R(0,0)} \right\rangle \sim \frac{1}{|t - x|^{\frac{1}{\pi T}}} \ .$$

It is tempting to conclude that the fermion correlation function $\Psi(x) = e^{\mathbf{i}k_F x}\Psi_R(x) +$ ... is

$$\left\langle \Psi(x,t)^\dagger \Psi(0,0) \right\rangle \overset{?}{\sim} \frac{\sin(k_F x)}{|t - x|^{\frac{1}{\pi T}}}$$

The fourier transform of this has a broad power-law peak (a branch point) instead of a pole; the spectral density has no delta function, but rather goes like

$$\mathrm{Im}G(\omega,k) \stackrel{?}{\sim} |\omega - v_F(k-k_F)|^{\frac{1}{\pi T}+1}\theta(\omega - v_F(k-k_F)),$$

and hence there is no jump at $k = k_F$.

This is not quite the correct story however, since the operator $e^{\mathbf{i}\phi_R}$ is not quite the operator which creates an electron, except at $T = \frac{1}{\pi}$ (for example, it is not a fermionic object!). Instead what happens is that the interactions mix the R and L operators; the operator which creates the rightmoving electron is actually the linear(-in-the-exponent) combination:

$$\Psi_R = e^{\mathbf{i}\alpha_+\phi_R + \mathbf{i}\alpha_-\phi_L}, \qquad \text{with } \alpha_\pm = \frac{(K\pm 1)^2}{4K}$$

(recall that $K = \frac{1}{\pi T}$); it has total dimension $\Delta = \frac{1}{4}\left(K + \frac{1}{K}\right)$. The statement about the pole being replaced by a branch point is correct.

(Pieces of) the fermion Green's function are (relatively) directly measured in angle-resolved photoemission spectroscopy (ARPES) experiments. Other combinations of them can be measured by scanning-tunneling microscopy (STM).

[Fradkin, 2d ed. p. 170] There is a lot more to say about this subject. In particular, one can understand in great detail various ordering instabilities of interacting fermions in one dimension using the technology we have developed.

7.4.3 Application 2: sine-Gordon and Thirring

Let us reconsider the scale invariant theory described by the free scalar again.

$$\mathcal{L} = \frac{T}{2}\left(\partial_\mu\phi\right)^2 + g_n : \cos n\phi :$$

I allow the possibility of a cosine term. For the case of the Luttinger liquid we found that umklapp produced $n = 2$. There are several important points to make here:

(-1) In the absence of lattice effects or a background gauge field, there is a $\mathsf{U}(1)\times\mathsf{U}(1)$ symmetry acting on ϕ_L and ϕ_R independently.

(0) The theory is scale invariant for any T: there is a *line of fixed points* parametrized by T. That is, T is the coefficient of an *exactly marginal operator*. This is weird and the only examples in $D > 1+1$ (that I know) require supersymmetry.

(1) by changing T we can study interacting fermion systems.

(2) the fermion mass corresponds to a $\cos\phi$ term.

(3) These $\cos n\phi$ terms have a dimension that varies with T. This means that as we vary T they can go from irrelevant to relevant. If they are relevant they produce a mass for everybody (You can see this by Taylor expanding about the minimum of the potential). This happens when

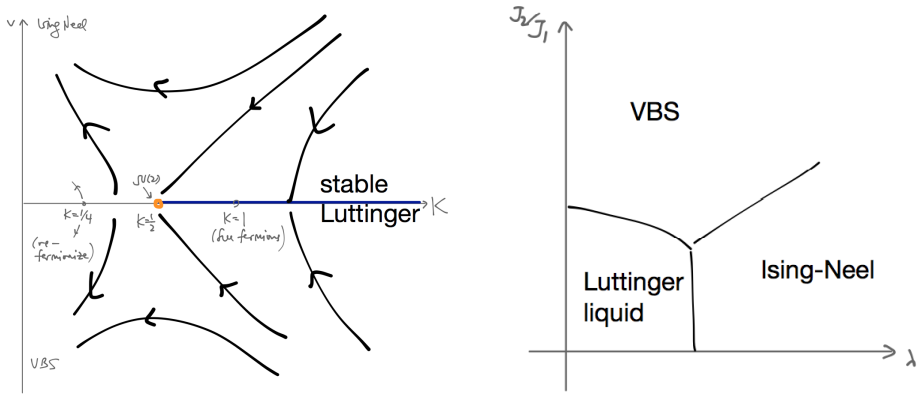
$$\Delta_n = \frac{n^2}{2\pi T} = K\frac{n^2}{2} \leq 2$$

that is, for $T > T_c = \frac{n^2}{4\pi}$ (or $K < K_c = \frac{n^2}{2}$)

Brief comments on phase diagram. [Sachdev Ch. 20 – beware that $\phi_{\text{Sachdev}} = 2\phi_{\text{here}}$] The RG equations (for $n = 1$) are

$$\frac{dv}{d\ell} = (2 - 4K)v, \quad \frac{dK}{d\ell} = -\delta v^2$$

where $\delta > 0$ is some non-universal quantity. This leads to the following RG flow diagram in the space of these couplings (from Sachdev chapter 20):

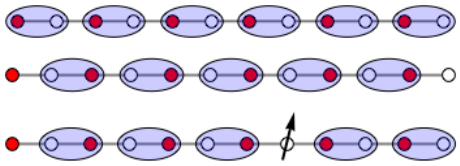


The figure at right is the phase diagram we can then infer for the model

$$\mathbf{H} = -J_1 \sum_{\langle ij \rangle} (\mathbf{X}_i\mathbf{X}_j + \mathbf{Y}_i\mathbf{Y}_j + \lambda\mathbf{Z}_i\mathbf{Z}_j) - J_2 \sum_{\langle\langle ij \rangle\rangle} (\mathbf{X}_i\mathbf{X}_j + \mathbf{Y}_i\mathbf{Y}_j + \mathbf{Z}_i\mathbf{Z}_j) \ .$$

where $\langle\langle ij \rangle\rangle$ denotes next-nearest neighbors. The simpler model we've been discussing moves along the λ axis.

• When $v \to -\infty$, we can find the ground-states by making the $-|v| \cos 2\phi$ happy first. The minima occur at $\phi = (n + \frac{1}{2})\pi$, $n \in \mathbb{Z}$. They can be distinguished by the value of $\sin \phi_i = (-1)^n = \langle(-1)^i \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}\rangle$. This is described by singlets on alternating bonds. This is called a *valence bond*



solid state. The low energy excitations in this
state are interesting – they are not just the gapped ϕ excitations rolling in the minima of the cosine potential. Rather, the domain walls between the two groundstates (created in the CFT by $e^{\mathbf{i}\frac{1}{2}\phi}$!) have lower energy. We can see this explicitly at the special point $K = \frac{1}{4}$ (which is $< K_c$) by 'refermionizing'. The previous original fermions where created by $\psi_L^\dagger = e^{\mathbf{i}\phi_R} = e^{\mathbf{i}\frac{1}{2}(\phi+\Theta)}$. Introduce new fermions

$$\tilde{\psi}_L^\dagger \equiv e^{\mathbf{i}\frac{1}{2}\left(2\phi+\frac{\Theta}{2}\right)}, \tilde{\psi}_R^\dagger \equiv e^{\mathbf{i}\frac{1}{2}\left(2\phi-\frac{\Theta}{2}\right)}.$$

At $K = \frac{1}{4}$ these are free fermions, and the $v\cos 2\phi$ term is just a mass term $v\tilde{\psi}_L^\dagger\tilde{\psi}_R + h.c.$. Interestingly, there is a regime where the quantum of the boson can *decay* into a pair of these (fermionic) domain walls.

• The phase at $v \to +\infty$ is similar. Minimizing $+|v|\cos 2\phi$ gives $\phi = n\pi$ which are distinguished by $\cos\phi = (-1)^n = \langle(-1)^i \mathbf{Z}_i\rangle$. This is 'Ising Neel' order, that is: staggered Ising spins. (Neel order refers to staggered magnetization in an arbitrary direction, spontaneously breaking $\mathsf{SU}(2)$ spin rotation symmetry.)

The two phases at large $|v|$ are related by the relabelling $v \to -v, \phi \to \phi + \pi/2$. In terms of the bosons or fermions both of these large $|v|$ states are charge density waves, where the density of particles breaks the translation symmetry down to a smaller discrete subgroup.

Most continuous phase transitions occur by tuning the coefficient of a relevant operator to zero (recall the TFIM example where we had to tune $g \to g_c$ to get massless fermions). This is *not* what happens in the XY model; rather, we are varying a *marginal* parameter and the dimensions of other operators depend on it and become relevant at some critical value of that marginal parameter. This leads to very weird scaling near the transition, of the form $e^{-\frac{a}{\sqrt{K-K_c}}}$ (for example, in the correlation length, the exponential arises from inverting expressions involving $G_R(z) = -\frac{1}{4\pi K}\log z$) – it is sometimes called an 'infinite order' phase transition, because all derivatives of such a function are continuous.

In the high-energy literature, this is called duality between the Thirring model and the sine-Gordon model.

8 Saddle points, non-perturbative field theory and resummations

8.1 Instantons in the Abelian Higgs model in $D = 1 + 1$

[Coleman p. 302-307] Consider the $\mathbb{CP}^{N-1}$ model in $D = 1 + 1$ again. What is the force between two distant (massive) z-particles? According to (4.39), the force from σ exchange is short-ranged: $\Pi(q \to 0) = \frac{4\pi}{N}$. But the Coulomb force, from A in $D = 1 + 1$ is independent of separation (*i.e.* the potential $\int đp\, \frac{e^{ipx}}{p^2} \sim x$ is linear). This means confinement.

Let's think more about abelian gauge theory in $D = 1 + 1$. Consider the case of $N = 1$. This could be called the $\mathbb{CP}^0$ model, but it is usually called the Abelian Higgs model.

$$L = \frac{1}{4e^2}F^2 + D_\mu z^\dagger D^\mu z + \frac{\kappa}{4}(z^\dagger z)^2 + \frac{\mu^2}{2}z^\dagger z + \theta\frac{F}{2\pi}.$$

What would a classical physicist say is the phase diagram of this model as we vary μ^2? For $\mu^2 > 0$, it is 2d scalar QED. There is no propagating photon, but (as we just discussed) the model confines because of the Coulomb force. The spectrum is made of boundstates of zs and $z^\dagger$s, which are stable because there is no photon for them to decay into. For $\mu^2 < 0$, it looks like the potential wants $|z|^2 = \mu^2/\kappa \equiv v^2$ in the groundstate. This would mean that A_μ eats the phase of z, gets a mass (a massive vector in $D = 1 + 1$ has a propagating component); the radial excitation of z is also massive. In such a Higgs phase, external charges don't care about each other, the force is short-ranged.

Not all of the statements in the classical, shaded box are correct quantumly. In fact, even at $\mu^2 < 0$, external charges are still confined (but with a different string tension than $\mu^2 > 0$). Non-perturbative physics makes a big difference here.

Let's try to do the euclidean path integral at $\mu^2 < 0$ by saddle point. This means we have to find minima of

$$S_E^0 \equiv \int \left(\frac{1}{4e^2}F^2 + D_\mu z^\dagger D^\mu z + \frac{\kappa}{4}(z^\dagger z - v^2)^2\right) d^2x.$$

(Ignore θ for now, since it doesn't affect the EOM.) Where have you seen this before?

This is exactly the functional we had to minimize in §10.1 to find the (Abrikosov-Nielsen-Olesen) vortex solution of the Abelian Higgs model. There we were thinking about a 3+1 D field theory, and we found a static configuration, translation invariant

in one spatial direction, localized in the two remaining directions. Here we have only two dimensions. The same solution of the equations now represents an *instanton* – a solution of the euclidean equations of motion, localized in euclidean spacetime. Here's a quick review of the solution: Choosing polar coordinates about some origin (more on this soom), the solution has (in order that $V(\rho)$ goes to zero at large r)

$$z(r,\theta) \stackrel{r\to\infty}{\to} g(\theta)v,$$

where $g(\theta)$ is a phase. We can make the $|Dz|^2$ term happy by setting

$$A \stackrel{r\to\infty}{\to} -\mathbf{i}g\partial_\mu g + \mathcal{O}(r^{-2}).$$

Then the F^2 term is automatically happy.

What are the possible $g(\theta)$? g is a map from the circle at infinity to the circle of phases $g : S^1 \to S^1$. Such maps are classified by a winding number, $Q \in \mathbb{Z}$. A representative of each class is $g(\theta) = e^{\mathbf{i}Q\theta}$. This function gives

$$\int_{\text{spacetime}} \frac{F}{2\pi} = \oint \frac{A}{2\pi} = -\mathbf{i}\int_0^{2\pi} \text{đ}\theta e^{-\mathbf{i}Q\theta}(+\mathbf{i}Q)e^{\mathbf{i}Q\theta} = Q.$$

The winding number determines the flux.

This means the partition function is

$$Z = \int [dAdz]e^{-S[A,z]} = \sum_{Q_T\in\mathbb{Z}} e^{\mathbf{i}\theta Q_T} Z_{Q_T} \simeq \sum_{Q_T} e^{\mathbf{i}\theta Q_T} e^{-S_0^{Q_T}} \frac{1}{\det S''_{Q_T}}.$$

In the last step I made a caricature of the saddle point approximation. Notice the dependence of the instanton ($Q \neq 0$) contributions: if we scale out an overall coupling (by rescaling fields) and write the action as $S[\phi] = \frac{1}{g^2}S[\phi, \text{ratios of couplings}]$, then $e^{-S_0} = e^{-\frac{1}{g^2}S[\underline{\phi},\text{ratios}]}$ is non-analytic at $g = 0$ – all the terms of its taylor expansion vanish at $g = 0$. This is not something we could ever produce by perturbation series, it is *non-perturbative*. Notice that it is also *small* at weak coupling. However, sometimes it is the leading contribution, *e.g.* to the energy of a metastable vacuum. (For more on this, see Coleman.)

To do better, we need to understand the saddle points better.

1. First, in the instanton solution we found, we picked a center, the location of the core of the vortex. But in fact, there is a solution for any center x_0^μ, with the same action. This means the determinant of S'' actually has a zero! The resolution is simple: There is actually a *family* of saddles, labelled by the *collective coordinate* x_0^μ. We just have to *do* the integral over these coordinates. The result is simple:

it produces a factor of $\int d^D x_0 = VT$ where VT is the volume of spacetime. The contribution of one instanton to the integral is then

$$Ke^{-S_0}e^{\mathbf{i}\theta}VT$$

for some horrible constant K.

2. Second, since the vortex solution is localized, we can make arbitrarily-close-to-solutions by introducing multiple vortices with their respective centers arbitrarily far from each other. The Q_T is actually the sum of the instanton numbers. If they are far enough apart, their actions also add. Each center has its own collective coordinate and produces its own factor of VT.

3. We can also have anti-instantons. This just means that individual Qs can be negative.

So we are going to approximate our integral by a dilute gas of instantons and anti-instantons. Their actions add. A necessary condition for this to be a good idea is that $VT \gg$ (core size)^2. $e^{\mathbf{i}\theta}$ is the instanton fugacity.

$$\begin{aligned} Z = \mathrm{Tr}\ e^{-TH} &\overset{T\to\infty}{\simeq} \sum_{n,\bar{n}} \left(Ke^{-S_0}\right)^{n+\bar{n}} (VT)^{n+\bar{n}} e^{\mathbf{i}(n-\bar{n})\theta} \frac{1}{n!\bar{n}!} \\ &= \left(\sum_n \frac{1}{n!}\left(Ke^{-S_0}VT\right)^n e^{\mathbf{i}n\theta}\right) \times (h.c.) \\ &= e^{VTKe^{-S_0}e^{\mathbf{i}\theta}+h.c.} = e^{VT2Ke^{-S_0}\cos\theta}. \end{aligned} \tag{8.1}$$

We should be happy about this answer. Summing over the dilute gas of instantons gives an extensive contribution to the free energy. The free energy per unit time in the euclidean path integral is the groundstate energy density:

$$Z = \mathrm{Tr}\ e^{-TH} \overset{T\to\infty}{\simeq} e^{-TV\mathcal{E}(\theta)}, \quad \Longrightarrow \quad \mathcal{E}(\theta) = -2K\cos\theta e^{-S_0}.$$

We can also calculate the expected flux:

$$\left\langle \frac{\int F}{2\pi} \right\rangle = \sum_Q Q e^{\mathbf{i}\theta Q} e^{-S} \sum_Q e^{\mathbf{i}\theta Q} e^{-S} = -\mathbf{i}\partial_\theta \ln Z(\theta) = 2KV\sin\theta e^{-S_0}.$$

Therefore, when $\theta \neq 0 \bmod \pi$, there is a nonzero electric field in the vacuum: $\langle F_{01}\rangle = E \neq 0$. It is uniform.

A small variation of this calculation gives the force between external charges:

$$\left\langle W \left[\begin{array}{c} \square \updownarrow T' \\ \overleftrightarrow{L}' \end{array}\right] \right\rangle = \left\langle e^{\mathbf{i}\frac{q}{e}\oint_\square A_\mu dx^\mu} \right\rangle = \left\langle e^{\mathbf{i}\frac{q}{e}\int_\blacksquare F} \right\rangle$$

This has the effect of shifting the value of θ on the *inside* of the loop to $\theta_{\rm in} \equiv \theta + \frac{q}{e} 2\pi$. So the answer in the dilute instanton gas approximation is

$$\left\langle W \left[\begin{array}{c} \square \updownarrow T' \\ \leftrightarrow L' \end{array} \right] \right\rangle = \frac{\exp\left(2Ke^{S_0} \left(\underbrace{(LT - L'T')\cos\theta}_{\text{inside}} + \underbrace{L'T'\cos\theta_{\rm in}}_{\text{outside}} \right) \right)}{e^{2Ke^{-S_0} LT\cos\theta}} = e^{-T'V(L')}$$

with

$$V(L') = L' 2Ke^{-S_0} \left(\cos\theta - \cos\left(\theta + 2\pi \frac{q}{e} \right) \right)$$

which is linear in the separation between the charges – linear confinement, except when $q = ne, n \in \mathbb{Z}$.

Here's how to think about this result. For small $\theta, q/e$, the potential between charges is

$$V(L') \stackrel{\theta \ll 1}{\simeq} L' K e^{-S_0} \left(\left(\theta + 2\pi \frac{q}{e} \right)^2 - \theta^2 \right)$$

and the energy and flux are

$$\mathcal{E}(\theta) \stackrel{\theta \ll 1}{\simeq} 2Ke^{-S_0}\theta^2 + \text{const}, \qquad \langle F \rangle \stackrel{\theta \ll 1}{\simeq} 4\pi K e^{-S_0}\theta.$$

θ is like the charge on a pair of parallel capacitor plates at $x = \infty$. Adding charge and anticharge changes the electric field in between, and the energy density is quadratic in the field, $U \propto E^2$. But what happens when $q = ne$? Notice that the potential is actually periodic in $q \to q + ne$. If $L' > \frac{1}{2\mu}$ (μ is the mass of the z excitations), then the energy can be decreased by pair-creating a z and $z^\dagger$, which then fly to the capacitor plates and discharge them, changing $\theta \to \theta - 2\pi$.

Comments about $D = 4$. Some of the features of this story carry over to gauge theory in $D = 3 + 1$. Indeed there is a close parallel between the $\theta \int_2 F$ term and the $\theta \int_4 F \wedge F$ term. In 4d, too, there are solutions of the euclidean equations (even in pure Yang-Mills theory) which are localized in spacetime. (The word *instanton* is sometimes used to refer to these solutions, even when they appear in other contexts than euclidean saddle points. These solutions were found by Belavin, Polyakov, Schwartz and Tyupin.) Again, the gauge field looks like a gauge transformation at ∞:

$$A \stackrel{r \to \infty}{\to} -\mathbf{i} g \partial_\mu g + \mathcal{O}(r^{-\#}).$$

Now g is a map from the 3-sphere at infinity (in euclidean 4-space) to the gauge group, $g : S^3 \to \mathsf{G}$. Any simple Lie group has an $\mathsf{SU}(2) \simeq S^3$ inside, and there is an integer classification of such maps. So again there is a sum over $Q \in \mathbb{Z}$. However: the calculation leading to confinement does *not* go through so simply. The 4d θ term does

not produce a nonzero electric field in the vacuum, and an external charge isn't like a capacitor plate. As Coleman says, whatever causes confinement in 4d gauge theory, it's not instantons.

8.2 Blobology (aka Large Deviation Theory)

Many bits of the following discussion are already familiar, but I like the organization.

Feynman diagrams from the path integral. Now that we are using path integrals all the time, the diagrammatic expansion is much less mysterious (perhaps we should have started here, like Zee does? maybe next time). Much of what we have to say below is still interesting for QFT in $0+0$ dimensions, which means integrals. If everything is positive, this is probability theory. Suppose we want to do the integral

$$Z(J) = \int_{-\infty}^{\infty} dq\ e^{-\frac{1}{2}m^2q^2 - \frac{g}{4!}q^4 + Jq} \equiv \int dq\ e^{-S(q)}\ . \tag{8.2}$$

It is the path integral for ϕ^4 theory with fewer labels. For $g = 0$, this is a gaussian integral which we know how to do. For $g \neq 0$ it's not an elementary function of its arguments. We can develop a (non-convergent!) series expansion in g by writing it as

$$Z(J) = \int_{-\infty}^{\infty} dq\ e^{-\frac{1}{2}m^2q^2 + Jq} \left(1 - \frac{g}{4!}q^4 + \frac{1}{2}\left(-\frac{g}{4!}q^4\right)^2 + \cdots\right)$$

and integrating term by term. And the term with q^{4n} (that is, the coefficient of g^n) is

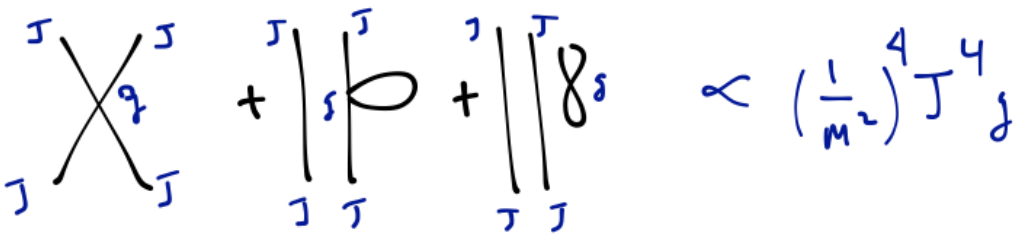
$$\int_{-\infty}^{\infty} dq\ e^{-\frac{1}{2}m^2q^2 + Jq} q^{4n} = \left(\frac{\partial}{\partial J}\right)^{4n} \int_{-\infty}^{\infty} dq\ e^{-\frac{1}{2}m^2q^2 + Jq} = \left(\frac{\partial}{\partial J}\right)^{4n} e^{\frac{1}{2}J\frac{1}{m^2}J} \sqrt{\frac{2\pi}{m^2}}\ .$$

So:

$$Z(J) = \sqrt{\frac{2\pi}{m^2}} e^{-\frac{g}{4!}\left(\frac{\partial}{\partial J}\right)^4} e^{\frac{1}{2}J\frac{1}{m^2}J}\ .$$

This is a double expansion in powers of J and powers of g. The process of computing the coefficient of $J^n g^m$ can be described usefully in terms of diagrams. There is a factor of $1/m^2$ for each line (the propagator), and a factor of $(-g)$ for each 4-point vertex (the coupling), and a factor of J for each external line (the source). For example, the coefficient of gJ^4 comes from:

$$\sim \left(\frac{1}{m^2}\right)^4 gJ^4.$$



There is a symmetry factor which comes from expanding the exponential: if the diagram has some symmetry preserving the external labels, the multiplicity of diagrams does not completely cancel the $1/n!$.

As another example, consider the analog of the two-point function:

$$G \equiv \langle q^2 \rangle |_{J=0} = \frac{\int dq\ q^2\ e^{-S(q)}}{\int dq\ e^{-S(q)}} = -2\frac{\partial}{\partial m^2} \log Z(J=0).$$

In perturbation theory this is:

$$G \simeq \quad + \tfrac{1}{2} \quad + \tfrac{1}{4} \quad + \tfrac{1}{8} \quad + \tfrac{1}{6} \quad + \mathcal{O}(g^3)$$

$$= m^{-2}\left(1 \quad - \frac{1}{2}gm^{-2} \quad + \quad \frac{2}{3}g^2m^{-4} \quad + \mathcal{O}(g^3)\right) \tag{8.3}$$

Brief comments about large orders of perturbation theory.

- How do I know the perturbation series about $g = 0$ doesn't converge? One way to see this is to notice that if I made g even infinitesimally negative, the integral itself would not converge (the potential would be unbounded below), and $Z_{g=-|\epsilon|}$ is not defined. Therefore Z_g as a function of g cannot be analytic in a neighborhood of $g = 0$. This argument is due to Dyson.

- The expansion of the exponential in the integrand is clearly convergent for each q. The place where we went wrong is exchanging the order of integration over q and summation over n.

- The integral actually does have a name – it's a Bessel function:

$$Z(J=0) = \frac{2}{\sqrt{m^2}}\sqrt{\rho}e^{\rho}K_{\frac{1}{4}}(\rho), \qquad \rho \equiv \frac{3m^4}{4g}$$

  (for $\mathrm{Re}\sqrt{\rho} > 0$), as `Mathematica` will tell you. Because we know about Bessel functions, in this case we can actually figure out what happens at strong coupling, when $g \gg m^4$, using the asymptotics of the Bessel function.

- In this case, the perturbation expansion too can be given a closed form expression:

$$Z(0) \simeq \sqrt{\frac{2\pi}{m^2}} \sum_n \frac{(-1)^n}{n!} \frac{2^{2n+\frac{1}{2}}}{(4!)^n} \Gamma\left(2n+\frac{1}{2}\right) \left(\frac{g}{m^4}\right)^n . \tag{8.4}$$

- The expansion for G is of the form

$$G \simeq m^{-2} \sum_{n=0}^{\infty} c_n \left(\frac{g}{m^4}\right)^n .$$

  When n is large, the coefficients satisfy $c_{n+1} \overset{n\gg 1}{\simeq} -\frac{2}{3} n c_n$ (you can see this by looking at the coefficients in (8.4)) so that $|c_n| \sim n!$. This factorial growth of the number of diagrams is general in QFT and is another way to see that the series does not converge.

- The fact that the coefficients c_n grow means that there is a *best* number of orders to keep. The errors start getting bigger when $c_{n+1}\left(\frac{g}{m^4}\right) \sim c_n$, that is, at order $n \sim \frac{3m^4}{2g}$. So if you want to evaluate G at this value of the coupling, you should stop at that order of n.

- A technique called *Borel resummation* can sometimes produce a well-defined function of g from an asymptotic series whose coefficients diverge like $n!$. The idea is to make a new series

$$B(z) \equiv \sum_{m=0} \frac{c_m}{n!} z^m$$

  whose coefficients are ensmallened by $n!$. Then to get back $Z(g)$ we use the identity

$$1 = \frac{1}{n!} \int_0^{\infty} dz e^{-z} z^n$$

  and do the Laplace transform of $B(z)$:

$$\int_0^{\infty} dz B(z) e^{-z/g} = \sum_{m=0} c_m \frac{\int_0^{\infty} dz e^{-z/g} z^m}{m!} = g \sum_{m=0}^{\infty} c_m g^m = g Z(g).$$

  This procedure requires both that the series in $B(z)$ converges and that the Laplace transform can be done. In fact this procedure works in this case.

  The existence of saddle-point contributions to $Z(g)$ which go like $e^{-a/g}$ imply that the number of diagrams at large order grows like $n!$. This is because they are associated with singularities of $B(z)$ at $z = a$; such a singularity means the sum of $\frac{c_n}{n!} z^n$ must diverge at $z = a$. (More generally, non-perturbative effects which go like $e^{-a/g^{1/p}}$ (larger if $p > 1$) are associated with (faster) growth like $(pn)!$. See this classic work.)

- The function $G(g)$ can be analytically continued in g away from the real axis, and can in fact be defined on the whole complex g plane. It has a branch cut on the negative real axis, across which its discontinuity is related to its imaginary part. The imaginary part goes like $e^{-\frac{a}{|g|}}$ near the origin and can be computed by a tunneling calculation.

  How did we know Z has a branch cut? One way is from the asymptotics of the Bessel function. But, better, *why* does Z satisfy the Bessel differential equation as a function of the couplings? The answer, as you'll check on the homework, is that the Bessel equation is a Schwinger-Dyson equation,

$$0 = \int_{-\infty}^{\infty} \frac{\partial}{\partial q} \left(\text{something} e^{-S(q)} \right)$$

  which results from demanding that we can change integration variables in the path integral.

For a bit more about this, you might look at sections 3 and 4 of this recent paper from which I got some of the details here. See also the giant book by Zinn-Justin. There is a deep connection between the large-order behavior of the perturbation series about the trivial saddle point and the contributions of non-trivial saddle points. The keywords for this connection are *resurgence* and *trans-series* and a starting references is here.

The Feynman diagrams we've been drawing all along are the same but with more labels. Notice that each of the qs in our integral could come with a label, $q \to q_a$. Then each line in our diagram would be associated with a matrix $\left(m^{-2}\right)_{ab}$ which is the inverse of the quadratic term $q_a m^2_{ab} q_b$ in the action. If our diagrams have loops we get free sums over the label. If that label is conserved by the interactions, the vertices will have some delta functions. In the case of translation-invariant field theories we can label lines by the conserved momentum k. Each comes with a factor of the free propagator $\frac{\mathbf{i}}{k^2+m^2+\mathbf{i}\epsilon}$, each vertex conserves momentum, so comes with $\mathbf{i}g\delta^D\left(\sum k\right)(2\pi)^D$, and we must integrate over momenta on internal lines $\int \text{đ}^D k$.

Next, three general organizing facts about the diagrammatic expansion, two already familiar. In thinking about the combinatorics below, we will represent collections of Feynman diagrams by blobs with legs sticking out, and think about how the blobs combine. Then we can just renormalize the appropriate blobs and be done.

The following discussion will look like I am talking about a field theory with a single scalar field. But really each of the ϕs is a collection of fields and all the indices are too small to see. This is yet another example of coarse-graining.

1. **Disconnected diagrams exponentiate.** [Zee, I.7, Banks, chapter 3] Recall that the Feynman rules come with a (often annoying, here crucial) statement about symmetry factors: we must divide the contribution of a given diagram by the order of the symmetry group of the diagram (preserving various external labels). For a diagram with k identical disconnected pieces, this symmetry group includes the permutation group S_k which permutes the identical pieces and has $k!$ elements. (Recall that the origin of the symmetry factors is that symmetric feynman diagrams fail to completely cancel the $1/n!$ in the Dyson formula. For a reminder about this, see *e.g.* Peskin p. 93.) Therefore:

$$Z = \sum (\text{all diagrams}) = e^{\sum(\text{connected diagrams})} = e^{\mathbf{i}W}.$$

You can go a long way towards convincing yourself of this by studying the case where there are only two connected diagrams $A+B$ (draw whatever two squiggles you want) and writing out e^{A+B} in terms of disconnected diagrams with symmetry factors.

Notice that this relationship is just like that of the partition function to the (Helmholtz) free energy $Z = e^{-\beta F}$ (modulo the factor of $\mathbf{i}$) in statistical mechanics (and is the same as that relationship when we study the euclidean path integral with periodic boundary conditions in euclidean time). This statement is extremely general. It remains true if we include external sources:

$$Z[J] = \int [D\phi] e^{\mathbf{i}S[\phi]+\mathbf{i}\int \phi J} = e^{\mathbf{i}W[J]}.$$

Now the diagrams have sources J at which propagator lines can terminate; (the perturbation theory approximation to) $W[J]$ is the sum of all connected such diagrams. For example

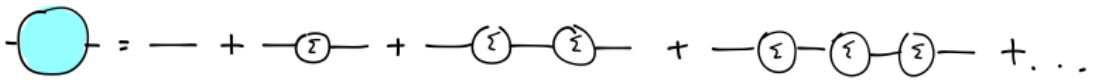
$$\langle\phi(x)\rangle = \frac{1}{Z}\frac{\delta}{\mathbf{i}\delta J(x)}Z = \frac{\delta}{\mathbf{i}\delta J(x)}\log Z = \frac{\delta}{\delta J(x)}W$$

$$\langle\mathcal{T}\phi(x)\phi(y)\rangle = \frac{\delta}{\mathbf{i}\delta J(x)}\frac{\delta}{\mathbf{i}\delta J(y)}\log Z = \frac{\delta}{\mathbf{i}\delta J(x)}\frac{\delta}{\mathbf{i}\delta J(y)}\mathbf{i}W\ .$$

(Note that here $\langle\phi\rangle \equiv \langle\phi\rangle_J$ depends on J. You can set it to zero if you want, but the equation is true for any J.) If you forget to divide by the normalization Z, and instead look at just $\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}Z$, you get *disconnected* quantities like $\langle\phi\rangle\langle\phi\rangle$ (the terminology comes from the diagrammatic representation). [48] The point in life of W is that by differentiating it with respect to J we can construct all the connected Green's functions.

[48] More precisely: $\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}Z = \frac{\delta}{\delta J(x)}\left(\langle\phi(x)\rangle_J Z\right) = \langle\phi(x)\rangle_J\langle\phi(y)\rangle_J Z + \langle\phi(x)\phi(y)\rangle_J Z.$

2. **Propagator corrections form a geometric series.** This one I don't need to say more about:



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3. **The sum of all connected diagrams is the Legendre transform of the sum of the 1PI diagrams.**

   [Banks, 3.8; Zee IV.3; Schwarz §34, Srednicki §21] A simpler way to say our third fact is

   $$\sum(\text{connected diagrams}) = \sum(\text{connected } \textit{tree} \text{ diagrams with 1PI vertices})$$

   where a *tree* diagram is one with no loops. But the description in terms of Legendre transform will be extremely useful. Along the way we will show that the perturbation expansion is a semi-classical expansion. And we will construct a useful object called the 1PI effective action Γ. The basic idea is that we can construct the actual correct correlation functions by making *tree* diagrams ($\equiv$ diagrams with no loops) using the 1PI effective action as the action.

   Notice that this is a very good reason to care about the notion of 1PI: if we sum all the tree diagrams using the 1PI blobs, we clearly are including all the diagrams. Now we just have to see what machinery will pick out the 1PI blobs. The answer is: Legendre transform. There are many ways to go about showing this, and all involve a bit of complication. Bear with me for a bit; we will learn a lot along the way.

   Def'n of ϕ_c, the 'classical field'. Consider the functional integral for a scalar field theory:

   $$Z[J] = e^{\mathbf{i}W[J]} = \int [D\phi] e^{\mathbf{i}\left(S[\phi] + \int J\phi\right)} \ . \tag{8.5}$$

   Define

   $$\phi_c(x) \equiv \frac{\delta W[J]}{\delta J(x)} = \frac{1}{Z}\int [D\phi] e^{\mathbf{i}\left(S[\phi]+\int J\phi\right)}\phi(x) = \langle 0|\, \hat{\phi}(x)\, |0\rangle \ . \tag{8.6}$$

   This is the vacuum expectation value of the field operator, in the presence of the source J. Note that $\phi_c(x)$ is a functional of J.

Warning: we are going to use the letter ϕ for many conceptually distinct objects here: the functional integration variable ϕ, the quantum field operator $\hat{\phi}$, the classical field ϕ_c. I will not always use the hats and subscripts.

Legendre Transform. Next we recall the notion of Legendre transform and extend it to the functional case: Given a function L of $\dot{q}$, we can make a new function H of p (the Legendre transform of L with respect to $\dot{q}$) defined by:

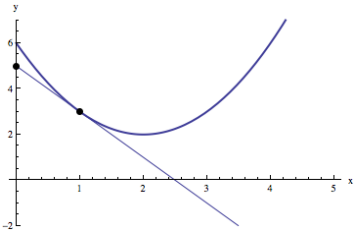
$$H(p,q) = p\dot{q} - L(\dot{q},q).$$

On the RHS here, $\dot{q}$ must be eliminated in favor of p using the relation $p = \frac{\partial L}{\partial \dot{q}}$. You've also seen this manipulation in thermodynamics using these letters:

$$F(T,V) = E(S,V) - TS, \quad T = \frac{\partial E}{\partial S}|_V \ .$$

The point of this operation is that it relates the free energies associated with different ensembles in which different variables are held fixed.

More mathematically, it encodes a function (at least one with nonvanishing second derivative, *i.e.* one which is convex or concave) in terms of its envelope of tangents. For further discussion of this point of view, look here.



Now the functional version: Given a functional $W[J]$, we can make a new associated functional Γ of the conjugate variable ϕ_c:

$$\Gamma[\phi_c] \equiv W[J] - \int J\phi_c.$$

Again, the RHS of this equation defines a functional of ϕ_c implicitly by the fact that J can be determined from ϕ_c, using (8.6)[49].

Interpretation of ϕ_c. How to interpret ϕ_c? It's some function of spacetime, which depends on the source J. **Claim:** It solves

$$-J(x) = \frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} \tag{8.7}$$

So, in particular, when $J = 0$, it solves

$$0 = \frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)}|_{\phi_c=\langle\phi\rangle} \tag{8.8}$$

[49] Come back later and worry about what happens if J is not determined uniquely.

– the extremum of the effective action is $\langle\phi\rangle$. This gives a classical-like equation of motion for the field operator expectation value in QFT.

Proof of (8.7): $$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = \frac{\delta}{\delta\phi_c(x)}\left(W[J] - \int dy J(y)\phi_c(y)\right)$$

What do we do here? We use the functional product rule – there are three places where the derivative hits:

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = \frac{\delta W[J]}{\delta\phi_c(x)} - J(x) - \int dy \frac{\delta J(y)}{\delta\phi_c(x)}\phi_c(y)$$

In the first term we must use the functional chain rule:

$$\frac{\delta W[J]}{\delta\phi_c(x)} = \int dy \frac{\delta J(y)}{\delta\phi_c(x)}\frac{\delta W[J]}{\delta J(y)} = \int dy \frac{\delta J(y)}{\delta\phi_c(x)}\phi_c(y).$$

So we have:

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = \int dy \frac{\delta J(y)}{\delta\phi_c(x)}\phi_c(y) - J(x) - \int dy \frac{\delta J(y)}{\delta\phi_c(x)}\phi_c(y) = -J(x). \tag{8.9}$$

Now $\phi_c|_{J=0} = \langle\phi\rangle$. So if we set $J = 0$, we get the equation (8.8) above. So (8.8) replaces the action principle in QFT – to the extent that we can calculate $\Gamma[\phi_c]$. (Note that there can be more than one extremum of Γ. That requires further examination.)

Next we will build towards a demonstration of the diagrammatic interpretation of the Legendre transform; along the way we will uncover important features of the structure of perturbation theory.

Semiclassical expansion of path integral. Recall that the Legendre transform in thermodynamics is the leading term you get if you compute the partition function by saddle point – the classical approximation. In thermodynamics, this comes from the following manipulation: the thermal partition function is:

$$Z = e^{-\beta F} = \mathrm{tr} e^{-\beta \mathbf{H}} = \int dE \underbrace{\Omega(E)}_{\text{(density of states with energy } E) = e^{S(E)}} e^{-\beta E} \overset{\text{saddle}}{\approx} e^{S(E_\star)-\beta E_\star}|_{E_\star \text{ solves } \partial_E S = \beta} .$$

The log of this equation then says $F = E - TS$ with S eliminated in favor of T by $T = \frac{1}{\partial_E S}|_V = \partial_S E|_V$, *i.e.* the Legendre transform we discussed above. In simple thermodynamics the saddle point approx is justified by the thermodynamic limit: the quantity in the exponent is extensive, so the saddle point is well-peaked. This part of the analogy will not always hold, and we will need to think about fluctuations about the saddle point.

Let's go back to (8.5) and think about its semiclassical expansion. If we were going to do this path integral by stationary phase, we would solve

$$0 = \frac{\delta}{\delta\phi(x)}\left(S[\phi] + \int \phi J\right) = \frac{\delta S}{\delta\phi(x)} + J(x)\ . \tag{8.10}$$

This determines some function ϕ which depends on J; let's denote it here as $\phi^{[J]}(x)$. In the semiclassical approximation to $Z[J] = e^{\mathbf{i}W[J]}$, we would just plug this back into the exponent of the integrand:

$$W_c[J] = \frac{1}{g^2\hbar}\left(S[\phi^{[J]}] + \int J\phi^{[J]}\right).$$

So in this approximation, (8.10) is exactly the equation determining ϕ_c. This is just the Legendre transformation of the original bare action $S[\phi]$ (I hope this manipulation is also familiar from stat mech, and I promise we're not going in circles).

Let's think about expanding $S[\phi]$ about such a saddle point $\phi^{[J]}$ (or more correctly, a point of stationary phase). The stationary phase (or semi-classical) expansion familiar from QM is an expansion in powers of $\hbar$ (WKB):

$$Z = e^{\mathbf{i}W/\hbar} = \int dx\, e^{\frac{\mathbf{i}}{\hbar}S(x)} = \int dx e^{\frac{\mathbf{i}}{\hbar}\left(S(x_0)+(x-x_0)\underbrace{S'(x_0)}_{=0}+\frac{1}{2}(x-x_0)^2S''(x_0)+\ldots\right)} = e^{\mathbf{i}W_0/\hbar+\mathbf{i}W_1+\mathbf{i}\hbar W_2+\ldots}$$

with $W_0 = S(x_0)$, and W_n comes from (the exponentiation of) diagrams involving n contractions of $\delta x = x - x_0$, each of which comes with a power of $\hbar$: $\langle \delta x \delta x\rangle \sim \hbar$.

Expansion in $\hbar$ = expansion in coupling. Is this semiclassical expansion the same as the expansion in powers of the coupling? Yes, if there is indeed a notion of "*the* coupling", *i.e.* only one for each field. Then by a rescaling of the fields we can put all the dependence on the coupling in front:

$$S = \frac{1}{g^2}s[\phi]$$

so that the path integral is

$$\int [D\phi]\ e^{\mathbf{i}\frac{s[\phi]}{\hbar g^2}+\int\phi J}.$$

(It may be necessary to rescale our sources J, too.) For example, suppose we are talking about a QFT of a single field $\tilde{\phi}$ with action

$$S[\tilde{\phi}] = \int\left(\left(\partial\tilde{\phi}\right)^2 - \lambda\tilde{\phi}^p\right).$$

Then define $\phi \equiv \tilde{\phi}\lambda^\alpha$ and choose $\alpha = \frac{1}{p-2}$ to get

$$S[\phi] = \frac{1}{\lambda^{\frac{2}{p-2}}} \int \left((\partial\phi)^2 - \phi^p \right) = \frac{1}{g^2} s[\phi].$$

with $g \equiv \lambda^{\frac{1}{p-2}}$, and $s[\phi]$ independent of g. Then the path-integrand is $e^{\frac{\mathbf{i}}{\hbar g^2} s[\phi]}$ and so g and $\hbar$ will appear only in the combination $g^2\hbar$. (If we have more than one coupling term, this direct connection must break down; instead we can scale out some overall factor from all the couplings and that appears with $\hbar$.)

Loop expansion = expansion in coupling. Now I want to convince you that this is also the same as the loop expansion. The first correction in the semi-classical expansion comes from

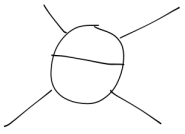
$$S_2[\phi_0, \delta\phi] \equiv \frac{1}{g^2} \int dxdy \delta\phi(x)\delta\phi(y) \frac{\delta^2 s}{\delta\phi(x)\delta\phi(y)}|_{\phi=\phi_0}.$$

For the accounting of powers of g, it's useful to define $\Delta = g^{-1}\delta\phi$, so the action is

$$g^{-2}s[\phi] = g^{-2}s[\phi_0] + S_2[\Delta] + \sum_n g^{n-2} V_n[\Delta].$$

With this normalization, the power of the field Δ appearing in each term of the action is correlated with the power of g in that term. And the Δ propagator is independent of g.

So use the action $s[\phi]$, in an expansion about $\phi_\star$ to construct Feynman rules for correlators of Δ: the propagator is $\langle \mathcal{T}\Delta(x)\Delta(y)\rangle \propto g^0$, the 3-point vertex comes from V_3 and goes like $g^{3-2=1}$, and so on. Consider a diagram that contributes to an E-point function (of Δ) at order g^n, for example this contribution to the $(E = 4)$-point function at order $n = 6 \cdot (3-2) = 6$:



With our normalization of Δ, the powers of g come only from the vertices; a degree k vertex contributes $k-2$ powers of g; so the number of powers of g is

$$n = \sum_{\text{vertices, } i} (k_i - 2) = \sum_i k_i - 2V \tag{8.11}$$

where

V = # of vertices (This does not include external vertices.)
We also define:
n = # of powers of g
L = # of loops = #of independent internal momentum integrals
I = # of internal lines = # of internal propoagators
E = # of external lines

Facts about graphs:

- The total number of lines leaving all the vertices is equal to the total number of lines:
$$\sum_{\text{vertices, } i} k_i = E + 2I. \tag{8.12}$$
So the number of internal lines is
$$I = \frac{1}{2}\left(\sum_{\text{vertices, } i} k_i - E\right). \tag{8.13}$$

- For a connected graph, the number of loops is
$$L = I - V + 1 \tag{8.14}$$
since each loop is a sequence of internal lines interrupted by vertices. (This fact is probably best proved inductively. The generalization to graphs with multiple disconnected components is $L = I - V + C$.)

We conclude that[50]

$$L \overset{(8.14)}{=} I - V + 1 \overset{(8.13)}{=} \frac{1}{2}\left(\sum_i k_i - E\right) - V + 1 = \frac{n-E}{2} + 1 \overset{(8.11)}{=} \frac{n-E}{2} + 1.$$

This equation says:

$L = \frac{n-E}{2} + 1$: More powers of g means (linearly) more loops.

[50] You should check that these relations are all true for some random example, like the one above, which has $I = 7, L = 2, \sum k_i = 18, V = 6, E = 4$. You will notice that Banks has several typos in his discussion of this in §3.4. His Es should be $E/2$s in the equations after (3.31).

Diagrams with a fixed number of external lines and more loops are suppressed by more powers of g. (By rescaling the external field, it is possible to remove the dependence on E.)

We can summarize what we've learned by writing the sum of connected graphs as

$$W[J] = \sum_{L=0}^{\infty} \left(g^2\hbar\right)^{L-1} W_L$$

where W_L is the sum of connected graphs with L loops. In particular, the order-$\hbar^{-1}$ (classical) bit W_0 comes from *tree graphs*, graphs without loops. Solving the classical equations of motion sums up the tree diagrams.

Diagrammatic interpretation of Legendre transform. $\Gamma[\phi]$ is called the 1PI effective action[51]. And as its name suggests, Γ has a diagrammatic interpretation: it is the sum of just the 1PI connected diagrams. (Recall that $W[J]$ is the sum of *all* connected diagrams.) Consider the (functional) Taylor expansion Γ_n in ϕ

$$\Gamma[\phi] = \sum_n \frac{1}{n!} \int \Gamma_n(x_1 ... x_n)\phi(x_1)...\phi(x_n) d^D x_1 \cdots d^D x_n \ .$$

The coefficients Γ_n are called 1PI Green's functions (we will justify this name presently). To get the full connected Green's functions, we sum all tree diagrams with the 1PI Green's functions as vertices, using the full connected two-point function as the propagators.

Perhaps the simplest way to arrive at this result is to consider what happens if we try to use Γ as the action in the path integral instead of S.

$$Z_{\Gamma,\hbar}[J] \equiv \int [D\phi] e^{\frac{\mathbf{i}}{\hbar}\left(\Gamma[\phi] + \int J\phi\right)}$$

By the preceding arguments, the expansion of $\log Z_\Gamma[J]$ in powers of $\hbar$, in the limit $\hbar \to 0$ is

$$\lim_{\hbar\to 0} \log Z_{\Gamma,\hbar}[J] = \sum_L \left(g^2\hbar\right)^{L-1} W_L^\Gamma \ .$$

The leading, tree level term in the $\hbar$ expansion, is obtained by solving

$$\frac{\delta\Gamma}{\delta\phi(x)} = -J(x)$$

[51] The 1PI effective action Γ must be distinguished from the *Wilsonian effective action* – the difference is that here we integrated over everybody, whereas the Wilsonian action integrates only high-energy modes. The different effective actions correspond to different choices about what we care about and what we don't, and hence different choices of what modes to integrate out.

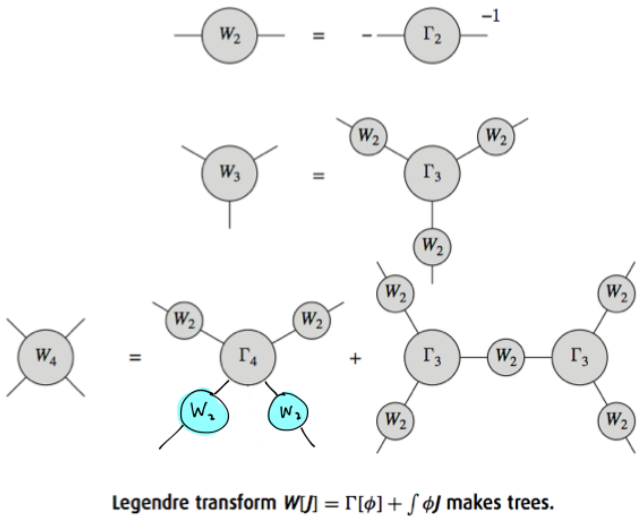


Figure 3: [From Banks, *Modern Quantum Field Theory*, slightly improved] W_n denotes the connected n-point function, $\left(\frac{\partial}{\partial J}\right)^n W[J] = \langle \phi^n \rangle$.

and plugging the solution into Γ; the result is

$$\left(\Gamma[\phi] + \int \phi J\right)_{\frac{\partial \Gamma}{\partial \phi(x)} = -J(x)} \overset{\text{inverse Legendre transf}}{\equiv} W[J].$$

This expression is the definition of the inverse Legendre transform, and we see that it gives back $W[J]$: the generating functional of connected correlators! On the other hand, the counting of powers above indicates that the only terms that survive the $\hbar \to 0$ limit are tree diagrams where we use the terms in the Taylor expansion of $\Gamma[\phi]$ as the vertices. This is exactly the statement we were trying to demonstrate: the sum of all connected diagrams is the sum of tree diagrams made using 1PI vertices and the exact propagator (by definition of 1PI). Therefore Γ_n are the 1PI vertices.

For a more arduous but more direct proof of this statement, see the problem set and/or Banks §3.5. There is an important typo on page 29 of Banks' book; it should say:

$$\frac{\delta^2 W}{\delta J(x) \delta J(y)} = \frac{\delta \phi(y)}{\delta J(x)} = \left(\frac{\delta J(x)}{\delta \phi(y)}\right)^{-1} \overset{(8.9)}{=} - \left(\frac{\delta^2 \Gamma}{\delta \phi(x) \delta \phi(y)}\right)^{-1}. \tag{8.15}$$

(where $\phi \equiv \phi_c$ here). You can prove this from the definitions above. Inverse here means in the sense of integral operators: $\int d^D z K(x,z) K^{-1}(z,y) = \delta^D(x-y)$. So we can write the preceding result more compactly as:

$$W_2 = -\Gamma_2^{-1}.$$

Here's two ways to think about why we get an inverse here: (1) diagrammatically, the 1PI blob is defined by removing the external propagators; but these external propagators are each W_2; removing two of them from one of them leaves -1 of them. You're on your own for the sign. (2) In the expansion of $\Gamma = \sum_n \int \Gamma_n \phi^n$ in powers of the field, the second term is $\int\int \phi\Gamma_2\phi$, which plays the role of the kinetic term in the effective action (which we're instructed to use to make tree diagrams). The full propagator is then the inverse of the kinetic operator here, namely Γ_2^{-1}. Again, you're on your own for the sign.

The idea to show the general case in Fig. 3 is to just compute W_n by taking the derivatives starting from (8.15): Differentiate again wrt J and use the matrix differentiation formula $dK^{-1} = -K^{-1}dKK^{-1}$ and the chain rule to get

$$W_3(x,y,z) = \int dw_1 \int dw_2 \int dw_3 W_2(x,w_1)W_2(y,w_2)W_2(z,w_3)\Gamma_3(w_1,w_2,w_3) \ .$$

To get the rest of the W_n requires an induction step.

This business is useful in at least two ways. First it lets us focus our attention on a much smaller collection of diagrams when we are doing our perturbative renormalization.

Secondly, this notion of effective action is extremely useful in thinking about the vacuum structure of field theories, and about spontaneous symmetry breaking. In particular, we can expand the functional in the form

$$\Gamma[\phi_c] = \int d^D x \left(-V_{\text{eff}}(\phi_c) + Z(\phi_c) \left(\partial\phi_c\right)^2 + ... \right)$$

(where the ... indicate terms with more derivatives of ϕ). In particular, in the case where ϕ_c is constant in spacetime we can minimize the *function* $V_{\text{eff}}(\phi_c)$ to find the vacuum. This is a lucrative endeavor which you get to do for homework.

8.3 Coleman-Weinberg potential

[Zee §IV.3, Xi Yin's notes §4.2] Let us now take seriously the lack of indices on our field ϕ, and see about actually evaluating more of the semiclassical expansion of the path integral of a scalar field (eventually we will specify $D = 3+1$):

$$Z[J] = e^{\frac{\mathrm{i}}{\hbar}W[J]} = \int [D\phi] e^{\frac{\mathrm{i}}{\hbar}\left(S[\phi] + \int J\phi\right)} \ . \tag{8.16}$$

To add some drama to this discussion consider the following: if the potential V in $S = \int \left(\frac{1}{2} \left(\partial\phi\right)^2 - V(\phi) \right)$ has a *minimum* at the origin, then we expect that the vacuum

has $\langle\phi\rangle = 0$. If on the other hand, the potential has a *maximum* at the origin, then the field will find a minimum somewhere else, $\langle\phi\rangle \neq 0$. If the potential has a discrete symmetry under $\phi \to -\phi$ (no odd powers of ϕ in V), then in the latter case ($V''(0) < 0$) this symmetry will be broken. If the potential is flat ($V''(0) = 0$) near the origin, what happens? Quantum effects matter.

The configuration of stationary phase is $\phi = \phi_\star$, which satisfies

$$0 = \frac{\delta\left(S + \int J\phi\right)}{\delta\phi(x)}|_{\phi=\phi_\star} = -\partial^2\phi_\star(x) - V'(\phi_\star(x)) + J(x) \ . \tag{8.17}$$

Change the integration variable in (8.16) to $\phi = \phi_\star + \varphi$, and expand in powers of the fluctuation φ:

$$\begin{aligned} Z[J] &= e^{\frac{\mathbf{i}}{\hbar}\left(S[\phi_\star]+\int J\phi_\star\right)} \int [D\varphi] e^{\frac{\mathbf{i}}{\hbar}\int d^Dx \frac{1}{2}\left((\partial\varphi)^2 - V''(\phi_\star)\varphi^2 + \mathcal{O}(\varphi^3)\right)} \\ &\overset{\text{IBP}}{=} e^{\frac{\mathbf{i}}{\hbar}\left(S[\phi_\star]+\int J\phi_\star\right)} \int [D\varphi] e^{-\frac{\mathbf{i}}{\hbar}\int d^Dx \frac{1}{2}\left(\varphi\left(\partial^2 + V''(\phi_\star)\right)\varphi + \mathcal{O}(\varphi^3)\right)} \\ &\approx e^{\frac{\mathbf{i}}{\hbar}\left(S[\phi_\star]+\int J\phi_\star\right)} \frac{1}{\sqrt{\det\left(\partial^2 + V''(\phi_\star)\right)}} \\ &= e^{\frac{\mathbf{i}}{\hbar}\left(S[\phi_\star]+\int J\phi_\star\right)} e^{-\frac{1}{2}\mathrm{tr}\log\left(\partial^2 + V''(\phi_\star)\right)}. \end{aligned}$$

In the second line, we integrated by parts to get the φ integral to look like a souped-up version of the fundamental formula of gaussian integrals – just think of $\partial^2 + V''$ as a big matrix – and in the third line, we did that integral. In the last line we used the matrix identity $\mathrm{tr}\log = \log\det$. Note that all the $\phi_\star$s appearing in this expression are functionals of J, determined by (8.17).

So taking logs of the BHS of the previous equation we have the generating functional:

$$W[J] = S[\phi_\star] + \int J\phi_\star + \frac{\mathbf{i}\hbar}{2}\mathrm{tr}\log\left(\partial^2 + V''(\phi_\star)\right) + \mathcal{O}(\hbar^2) \ .$$

To find the effective potential, we need to Legendre transform to get a functional of ϕ_c:

$$\phi_c(x) = \frac{\delta W}{\delta J(x)} \overset{\text{chain rule}}{=} \int d^Dz \frac{\delta\left(S[\phi_\star] + \int J\phi_\star\right)}{\delta\phi_\star(z)} \frac{\delta\phi_\star(z)}{\delta J(x)} + \phi_\star(x) + \mathcal{O}(\hbar) \overset{(8.17)}{=} \phi_\star(x) + \mathcal{O}(\hbar) \ .$$

The 1PI effective action is then:

$$\Gamma[\phi_c] \equiv W - \int J\phi_c = S[\phi_c] + \frac{\mathbf{i}\hbar}{2}\mathrm{tr}\log\left(\partial^2 + V''(\phi_c)\right) + \mathcal{O}(\hbar^2).$$

To leading order in $\hbar$, we just plug in the solution; to next order we need to compute the sum of the logs of the eigenvalues of a differential operator. This is challenging in

general. In the special case that we are interested in ϕ_c which is constant in spacetime, it is doable. This case is also often physically relevant if our goal is to solve (8.8) to find the groundstate, which often preserves translation invariance (gradients cost energy). If $\phi_c(x) = \phi$ is spacetime-independent then we can write

$$\Gamma[\phi_c(x) = \phi] \equiv \int d^D x \; V_{\text{eff}}(\phi).$$

The computation of the trace-log is doable in this case because it is translation invariant, and hence we can use fourier space. We do this next.

8.3.1 The one-loop effective potential

The tr in the one-loop contribution is a trace over the space on which the differential operator (≡big matrix) acts; it acts on the space of scalar fields φ:

$$\left(\left(\partial^2 + V''(\phi)\right)\varphi\right)_x = \sum_y \left(\partial^2 + V''(\phi)\right)_{xy} \varphi_y \equiv \left(\partial_x^2 + V''(\phi)\right)\varphi(x)$$

with matrix element $\left(\partial^2 + V''\right)_{xy} = \delta^D(x-y)\left(\partial_x^2 + V''\right)$. (Note that in these expressions, we've assumed ϕ is a background field, not the same as the fluctuation φ – this operator is linear. Further we've assumed that that background field ϕ is a constant, which greatly simplifies the problem.) The trace can be represented as a position integral:

$$\text{tr}\bullet = \int d^D x \, \langle x| \bullet |x\rangle$$

so

$$\begin{aligned}
\text{tr}\log\left(\partial^2 + V''(\phi)\right) &= \int d^D x \, \langle x| \log\left(\partial^2 + V''\right)|x\rangle \\
&= \int d^D x \int \text{đ}^D k \int \text{đ}^D k' \, \langle x|k'\rangle \langle k'| \log\left(\partial^2 + V''\right)|k\rangle \langle k|x\rangle \qquad \left(\mathbb{1} = \int \text{đ}^D k \, |k\rangle\langle k|\right) \\
&= \int d^D x \int \text{đ}^D k \int \text{đ}^D k' \, \langle x|k'\rangle \langle k'| \log\left(-k^2 + V''\right)|k\rangle \langle k|x\rangle \\
&\qquad\qquad \left(\langle k'| \log\left(-k^2 + V''\right)|k\rangle = \delta^D(k-k') \log\left(-k^2 + V''\right)\right) \\
&= \int d^D x \int \text{đ}^D k \log\left(-k^2 + V''\right), \qquad \left(\| \, \langle x|k\rangle \, \|^2 = 1\right)
\end{aligned}$$

The $\int d^D x$ goes along for the ride and we conclude that

$$V_{\text{eff}}(\phi) = V(\phi) - \frac{\mathbf{i}\hbar}{2} \int \text{đ}^D k \log\left(k^2 - V''(\phi)\right) + \mathcal{O}(\hbar^2).$$

What does it mean to take the log of a dimensionful thing? It means we haven't been careful about the additive constant (constant means independent of ϕ). And we don't need to be (unless we're worried about dynamical gravity); so let's choose the constant so that

$$V_{\text{eff}}(\phi) = V(\phi) - \frac{\mathbf{i}\hbar}{2}\int \text{đ}^D k \log\left(\frac{k^2 - V''(\phi)}{k^2}\right) + \mathcal{O}(\hbar^2). \tag{8.18}$$

$\boxed{V_{\mathbf{1\ loop}} = \sum_{\vec{k}} \frac{1}{2}\hbar\omega_{\vec{k}}}$. Here's the interpretation of the 1-loop potential: $V''(\phi)$ is the mass^2 of the field when it has the constant value ϕ; the one-loop term $V_{1\text{ loop}}$ is the vacuum energy $\int d^{D-1}\vec{k}\frac{1}{2}\hbar\omega_{\vec{k}}$ from the gaussian fluctuations of a field with that mass^2; it depends on the field because the mass depends on the field.

[Zee II.5.3] Why is $V_{1\text{ loop}}$ the vacuum energy? Recall that $k^2 \equiv \omega^2 - \vec{k}^2$ and $\text{đ}^D k = \text{đ}\omega\text{đ}^{D-1}\vec{k}$. Consider the integrand of the spatial momentum integrals: $V_{1\text{ loop}} = -\mathbf{i}\frac{\hbar}{2}\int \text{đ}^{D-1}\vec{k}\mathcal{I}$, with

$$\mathcal{I} \equiv \int \text{đ}\omega \log\left(\frac{k^2 - V''(\phi) + \mathbf{i}\epsilon}{k^2 + \mathbf{i}\epsilon}\right) = \int \text{đ}\omega \log\left(\frac{\omega^2 - \omega_k^2 + \mathbf{i}\epsilon}{\omega^2 - \omega_{k'}^2 + \mathbf{i}\epsilon}\right)$$

with $\omega_k = \sqrt{\vec{k}^2 + V''(\phi)}$, and $\omega_{k'} = |\vec{k}|$. The $\mathbf{i}\epsilon$ prescription is as usual inherited from the euclidean path integral. Notice that the integral is convergent – at large ω, the integrand goes like

$$\log\left(\frac{\omega^2 - A}{\omega^2 - B}\right) = \log\left(\frac{1 - \frac{A}{\omega^2}}{1 - \frac{B}{\omega^2}}\right) = \log\left(1 - \frac{A-B}{\omega^2} + \mathcal{O}\left(\frac{1}{\omega^4}\right)\right) \simeq \frac{A-B}{\omega^2}\,.$$

Integrate by parts:

$$\begin{aligned}\mathcal{I} = \int \text{đ}\omega \log\left(\frac{k^2 - V''(\phi) + \mathbf{i}\epsilon}{k^2 + \mathbf{i}\epsilon}\right) &= -\int \text{đ}\omega\omega\partial_\omega \log\left(\frac{\omega^2 - \omega_k^2}{\omega - \omega_{k'}}\right) \\ &= -2\int \text{đ}\omega\omega\left(\frac{\omega}{\omega^2 - \omega_k^2 + \mathbf{i}\epsilon} - (\omega_k \to \omega_{k'})\right) \\ &= -\mathbf{i}2\omega_k^2\left(\frac{1}{-2\omega_k}\right) - (\omega_k \to \omega_{k'}) = \mathbf{i}\left(\omega_k - \omega_{k'}\right).\end{aligned}$$

This is what we are summing (times $-\mathbf{i}\frac{1}{2}\hbar$) over all the modes $\int \text{đ}^{D-1}\vec{k}$.

8.3.2 Renormalization of the effective action

So we have a cute expression for the effective potential (8.18). Unfortunately it seems to be equal to infinity. The problem, as usual, is that we assumed that the parameters in

the bare action $S[\phi]$ could be finite without introducing any cutoff. Let us parametrize (following Zee §IV.3) the action as $S = \int d^D x \mathcal{L}$ with

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4!}\lambda\phi^4 - A(\partial\phi)^2 - B\phi^2 - C\phi^4$$

and we will think of A, B, C as *counterterms*, in which to absorb the cutoff dependence.

So our effective potential is actually:

$$V_{\text{eff}}(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4!}\lambda\phi^4 + B(\Lambda)\phi^2 + C(\Lambda)\phi^4 + \frac{\hbar}{2}\int^{\Lambda} đ^D k_E \log\left(\frac{k_E^2 + V''(\phi)}{k_E^2}\right) ,$$

(notice that A drops out in this special case with constant ϕ). We rotated the integration contour to euclidean space. This permits a nice regulator, which is just to limit the integration region to $\{k_E | k_E^2 \leq \Lambda^2\}$ for some big (Euclidean) wavenumber Λ.

Now let us specify to the case of $D = 4$, where the model with $\mu = 0$ is classically scale invariant. The integrals are elementary[52]

$$V_{\text{eff}}(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4!}\lambda\phi^4 + B(\Lambda)\phi^2 + C(\Lambda)\phi^4 + \frac{\Lambda^2}{32\pi^2}V''(\phi) - \frac{(V''(\phi))^2}{64\pi^2}\log\frac{\sqrt{e}\Lambda^2}{V''(\phi)}.$$

Notice that the leading cutoff dependence of the integral is Λ^2, and there is also a subleading logarithmically-cutoff-dependent term. ("log divergence" is certainly easier to say.)

Luckily we have two counterterms. Consider the case where V is a quartic polynomial; then V'' is quadratic, and $(V'')^2$ is quartic. In that case the two counterterms are in just the right form to absorb the Λ dependence. On the other hand, if V were sextic (recall that this is in the non-renormalizable category according to our dimensional analysis), we would have a fourth counterterm $D\phi^6$, but in this case $(V'')^2 \sim \phi^8$, and we're in trouble (adding a bare ϕ^8 term would produce $(V'')^2 \sim \phi^{12}$... and so on). We'll need a better way to think about such non-renormalizable theories. The better way (which we will return to in the next section) is simply to recognize that in non-renormalizable theories, *the cutoff is real* – it is part of the definition of the field theory. In renormalizable theories, we may pretend that it is not (though it usually is real there, too).

Renormalization conditions. Return to the renormalizable case, $V = \lambda\phi^4$ where we've found

$$V_{\text{eff}} = \phi^2\left(\frac{1}{2}\mu^2 + B + \lambda\frac{\Lambda^2}{64\pi^2}\right) + \phi^4\left(\frac{1}{4!}\lambda + C + \frac{\lambda^2}{16\pi^2}\log\frac{\phi^2}{\Lambda^2}\right) + \mathcal{O}(\lambda^3) .$$

[52] This is not the same as 'easy'. The expressions here assume that $\Lambda \gg V''$.

(I've absorbed an additive $\log\sqrt{e}$ in C.) The counting of counterterms works out, but how do we determine them? We need to impose *renormalization conditions*, *i.e.* specify some observable quantities to parametrize our model, in terms of which we can eliminate the silly letters in the lagrangian. We need two of these. Of course, what is observable depends on the physical system at hand. Let's suppose that we can measure some properties of the effective potential. For example, suppose we can measure the mass^2 when $\phi = 0$:

$$\mu^2 = \frac{\partial^2 V_{\text{eff}}}{\partial\phi^2}|_{\phi=0} \qquad \Longrightarrow \text{ we should set } B = -\lambda\frac{\Lambda^2}{64\pi^2}.$$

For example, we could consider the case $\mu = 0$, when the potential is flat at the origin. With $\mu = 0$, have

$$V_{\text{eff}}(\phi) = \left(\frac{1}{4!}\lambda + \frac{\lambda^2}{(16\pi)^2}\log\frac{\phi^2}{\Lambda^2} + C(\Lambda)\right)\phi^4 + \mathcal{O}(\lambda^3)\ .$$

And for the second renormalization condition, suppose we can measure the quartic term

$$\lambda_M = \frac{\partial^4 V_{\text{eff}}}{\partial\phi^4}|_{\phi=M}\ . \tag{8.19}$$

Here M is some arbitrarily chosen quantity with dimensions of mass. We run into trouble if we try to set it to zero because of $\partial_\phi^4(\phi^4\log\phi) \sim \log\phi$. So the coupling depends very explicitly on the value of M at which we set the renormalization condition. Let's use (8.19) to eliminate C:

$$\lambda(M) \stackrel{!}{=} 4!\left(\frac{\lambda}{4!} + C + \left(\frac{\lambda}{16\pi}\right)^2\left(\log\frac{\phi^2}{\Lambda^2} + c_1\right)\right)|_{\phi=M} \tag{8.20}$$

(where c_1 is a numerical constant that you should determine) to get

$$V_{\text{eff}}(\phi) = \frac{1}{4!}\lambda(M)\phi^4 + \left(\frac{\lambda(M)}{16\pi}\right)^2\left(\log\frac{\phi^2}{M^2} - c_1\right)\phi^4 + \mathcal{O}(\lambda(M)^3).$$

Here I used the fact that we are only accurate to $\mathcal{O}(\lambda^2)$ to replace $\lambda = \lambda(M) + \mathcal{O}(\lambda(M)^2)$ in various places. We can feel a sense of victory here: the dependence on the cutoff has disappeared. Further, the answer for V_{eff} does not depend on our renormalization point M:

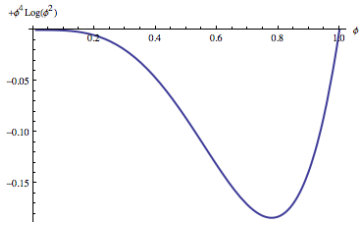
$$M\frac{d}{dM}V_{\text{eff}} = \frac{1}{4!}\phi^4\left(M\partial_M\lambda - \frac{2}{M}\frac{\lambda^2}{(16\pi^2)} + \mathcal{O}(\lambda^3)\right) = \mathcal{O}(\lambda^3) \tag{8.21}$$

which vanishes to this order from the definition of $\lambda(M)$ (8.20), which implies

$$M\partial_M\lambda(M) = \frac{3}{16\pi^2}\lambda(M)^2 + \mathcal{O}(\lambda^3) \equiv \beta(\lambda).$$

The fact (8.21) is sometimes called the *Callan-Symanzik equation*, the condition that $\lambda(M)$ must satisfy in order that physics be independent of our choice of renormalization point M.

So: when $\mu = 0$ is the $\phi \to -\phi$ symmetry broken by the groundstate? The effective potential looks like the figure at right for $\phi < M$. Certainly it looks like this will push the field away from the origin.



However, the minima lie in a region where our approximations aren't so great. In particular, the next correction looks like:

$$\lambda\phi^4\left(1 + \lambda\log\phi^2 + \left(\lambda\log\phi^2\right)^2 + ...\right)$$

– the expansion parameter is really $\lambda\log\phi$. (I haven't shown this yet, it is an application of the RG, below.) The apparent minimum lies in a regime where the higher powers of $\lambda\log\phi$ are just as important as the one we've kept.

RG-improvement. How do I know the good expansion parameter is actually $\lambda\log\phi/M$? The RG. Define $t \equiv \log\phi_c/M$ and $V_{eff}(\phi_c) = \frac{\phi_c^4}{4!}U(t,\lambda)$. We'll regard U as a running coupling, and t as the RG scaling parameter. Our renormalization conditions are $U(0,\lambda) = \lambda, Z(\lambda) = 1$, these provide initial conditions. At one loop in ϕ^4 theory, there are no anomalous dimensions, $\gamma(\lambda) = \frac{\partial}{\partial M}Z = \mathcal{O}(\lambda^2)$. This makes the RG equations quite simple. The running coupling U satisfies (to this order)

$$\frac{dU}{dt} = \beta(U) = \frac{3U^2}{16\pi^2}$$

which (with the initial condition $U(0,\lambda) = \lambda$) is solved by

$$U(\lambda,t) = \frac{\lambda}{1 - \frac{3\lambda t}{16\pi^2}}.$$

Therefore, the RG-improved effective potential is

$$V_{eff}(\phi_c) = \frac{\phi_c^4}{4!}U(t,\lambda) = \frac{1}{4!}\frac{\lambda\phi_c^4}{1 - \frac{3\lambda}{32\pi^2}\log\frac{\phi_c^2}{M^2}}.$$

The good news: this is valid as long as U is small, and it agrees with our previous answer, which was valid as long as $\lambda \ll 1$ *and* $\lambda t \ll 1$. The bad news is that there is no sign of the minimum we saw in the raw one-loop answer.

By the way, in nearly every other example, there will be wavefunction renormalization. In that case, the Callan-Syzmanzik (CS) equation we need to solve is

$$(-\partial_t + \beta\partial_\lambda + 4\gamma)\, U(t,\lambda) = 0$$

whose solution is

$$U(t,\lambda) = f(U(t,\lambda)) \exp\left(\int_0^t dt' 4\gamma(U(t',\lambda))\right), \quad \partial_t U(t,\lambda) = \beta(U),\ U(0,\lambda) = \lambda.$$

f can be determined by studying the CS equation at $t = 0$. For more detail, see E. Weinberg's thesis.

We can get around this issue by studying a system where the fluctuations producing the extra terms in the potential for ϕ come from some *other* field whose mass depends on ϕ. For example, consider a fermion field whose mass depends on ϕ:

$$S[\psi,\phi] = \int d^D x \bar{\psi}\left(\mathbf{i}\partial\!\!\!/ - m - g\phi\right)\psi$$

– then $m_\psi = m + g\phi$. The $\sum \frac{1}{2}\hbar\omega$s from the fermion will now depend on ϕ (the also have the opposite sign because they come from fermions), and we get a reliable answer for $\langle\phi\rangle \neq 0$ from this phenomenon of *radiative symmetry breaking*. In $D = 1+1$ this is a field theory description of the Peierls instability of a 1d chain of fermions (ψ) coupled to phonons (ψ). Notice that when ϕ gets an expectation value it gives a mass to the fermions. The microscopic picture is that the translation symmetry is spontaneously broken to a twice-as-big lattice spacing, alternating between strong and weak hopping matrix elements. This produces a gap in the spectrum of the tight-binding model. (For a little more, see Zee page 300.)

A second example where radiative symmetry breaking happens is scalar QED. There we can play the gauge coupling and the scalar self-coupling off each other. I'll say a bit more about this example as it's realized in condensed matter below.

Another example which has attracted a lot of attention is the Standard Model Higgs. Its expectation value affects the masses of many fields, and you might imagine this might produce features in its effective potential. Under various (strong) assumptions about what lies beyond the Standard Model, there is some drama here; I recommend Schwarz's discussion on page 748-750.

8.3.3 Useful properties of the effective action

[For a version of this discussion which is better in just about every way, see Coleman, *Aspects of Symmetry* §5.3.7. I also highly recommend all the preceding sections! And the ones that come after. This book is available electronically from the UCSD library.]

V_{eff} as minimum energy with fixed ϕ. Recall that $\langle\phi\rangle$ is the configuration of ϕ_c which extremizes the effective action $\Gamma[\phi_c]$. Even away from its minimum, the effective potential has a useful physical interpretation. It is the natural extension of the interpretation of the potential in classical field theory, which is: $V(\phi)$ = the value of the energy density if you fix the field equal to ϕ everywhere. Consider the space of states of the QFT where the field has a given expectation value:

$$|\Omega\rangle \quad \text{such that} \qquad \langle\Omega|\,\phi(x)\,|\Omega\rangle = \phi_0(x) \quad ; \tag{8.22}$$

one of them has the smallest energy. I claim that its energy is $V_{\text{eff}}(\phi_0)$. This fact, which we'll show next, has some useful consequences.

Let $|\Omega_{\phi_0}\rangle$ be the (normalized) state of the QFT which minimizes the energy subject to the constraint (8.22). The familiar way to do this (familiar from QM, associated with Rayleigh and Ritz)[53] is to introduce Lagrange multipliers to impose (8.22) and the normalization condition and extremize without constraints the functional

$$\langle\Omega|\,\mathbf{H}\,|\Omega\rangle - \alpha\left(\langle\Omega|\Omega\rangle - 1\right) - \int d^{D-1}\vec{x}\beta(\vec{x})\left(\langle\Omega|\,\phi(\vec{x},t)\,|\Omega\rangle - \phi_0(\vec{x})\right)$$

with respect to $|\Omega\rangle$ and the functions on space α, β. [54]

[53] The more familiar thing is to find the state which extremizes $\langle a|\,\mathbf{H}\,|a\rangle$ subject to the normalization condition $\langle a|a\rangle = 1$. To do this, we vary $\langle a|\,\mathbf{H}\,|a\rangle - E\left(\langle a|a\rangle - 1\right)$ with respect to both $|a\rangle$ and the Lagrange multiplier E. The equation from varying $|a\rangle$ says that the extremum occurs when $(\mathbf{H} - E)\,|a\rangle = 0$, *i.e.* $|a\rangle$ is an energy eigenstate with energy E. Notice that we could just as well have varied the simpler thing

$$\langle a|\,(\mathbf{H} - E)\,|a\rangle$$

and found the same answer.

[54] Here is the QM version (*i.e.* the same thing without all the labels): we want to find the extremum of $\langle a|\mathbf{H}|a\rangle$ with $|a\rangle$ normalized *and* $\langle a|\mathbf{A}|a\rangle = A_c$ some fixed number. Then we introduce two Lagrange multipliers E, J and vary without constraint the quantity

$$\langle a|\,(\mathbf{H} - E - J\mathbf{A})\,|a\rangle$$

(plus irrelevant constants). The solution satisfies

$$(\mathbf{H} - E - J\mathbf{A})\,|a\rangle = 0$$

so $|a\rangle$ is an eigenstate of the perturbed hamiltonian $\mathbf{H} - J\mathbf{A}$, with energy E. J is an auxiliary thing, which really depends on our choice A_c, via

$$A_c = \langle a|\mathbf{A}|a\rangle = -\frac{dE}{dJ}.$$

(If you like, we used the Feynman-Hellmann theorem, $\frac{dE}{dJ} = \langle\frac{d\mathbf{H}}{dJ}\rangle$.) The quantity we extremized is

$$\langle a|\mathbf{H}|a\rangle = E + JA_c = E - J\frac{dE}{dJ}\ .$$

This Legendre transform is exactly (the QM analog of) the effective potential.

Clearly the extremum with respect to α, β imposes the desired constraints. Extremizing with respect to $|\Omega\rangle$ gives:

$$\mathbf{H}\,|\Omega\rangle = \alpha\,|\Omega\rangle + \int d^{D-1}\vec{x}\beta(\vec{x})\phi(\vec{x},t)\,|\Omega\rangle \tag{8.23}$$

or

$$\left(\mathbf{H} - \int d^{D-1}\vec{x}\beta(\vec{x})\phi(\vec{x},t)\right)|\Omega\rangle = \alpha\,|\Omega\rangle \tag{8.24}$$

Note that α, β are functionals of ϕ_0. We can interpret the operator $\mathbf{H}_\beta \equiv \mathbf{H} - \int d^{D-1}\vec{x}\beta(\vec{x})\phi(\vec{x},t)$ on the LHS of (8.24) as the hamiltonian with a source β; and α is the groundstate energy in the presence of that source. (Note that that source is *chosen* so that $\langle\phi\rangle = \phi_0$ – it is a functional of ϕ_0.)

This groundstate energy is related to the generating functional $W[J=\beta]$ as we've seen several times – $e^{\mathbf{i}W[\beta]}$ is the vacuum persistence amplitude in the presence of the source

$$e^{\mathbf{i}W[\beta]} = \left\langle 0|\mathcal{T}e^{\mathbf{i}\int\beta\phi}|0\right\rangle = \langle 0_\beta|\,e^{-\mathbf{i}T\mathbf{H}_\beta}\,|0_\beta\rangle = e^{-\mathbf{i}\alpha T} \tag{8.25}$$

where T is the time duration. (If you want, you could imagine that we are adiabatically turning on the interactions for a time duration T.)

The actual extremal energy (of the unperturbed hamiltonian, with constrained expectation value of ϕ) is obtained by taking the overlap of (8.23) with $\langle\Omega|$ (really all the Ωs below are Ω_{ϕ_0}s):

$$\begin{aligned}
\langle\Omega|\,\mathbf{H}\,|\Omega\rangle &= \alpha\,\langle\Omega|\Omega\rangle + \int d^{D-1}\vec{x}\beta(\vec{x})\,\langle\Omega|\phi(\vec{x},t)|\Omega\rangle \\
&= \alpha + \int d^{D-1}\vec{x}\beta(\vec{x})\phi_0(\vec{x}) \\
&\overset{(8.25)}{=} \frac{1}{T}\left(-W[\beta] + \int d^D x\beta(\vec{x})\phi_0(\vec{x})\right) \\
&\overset{\text{Legendre}}{=} -\frac{1}{T}\Gamma[\phi_0] \overset{\phi=\phi_0,\,\text{const}}{=} \int d^{D-1}\vec{x}V_{\text{eff}}(\phi_0).
\end{aligned}$$

Cluster decomposition. The relationship (8.25) between the generating functional $W[J]$ (for time-independent J) and the energy in the presence of the source is very useful. (You've previously used it on the homework to compute the potential between static sources, and to calculate the probability for pair creation in an electric field.) Notice that it gives an independent proof that W only gets contributions from connected amplitudes. Amplitudes with n connected components, $\underbrace{\langle....\rangle\,\langle...\rangle\,\langle...\rangle}_{n\text{ of these}}$, go like T^n (where T is the time duration) at large T. Since $W = -E_J T$ goes like T^1, we conclude that it has one connected component (terms that went like $T^{n>1}$ would

dominate at large T and therefore must be absent). This *extensivity* of W in T is of the same nature as the extensivity in volume of the free energy in thermodynamics.

[Brown, 6.4.2] Another important reason why W must be connected is called the *cluster decomposition property*. Consider a source which has the form $J(x) = J_1(x) + J_2(x)$ where the two parts have support in widely-separated (spacelike separated) spacetime regions. If all the fields are massive, 'widely-separated' means precisely that the distance between the regions is $R \gg 1/m$, much larger than the range of the interactions mediated by ϕ. In this case, measurements made in region 1 cannot have any effect on those in region 2, and they should be uncorrelated. If so, the probability amplitude factorizes

$$Z[J_1 + J_2] = Z[J_1]Z[J_2]$$

which by the magic of logs is the same as

$$W[J_1 + J_2] = W[J_1] + W[J_2].$$

If W were not connected, it would not have this additive property.

There are actually some exceptions to cluster decomposition arising from situations where we prepare an initial state (it could be the groundstate for some hamiltonian) in which there are correlations between the excitations in the widely separated regions. Such a thing happens in situations with spontaneous symmetry breaking, where the value of the field is the same everywhere in space, and therefore correlates distant regions.

Convexity of the effective potential. Another important property of the effective potential is $V''_{\text{eff}}(\phi) > 0$ – the effective potential is *convex* (sometimes called 'concave up'). We can see this directly from our previous work. Most simply, recall that the functional Taylor coefficients of $\Gamma[\phi]$ are the 1PI Green's functions; V_{eff} is just Γ evaluated for constant ϕ, *i.e.* zero momentum; therefore the Taylor coefficients of V_{eff} are the 1PI Green's functions at zero momentum. In particular, $V''_{\text{eff}}(\phi) = \langle \phi_{k=0}\phi_{k=0} \rangle$: the ground state expectation value of the square of a hermitian operator, which is

positive.[55] [56]

On the other hand, it seems that if $V(\phi)$ has a maximum, or even any region of field space where $V''(\phi) < 0$, we get a *complex* one-loop effective potential (from the log of a negative V''). What gives? One resolution is that in this case the minimum energy state with fixed $\langle\phi\rangle$ is not a ϕ eigenstate.

For example, consider a quartic potential $\frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4$ with $m^2 < 0$, with minima at $\phi_\pm \equiv \pm\sqrt{\frac{6|m|^2}{g}}$. Then for $\langle\phi\rangle \in (\phi_-, \phi_+)$, rather we can lower the energy below $V(\phi)$ by considering a state

$$|\Omega\rangle = c_+ |\Omega_+\rangle + c_- |\Omega_-\rangle , \quad \langle\Omega|\phi|\Omega\rangle = |c_+|^2\phi_+ + |c_-|^2\phi_-.$$

The one-loop effective potential at ϕ only knows about some infinitesimal neighborhood of the field space near ϕ, and fails to see this non-perturbative stuff. In fact, the correct effective potential is exactly flat in between the two minima. More generally, if the two minima have unequal energies, we have

$$V_{\text{eff}} = \langle\Omega| \mathbf{H} |\Omega\rangle = |c_+|^2 V(\phi_+) + |c_-|^2 V(\phi_-)$$

– the potential interpolates *linearly* between the energies of the two surrounding minima.

The imaginary part of $V_{1\text{ loop}}$ is a decay rate. If we find that the (perturbative approximation to) effective potential $E \equiv V_{1\text{ loop}}$ is complex, it means that the amplitude for our state to persist is not just a phase:

$$\mathcal{A} \equiv \langle 0| e^{-\mathbf{i}T\mathbf{H}} |0\rangle = e^{-\mathbf{i}E\mathcal{V}T}$$

[55] More explicitly: Begin from $V_{\text{eff}} = -\frac{\Gamma}{\mathcal{V}}$.

$$\frac{\partial}{\partial\phi_0} V_{\text{eff}}(\phi_0) = -\int \frac{d^D x}{\mathcal{V}} \frac{\delta}{\delta\phi(x)} \frac{\Gamma[\phi]}{\mathcal{V}}|_{\phi(x)=\phi_0} \overset{(8.7)}{=} -\frac{1}{\mathcal{V}} \int \frac{d^D x}{\mathcal{V}} \left(-J(x)\right)|_{\phi(x)=\phi_0}.$$

In the first expression here, we are averaging over space the functional derivative of Γ. The second derivative is then

$$\left(\frac{\partial}{\partial\phi_0}\right)^2 V_{\text{eff}}(\phi_0) = \frac{1}{\mathcal{V}} \int \frac{d^D y}{\mathcal{V}} \frac{\delta}{\delta\phi(y)} \int \frac{d^D x}{\mathcal{V}} \left(J(x)\right)|_{\phi(x)=\phi_0} = +\frac{1}{\mathcal{V}^3} \int_y \int_x \frac{\delta J(x)}{\delta\phi(y)}|_{\phi(x)=\phi_0}$$

Using (8.15), this is

$$V''_{\text{eff}} = +\frac{1}{\mathcal{V}^3} \int_y \int_x \left(W_2^{-1}\right)_{xy}$$

– the inverse is in a matrix sense, with x, y as matrix indices. But W_2 is a positive operator – it is the groundstate expectation value of the square of a hermitian operator.

[56] In fact, the whole effective action $\Gamma[\phi]$ is a convex functional: $\frac{\delta^2\Gamma}{\delta\phi(x)\delta\phi(y)}$ is a positive integral operator. For more on this, I recommend Brown, *Quantum Field Theory*, Chapter 6.

has a modulus different from one ($\mathcal{V}$ is the volume of space). Notice that the $|0\rangle$ here is our perturbative approximation to the groundstate of the system, which is *wrong* in the region of field space where $V'' < 0$. The modulus of this object is

$$P_{\text{no decay}} = \| \mathcal{A} \|^2 = e^{-\mathcal{V}T 2\text{Im}\, E}$$

– we can interpret $2\text{Im}\, E$ as the (connected!) decay probability of the state in question per unit time per unit volume. (Notice that this relation means that the imaginary part of $V_{\text{1-loop}}$ had *better* be positive, so that the probability stays less than one! In the one-loop approximation, this is guaranteed by the correct $\mathbf{i}\epsilon$ prescription.)

For more on what happens when the perturbative answer becomes complex and non-convex, and how to interpret the imaginary part, see this paper by E. Weinberg and Wu.
