University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 10 ('Final exam')

Due 12:30pm Wednesday, June 12, 2019

1. **Kondo problem problem.** At last, the long-awaited Kondo problem problem. Consider a spinful Fermi liquid (treat it as free) in d dimensions coupled to a single spin s at the origin by the Kondo interaction

$$H_K = J_K \mathbf{c}^\dagger \vec{\sigma} \mathbf{c} \cdot \vec{\mathbf{S}}.$$

   (a) Make a coherent state path integral representation for both the fermions and the spin. Write the action in Euclidean time.

   (b) Find the Feynman rules for perturbation theory about $J_K = 0$: the propagator for ψ (the fermion coherent state variable), the propagator for z (the spin coherent state variable, $\vec{n} = z^\dagger \vec{\sigma} z$), and the interaction vertex.

   (c) Find the one-loop beta function for the Kondo coupling.

   Two hints about how to proceed: (1) Recall from our previous discussion the methods for doing momentum integrals over functions peaked on a round Fermi surface. (2) Integrate out a shell of momentum modes with $|k| \in (\Lambda/b, \Lambda)$, where Λ is a UV cutoff (the bandwidth), and b is the RG parameter.

   (d) Solve the beta function equation for the running of the coupling, $J(b)$

   (e) Some poetry: at finite temperature, the system explores states in a shell of width T around the Fermi surface. In your solution for $J(b)$ make the replacement $b = \Lambda/(\Lambda/b) = T/T_F$ to find $J_{\text{eff}}(T)$. Assuming that the bare J_K is antiferromagnetic, find the *Kondo temperature* T_K defined by $1 = J_{eff}(T_K)$.

   (f) What happens when the Kondo coupling becomes strong? Unlike QCD, here we can answer this question. Study the limit of the hamiltonian where J_K is the largest scale (so we may ignore the kinetic terms of the fermions at leading order) and find the groundstate.

Below is a collection of problems on CFT (alas, they are all in $D = 2$ dimensions). Do as many as you can.

2. **Bosonization exercise: non-chiral boson.**

In a model where the Hilbert space is made from bosons, there are no local fermionic operators, no matter how much coarse-graining we do or how many dualities we employ. (This statement is an example of imposing the condition that the effective field theory is *emergable* from the microscopic model, as advanced by T. Senthil in various recent talks.) This means, in particular, that if we implement the full bosonization map, starting from bosons, the sign of the fermion operators must be ambiguous: there must be a $\mathbb{Z}_2$ gauge redundancy acting on them. This means that we must sum over their boundary conditions (as we did in the TFIM) and project onto invariant states. (In string theory this process is called the *GSO projection*.)

(a) Consider the CFT of a massless periodic scalar $\phi \equiv \phi + 2\pi$ in $D = 1+1$, with action

$$S[\phi] = \frac{R^2}{8\pi} \int \mathrm{d}x\mathrm{d}t \partial_\mu \phi \partial^\mu \phi. \tag{1}$$

Show that the partition function on a torus of modular parameter $q = e^{2\pi \mathbf{i} \tau}$ [1] is

$$\begin{aligned} Z_R(\tau, \bar{\tau}) &\equiv \mathrm{tr}\, q^{\mathbf{L}_0 - \frac{1}{24}} \bar{q}^{\tilde{\mathbf{L}}_0 - \frac{1}{24}} \\ &= \frac{1}{|\eta|^2} \sum_{n,m \in \mathbb{Z}} q^{\frac{1}{2}\left(\frac{n}{R} + \frac{mR}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{n}{R} - \frac{mR}{2}\right)^2}, \end{aligned} \tag{2}$$

where the Dedekind eta function is

$$\eta(\tau) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

Here $\mathbf{L}_0$ ($\tilde{\mathbf{L}}_0$) is the part of the Hamiltonian acting on ϕ_R (ϕ_L). The $-\frac{1}{24}$s are significant but you may regard them as decorative.

(b) Note that this function is manifestly invariant under the T-duality map:

$$Z_R = Z_{2/R}.$$

[1] If you don't like this fancy language, just consider $\tau = \mathbf{i}/2\pi T$, so this is just the thermal partition sum on a spatial circle of radius $L = 1$:

$$Z_R(T) = \mathrm{tr} e^{-\frac{1}{T}\left(\mathbf{L}_0 + \tilde{\mathbf{L}}_0 - E_0\right)} = e^{E_0/T} \mathrm{tr} e^{-\mathbf{H}/T}.$$

There is a little bit more information if we keep complex τ.

(c) Consider the special radius $R = 1$ (which is equivalent to $R = 2$ by T-duality). Show that at this special radius (not the same as the self-dual radius $R = \sqrt{2}$ from the previous problem!) the partition function is

$$Z_1(\tau, \bar{\tau}) = \frac{1}{2}\frac{1}{|\eta|^2}\left(\left|\sum_n q^{n^2/2}\right|^2 + \left|\sum_n (-1)^n q^{n^2/2}\right|^2 + \left|\sum_n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}\right|^2\right) .$$

(d) Show that this last form of Z is the partition function of a 2d Dirac fermion *coupled to a* $\mathbb{Z}_2$ *gauge field*, *i.e.* with the transformation $\psi \to -\psi$ regarded as a gauge redundancy. So we must sum over boundary conditions (the value of the $\mathbb{Z}_2$ Wilson line) and project onto gauge invariant configurations. The partition sum is therefore:

$$Z_F(\tau, \bar{\tau}) = \frac{1}{2}\left(\mathrm{tr}_{PBC} q^{\mathbf{L}_0 - \frac{1}{24}} \bar{q}^{\tilde{\mathbf{L}}_0 - \frac{1}{24}} \frac{1}{2}\left(1 - (-1)^F\right) + \mathrm{tr}_{APBC} q^{\mathbf{L}_0 - \frac{1}{24}} \bar{q}^{\tilde{\mathbf{L}}_0 - \frac{1}{24}} \frac{1}{2}\left(1 - (-1)^F\right)\right) .$$

Here $\frac{1}{2}\left(1 - (-1)^F\right)$ is the projector onto states invariant under the gauge transformation.

[Hints: (i) $\mathrm{tr}_{PBC} (-1)^F q^{\mathbf{L}_0 - \frac{1}{24}} \bar{q}^{\tilde{\mathbf{L}}_0 - \frac{1}{24}} = 0$ because there is an unsaturated fermion zero-mode.

(ii) The sums in the squares are various theta functions, specifically:

$$\theta_3(\tau) = \vartheta_{00}(0|\tau) = \sum_n q^{n^2/2}$$

$$\theta_4(\tau) = \vartheta_{01}(0|\tau) = \sum_n (-1)^n q^{n^2/2}$$

$$\theta_2(\tau) = \vartheta_{10}(0|\tau) = \sum_n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}$$

which can be expressed also as infinite products (instead of infinite sums), as described *e.g.* on page 215 of Polchinski vol I. Rewrite Z_1 using the product forms of the theta functions.]

(e) (Challenge problem) Show that $Z_R(\tau, \bar{\tau})$ is *modular invariant* – that is, that it does not change if we reparametrize the complex structure of our torus by an $\mathsf{SL}(2, \mathbb{Z})$ transformation:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1.$$

[Use the Poisson resummation formula, or look up the modular transformation rules for the theta functions.]

3. **Non-abelian bosonization.**

This problem is a continuation of the problem on the beta function of the non-linear sigma model. In addition to the kinetic term associated with the metric, we may add a term which is antisymmetric in derivatives:

$$S_G = \frac{1}{4\pi}\int d^2\sigma \left(G_{ab}(X)\partial_\alpha X^a \partial^\alpha X^b\right)$$

$$S_B = \frac{1}{4\pi}\int d^2\sigma \left(B_{ab}(X)\epsilon^{\alpha\beta}\partial_\alpha X^a \partial_\beta X^b\right) = \frac{1}{4\pi}\int_{\text{worldsheet}} B.$$

So $B_{ab}(X)dX^a \wedge dX^b$ is a two-form on the target space.[2]

Now consider the special case of a non-linear sigma model in $D = 1+1$ whose target space is the group $\mathsf{SU}(2)$ with radius R, i.e.

$$S_G = \frac{R^2}{8\pi}\int d^2\sigma \mathrm{tr} g^{-1}\partial_\alpha g g^{-1}\partial^\alpha g$$

and consider $B_{ab}(X)$ such that

$$H = dB = k\epsilon_{abc}dX^a \wedge dX^b \wedge dX^c$$

where ϵ_{abc} are the structure constants of $\mathsf{SU}(2)$. The relation between g and X is $g = e^{\mathbf{i}T^a X^a}$. Notice that this preserves the $\mathsf{SU}(2)$ symmetry.

(a) Argue that k must be quantized (hint: write S_B as a 3-dimensional integral). Therefore it cannot run under the RG.

(b) We saw previously that in the absence of S_B, S_Φ, the beta function for the metric depended on the curvature. If $B = \Phi = 0$, what happens in the infrared? Hint: $\mathsf{SU}(2) \simeq S^3$.

(c) How does the coupling k affect the beta function for the metric (work to second order in k)? Argue for the existence of a fixed point.

(d) [optional] Find $\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R$ conserved currents in terms of the group-valued variables $g(\sigma)$.

4. **Linear dilaton CFT.**

[2] Another coupling we can add is a coupling to the background curvature of the worldsheet:

$$S_\Phi = \int d^2\sigma R^{(2)}\Phi(X).$$

If the *dilaton field*, $\Phi(X)$ is constant, this is a total derivative.

The *linear dilaton theory* is a 2d CFT made from a free boson in the presence of a 'background charge.' This means that the boson X has some funny coupling to the worldsheet gravity, which can be described by a linear dilaton term in the action $S_\Phi = \int d^2\sigma \Phi(X) R^{(2)}$, $\Phi(X) = QX$, Q is a constant. On a flat worldsheet, the quantization of X proceeds as before. In that case, the only difference from the ordinary free boson is that the stress tensor (which is sensitive to how the theory is coupled to gravity) has the form

$$T_Q = -\frac{1}{\alpha'} : \partial X \partial X : + V \partial^2 X$$

(where $V \sim Q$).

[Optional: relate V to Q.]

(a) Verify that T_Q has the right OPE with itself to be the stress tensor for a CFT. Compute the Virasoro central charge for the linear dilaton theory. (Check that the answer reduces to the free boson answer when $V = 0$.)

(b) Compute the scaling dimension of the operator $: e^{ikX} :$ in the linear dilaton theory.

5. **The stress tensor is not a conformal primary if $c \neq 0$.**

(a) For any 2d CFT, use the general form of the TT OPE to show that the transformation of T under an infinitesimal conformal transformation $z \mapsto z + \xi(z)$ is

$$\delta_\xi T(w) = (\xi \partial + 2\partial \xi) T(w) + \frac{c}{12} \partial^3 \xi. \qquad (1)$$

(b) Consider the *finite* conformal transformation $z \mapsto f(z)$. Show that (1) is the infinitesimal version of the transformation law

$$T(z) \mapsto (\partial f)^2 T\left(f(z)\right) + \frac{c}{12} \{f, z\}$$

where

$$\{f, z\} \equiv \frac{\partial f \partial^3 f - \frac{3}{2} (\partial^2 f)^2}{(\partial f)^2}$$

is called a *Schwarzian derivative.*

[Optional: verify that this extra term does the right thing when composing two maps $z \to f(z) \to g(f(z))$.]

(c) Given that the conformal map from the cylinder to the plane is $z = e^{-iw}$, show that (b) means that

$$T_{\text{cyl}}(w)(dw)^2 = \left(T_{\text{plane}}(z) - \frac{c}{24} \right) (dz)^2.$$

Use this relation to show that the Hamiltonian on the cylinder

$$H = \int \frac{d\sigma}{2\pi} T_{\tau\tau}$$

is

$$H = L_0 + \tilde{L}_0 - \frac{c + \bar{c}}{24}.$$

Comment: After all this complication, the result has a very simple physical interpretation: when putting a CFT on a cylinder, the scale invariance is spontaneously broken by the fact that the cylinder has a *radius*, *i.e.* the cylinder introduces a (worldsheet) length scale into the problem. The term in the energy extensive in the radius of the cylinder but not the length (and proportional to c) is actually experimentally observable sometimes.

6. [This problem is a repeat from HW09.] **OPE and mode commutator algebra.** [Bonus tedium] This is a continuation of the problem on the previous homework about the free boson at the $\mathfrak{SU}(2)$ radius. Defining modes of the current (as usual for a dimension 1 operator) by

$$J^a(z) = \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1}$$

show (from the current-current OPE) that

$$[J^a_m, J^b_n] = i\epsilon^{abc} J^c_{m+n} + mk\delta^{ab}\delta_{m+n}$$

with $k = 1$, which is an algebra called Affine $SU(2)$ at level $k = 1$. Note that the $m = 0$ modes satisfy the ordinary $SU(2)$ lie algebra.

7. **Constraints from Unitarity.** Show that in a unitary CFT, $c > 0$, and $h \geq 0$ for all primaries. Hint: consider $\langle\phi| [L_n, L_{-n}] |\phi\rangle$.