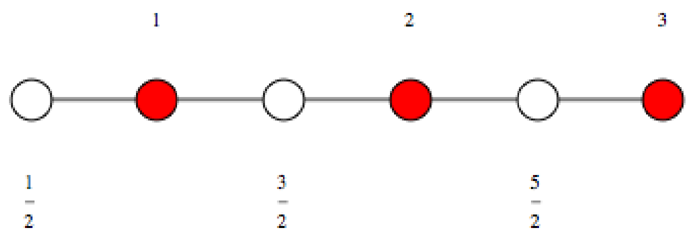
University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 8 – Solutions

Due 12:30pm Wednesday, May 29, 2019

1. **Brain-warmer on counting states.**

   Consider the gauge theory description of the transverse field Ising model with $N = 3$ sites.

   With open boundary conditions, find an explicit expression for the gauge transformation $\{s_{\bar{j}}\}$ which eliminates the original $\mathbf{X}_j$ variables (sets them to 1).

   Since $\mathbf{X}_{\bar{j}} \to s_{\bar{j}-\frac{1}{2}} \mathbf{X}_{\bar{j}} s_{\bar{j}+\frac{1}{2}}$, we require

   $$1 = s_{\frac{1}{2}} \mathbf{X}_1 s_{\frac{3}{2}}, \; 1 = s_{\frac{3}{2}} \mathbf{X}_2 s_{\frac{5}{2}}, \; 1 = s_{\frac{5}{2}} \mathbf{X}_3$$

   – there is no site to the right of $\mathbf{X}_3$. So set $s_{5/2} = \mathbf{X}_3$, $s_{3/2} = \mathbf{X}_2\mathbf{X}_3$, $s_{\frac{1}{2}} = \mathbf{X}_1\mathbf{X}_2\mathbf{X}_3$ and we're done.

   With closed boundary conditions, show that you cannot eliminate $\prod_j \mathbf{X}_j$. Find an expression for the gauge choice which eliminates all but one of the spins.

   Nothing we do with our silly ss can eliminate $\prod_j \mathbf{X}_j$ because it is gauge invariant.

2. **Non-linear sigma models on more general spaces.** [Some knowledge of differential geometry is helpful here.]

   In lecture we considered the 2d non-linear sigma model whose target space was a round 2-sphere, motivated by the low-energy physics of antiferromagnets. At weak coupling (large radius of sphere, which means large spin), we saw that the sphere wants to shrink in the IR.

   Consider now a 2d non-linear sigma model (NLSM) whose target space is a more general manifold X with Riemannian metric $ds^2 = L^2 g_{ij}(x) dx^i dx^j$. Assume that the space is *big*, in the sense that we will treat the parameter L^{-1} as a small parameter, and *smooth* in the sense that we can Taylor expand around any point.

The NLSM is a field theory whose fields $x^i(\sigma)$ are maps from spacetime (here 2d flat space) to the *target space* X. The simplest action is

$$S[x(\sigma)] = \int d^2\sigma L^2 g_{ij}(x)\partial_{\sigma^\mu} x^i \partial_{\sigma^\nu} x^j \eta^{\mu\nu}$$

where $\eta^{\mu\nu}$ is the flat metric on the 2d spacetime 'worldsheet'.

$D = 2$ is special because the free scalar field $x(\sigma)$ is dimensionless. As long as g_{ij} is nonsingular, in the limit $L \to \infty$, the local coordinate field becomes free.

Regard $g_{ij}(x)$ as a coupling *function*. What is the leading beta function (actually beta functional) for this set of couplings?

Hint: use the fact that the answer must be covariant under changes of coordinates on X plus dimensional analysis.

The beta function vanishes when the target-space metric is flat, since the theory is free. We can organize the perturbation expansion as an expansion in powers of the curvature. It must be a two-index symmetric tensor. Therefore the leading term is:

$$\beta_{\mu\nu} = \partial_t g_{\mu\nu} \propto \mathcal{R}_{\mu\nu} + \mathcal{O}\left(\mathcal{R}^2\right)$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci curvature of the target space. To match the coefficient, compare to the case where the target space is S^2, which we did in lecture. For an S^2 of radius R, $\mathcal{R}_{ij} = \frac{1}{R^2} g_{ij}$. We found

$$L(b) = \frac{1}{2}\left(g^2 + \frac{g^4}{2\pi}\log b + \cdots\right)^{-1} \partial_\mu \hat{n}^a \partial^\mu \hat{n}.$$

Identifying

$$g_{ij}(b)\partial X^i \partial X^j = g^2(b)\partial_\mu \hat{n}^a \partial^\mu \hat{n}$$

and $g = \frac{1}{R}$, we have

$$\beta_{g_{ij}} = \partial_{\log b} g_{ij}(b) = \frac{1}{2\pi R^2} g_{ij} = \frac{1}{2\pi}\mathcal{R}_{ij}.$$

To give a hint toward a harder-working answer, expand the metric about a point in field space using Riemann normal coordinates, so that the action looks like

$$L = (\partial x)^2 + R_{ijkl} x^i x^j \partial_\alpha x^k \partial^\alpha x^l + \ldots.$$

This is a quartic interaction in which we may do perturbation theory.

A comment on dimensional analysis: in a non-linear sigma model, there are actually two independent sets of engineering dimensions, one set for the base space $(\sigma, \partial_\sigma ...)$ – this is ordinary dim'l analysis for a QFT in $D = 1+1$ – and another set for the target space $(X, \mathcal{R}_{ij}...)$. By writing the action

$$S = \int d^2\sigma \partial X^i \partial X^j g_{ij}$$

I am treating the target space coordinates as dimensionless. Alternatively, we could introduce a quantity with units of target-space-length^{-2} in front of the action (called $\frac{1}{4\pi\alpha'}$) and write

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial X^i \partial X^j g_{ij}.$$

The previous action works in target-space units where $\frac{1}{4\pi\alpha'} = 1$. This quantity is a string tension.

3. **Haldane phase from the path integral.**

   Consider the $D = 1+1$ nonlinear sigma model with target space S^2 at $\theta = 2\pi$. Recall that this describes a spin-one antiferromagnetic chain. The θ term is a total derivative in the action, so it can manifest itself when we study the path integral on a spacetime with boundary.

   (a) Put this field theory on the half-line $x > 0$. Suppose that the boundary conditions respect the $\mathsf{SO}(3)$ symmetry, so that the boundary values $\vec{n}(\tau, x = 0)$ are free to fluctuate. By remembering that the θ-term is a total derivative, and considering the strong-coupling (IR) limit, $g \to \infty$, show that there is a spin-$\frac{1}{2}$ at the boundary. (Hint: Recall the coherent state path integral for a spin-$\frac{1}{2}$.)

   See after my answer to the next part.

   (b) Now cut the path integral open at some fixed euclidean *time* $\tau = 0$. (Consider periodic boundary conditions in space.) Such a path integral computes the groundstate wavefunction, as a function of the boundary values of the fields, $\vec{S}(x, \tau = 0)$. Find the groundstate wavefunctional is $\Psi[\vec{n}(x, \tau = 0)]$ in the strong coupling limit $g \to \infty$ (where the gap is big).

Take $D = 1+1$, $G = \mathrm{SO}(3)$, and let's study a field variable which is a 3-component unit vector $\hat{n} \in S^2$. The fact that $\pi_2(S^2) = \mathbb{Z}$ will play an important role. Think of this $\vec{n}$ as arising from coherent-state quantization of a spin chain. So take the (imaginary-time) action to be

$$S = \int \mathrm{d}\tau \mathrm{d}x \left(\frac{1}{g^2} \partial_\mu \hat{n} \cdot \partial^\mu \hat{n} + \mathbf{i} \frac{\theta}{4\pi} \epsilon_{abc} n^a \partial_\tau n^b \partial_x n^c \right).$$

We will focus on $\theta \in 2\pi\mathbb{Z}$. Recall that in this case the model has a gap. We would like to understand what is different between $\theta = 0$ and $\theta = 2\pi$.

Recall the role of the θ term: on a closed spacetime manifold M_D

$$Z_\theta(M_D) \equiv \int [Dn] e^{-S} = \sum_{n \in \pi_2(S^2)} e^{\mathbf{i}\theta n} Z_n$$

and $Z_\theta(M_D) = Z_{\theta+2\pi}(M_D)$. In particular, we can take $M_D = S^1 \times N_{D-1}$ to compute the partition function on any spatial manifold N_{D-1}. This means the bulk spectrum is periodic in θ with period 2π.

In contrast to the case of a closed manifold, if we compute the path integral on an (infinite) cylinder (*i.e.* with two boundaries, at $\tau = \pm\infty$),

$$\int_{\substack{\hat{n}(x,\tau=\infty)=\hat{n}(x) \\ \hat{n}(x,\tau=-\infty)=\hat{n}'(x)}} [D\hat{n}(x,\tau)] e^{-S[n(x,\tau)]} = \langle \hat{n}(x)|0\rangle \langle 0|\hat{n}'(x)\rangle \tag{1}$$

then θ does matter, not just mod 2π. Notice that in expressions for functionals like $S[n(x,\tau)]$ I am writing the arguments of the function n to emphasize whether it is a function at fixed euclidean time or not. The fact that the theta term is a total derivative $\mathbf{i}\theta\mathcal{H} = \mathbf{i}\theta \int_{\partial M_D} w$ means that the euclidean action here is

$$S[n(x,\tau)] = \int_{M_D} \mathrm{d}\tau \mathrm{d}x \frac{1}{g^2} \partial_\mu \hat{n} \cdot \partial^\mu \hat{n} + \mathbf{i}\theta \int \mathrm{d}x \left(w(n(x)) - w(n'(x)) \right).$$

The θ term only depends on the boundary values, and comes out of the integral in (1).

If we also take $g \to \infty$ there is nothing left in the integral and we can factorize the expression (1) to determine:

$$|0\rangle_{\theta=0} \propto \int [D\hat{n}(x)] \, |\hat{n}(x)\rangle = \prod_x |\ell = 0\rangle_x$$

is a product state; on the far RHS here we have a product of local singlets: $\int d\hat{n} \, |\hat{n}\rangle = |\ell = 0\rangle$ at each site. Here we used $\langle \hat{n}(x)|\hat{n}'(x)\rangle = \delta[n - n']$.

$$|0\rangle_{\theta=2\pi} \propto \int [D\hat{n}(x)] e^{\mathbf{i}\frac{\theta}{2\pi}\int \mathrm{d}x w(\hat{n}(x))} \, |\hat{n}(x)\rangle = \int [D\vec{n}(x)] e^{\mathbf{i}\mathcal{W}[\vec{n}]} \, |\vec{n}(x)\rangle$$

Here $\int \mathrm{d}x w(\hat{n}(x)) = \mathcal{W}_0[\hat{n}]$ is the $D = 0 + 1$ WZW term; the role of the usually-fictitious extra dimension in writing this WZW term is now being played by the *real* euclidean time (and the role of time is played by space).

So what we found was that the groundstate wavefunctional is $\Psi[n] \propto e^{\mathbf{i}\theta Q[n]}$ where

$$Q[n] = \frac{1}{4\pi} \int_{bulk} n \wedge dn \wedge dn$$

as a functional of the boundary configuration $n(x)$. But this Q is exactly the WZW term $W_0[n]$ for the boundary configuration $n(x)$ just regard the euclidean time direction as the fictitious extra dimension, and the bulk field as the arbitrary extension of the boundary field.

By the usual Witten argument, the coefficient of this term must be quantized as long as the $\mathsf{SU}(2)$ symmetry is preserved, so a phase with a different coefficient of the WZW term must be separated by a phase transition.

And indeed when $s = 1$, so that $\theta = 2\pi s = 2\pi$, we get the minimal nonzero coefficient. (If we plug in $s = 1/2$ we get a fractional coefficient, but this is OK because in that case the bulk is gapless.)

Here is a paper where this same strategy is applied to bosonic SPTs in 2 and 3 dimensions.

For a more concrete statement, consider breaking the $\mathsf{SU}(2)$ symmetry down to an easy-plane $\mathsf{U}(1)$ symmetry, meaning we add $\Delta H = \sum_x J n_z^2$ which pins the field to the equator. In that case, the WZW term reduces to

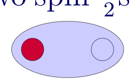
$$\theta \int dx \frac{1}{2}(1 - \cos\theta)\partial_x \varphi = \frac{1}{2} 2\pi s \int dx \partial_x \varphi$$

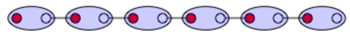
which measures the *winding* of the φ variable around the boundary. When $s \in \mathbb{Z}$, the wavefunction is

$$\Psi[n] = e^{\mathbf{i}\pi s \Delta\varphi} = (-1)^{\text{winding of } \varphi \,:\, S^1 \to S^1} \ .$$

(Again, for s a half-integer, the bulk is gapped, so we don't care what happens.) As long as we preserve time-reversal symmetry (so the wavefunction is real), we cannot continuously interpolate between this wavefunction and the trivial paramagnet with $\Psi[n] = 1$.

In the other part of the problem, the boundary is in space. With free ($\mathsf{SO}(3)$-symmetric) boundary conditions on the field at the boundary, the boundary action is exactly the WZW term, which says that there is an edge spin of spin $\frac{1}{2}$, despite the fact that the degrees of freedom only transform under $\mathsf{SO}(3)$.

Here is a pictorial understanding of these spin-$\frac{1}{2}$ edge states. We can make a spin-1 by tensoring together two spin-$\frac{1}{2}$s, and projecting onto the triplet; represent this by the following picture:  . Consider a chain of spin-1s and make singlets between the right spin-$\frac{1}{2}$ in each site with the left spin-$\frac{1}{2}$ of the neighboring site, like this:



This is called the AKLT state, and it's in the same phase as the groundstate of the spin-1 Heisenberg chain. You can see with your eyes the dangling spin-$\frac{1}{2}$s.