University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 7

Due 12:30pm Monday, May 20, 2019

1. **Spin and statistics of a dyon.**

   (a) Consider a magnetic monopole of strength g, so that $\vec{B} = \frac{\hat{r}g}{2r^2}$, and $\oint \vec{B} \cdot d\vec{A} = 2\pi g$. Now consider a particle of charge q in this field. Show that the usual angular momentum $m\vec{r} \times \vec{v}$ is not conserved (the EM field carries angular momentum). Show that instead the quantity

   $$\vec{L} = m\vec{r} \times \vec{v} - qg\hat{r}$$

   is conserved. Suppose there is a bound state of two such particles with the minimal charges satisfying Dirac quantization. Interpret the extra term as a contribution to the intrinsic spin of the dyon.

   (b) To confirm that the dyon has fermionic statistics, consider the wavefunction of two such dyons, $\psi(x_1, x_2)$. The exchange of the two dyons can be accomplished by a π-rotation about $\vec{x}_1$, followed by a translation by $\vec{x}_1 - \vec{x}_2$. By analyzing the Aharonov-Bohm phases, show that this process produces a phase Φ

   $$\psi(x_2, x_1) = e^{\mathbf{i}\Phi}\psi(x_1, x_2)$$

   with $\Phi = \pi$ in the case where q, g saturate the Dirac quantization condition.

2. **Jordan-Wigner.**

   Solve the following spin chains using the mapping to Majorana fermions.

   (a) **XY-model.**

   $$\mathbf{H} = -J \sum_j \left(\mathbf{Z}_j \mathbf{Z}_{j+1} + \mathbf{Y}_j \mathbf{Y}_{j+1} \right)$$

   This model has a U(1) symmetry which rotates $\mathbf{Z}$ into $\mathbf{Y}$, *i.e.* acting by $U = e^{\mathbf{i}\alpha \mathbf{X}}$. How does it act on the fermions?

(b) **Solve an interacting fermion system.**

$$\mathbf{H}_{\text{int}} = -J \sum_j \left(\mathbf{X}_j \mathbf{X}_{j+1} + \mathbf{Y}_j \mathbf{Y}_{j+1} \right) \tag{1}$$

This model is in fact related by a basis rotation ($\mathbf{U} = \prod_j e^{\mathbf{i}\frac{\pi}{4}\mathbf{Y}_j}$) to the one in part 2a.

But if you directly use the mapping we introduced in class in these variables, you'll find quartic terms in the fermions.

The basis transformation above therefore solves this interacting fermion system.

How does the $\mathsf{U}(1)$ symmetry of (1) act on these fermion variables?

(c) **A spin chain with a non-onsite Ising symmetry.**

Consider the Hamiltonian

$$\mathbf{H} = -J \sum_j \left(\mathbf{X}_j + \lambda \mathbf{Z}_{j-1} \mathbf{X}_j \mathbf{Z}_{j+1} \right)$$

i. [Slightly more optional] Show that when $\lambda = -1$ this model is invariant under the action of

$$\mathbf{S}_1 \equiv \prod_j \mathbf{X}_j \prod_j e^{\mathbf{i}\frac{\pi}{4}\mathbf{Z}_j \mathbf{Z}_{j+1}}. \tag{2}$$

This symmetry is "not-onsite" in that its action on the spin at site j depends on the state of the neighboring sites.

ii. Solve this model by Jordan-Wigner. Show that the spectrum is gapless and that each momentum state is doubly-degenerate.

iii. [Challenge problem] The previous part shows that this model produces *two* massless majorana fermions of each chirality. Find the action of the $\mathbb{Z}_2$ symmetry (2) on these fermions.

iv. [Challenge problem] Consider the effect of adding the ferromagnetic term $\sum_j \mathbf{Z}_j \mathbf{Z}_{j+1}$ on this system. Is it invariant under the symmetry?

In this problem we consider adding an extra term:

$$\mathbf{H}_2 = -J \sum_j \left(g_x \mathbf{X}_j + g_z \mathbf{Z}_j \mathbf{Z}_{j+1} + \tilde{g}_x \mathbf{Z}_{j-1} \mathbf{X}_j \mathbf{Z}_{j+1} \right) \ .$$

When $\tilde{g}_x = -g_x$, this hamiltonian has the symmetry

$$\mathbf{S}_1 = \left(\prod_j \mathbf{X}_j \right) \left(\prod_l e^{\mathbf{i} Q_{l,l+1}} \right)$$

where $e^{\mathbf{i} Q_{l,l+1}} = \sqrt{\mathbf{Z}_l \mathbf{Z}_{l+1}} = e^{\frac{\mathbf{i}\pi}{4}(1 - \mathbf{Z}_l \mathbf{Z}_{l+1})}$.

(d) **Kitaev-honeycomb-model-like chain** [optional]

Consider

$$\mathbf{H}_{\mathrm{K}} = \sum_j \left(\mathbf{X}_{2j}\mathbf{X}_{2j+1} + \mathbf{Y}_{2j}\mathbf{Y}_{2j-1}\right)$$

where the bonds alternate between XX interactions and YY interactions. There are now two sites per unit cell, which means that the solution in terms of momentum-space fermion operators will involve two bands. Find their dispersion.

3. **Homage to Onsager.** [optional]

Show that the groundstate energy of Ising chan with $N \gg 1$ sites may be written as

$$E_0(g) = -NJ \int_0^\pi \frac{dk}{2\pi} \epsilon_k$$

where ϵ_k is the dispersion we derived for the fermions.

Show that this can be written as

$$\frac{1}{NJ} E_0(g) = -\frac{2}{\pi}(1+g)E(\pi/2, \sqrt{1-\gamma^2}), \quad \gamma = \left|\frac{1-g}{1+g}\right|$$

(notice that this expression is manifestly self-dual) where $E(\pi/2, x)$ is the elliptic integral

$$E(\pi/2, x) \equiv \int_0^{\pi/2} d\theta \sqrt{1 - x^2 \sin^2\theta}.$$

Expand this result in $g - g_c$.

Use the quantum-to-classical mapping to infer the critical behavior of the 2d (classical) Ising model.

4. **Heisenberg chain**

Consider the Heisenberg hamiltonian

$$\mathbf{H} = -J \sum_j \left(\mathbf{X}_j\mathbf{X}_{j+1} + \mathbf{Y}_j\mathbf{Y}_{j+1} + v\mathbf{Z}_j\mathbf{Z}_{j+1}\right) \ .$$

When $v = 1$ there is $\mathsf{SU}(2)$ symmetry. What are the generators?

On the previous problem set we successfully fermionized the model with $v = 0$. Fermionize the v term.

Take the continuum limit.