University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 7 – Solutions

Due 12:30pm Monday, May 20, 2019

1. **Spin and statistics of a dyon.**

   (a) Consider a magnetic monopole of strength g, so that $\vec{B} = \frac{\hat{r}g}{2r^2}$, and $\oint \vec{B}\cdot d\vec{A} = 2\pi g$. Now consider a particle of charge q in this field. Show that the usual angular momentum $m\vec{r}\times\vec{v}$ is not conserved (the EM field carries angular momentum). Show that instead the quantity

   $$\vec{L} = m\vec{r}\times\vec{v} - \frac{qg\hat{r}}{2}$$

   is conserved. Suppose there is a bound state of two such particles with the minimal charges satisfying Dirac quantization. Interpret the extra term as a contribution to the intrinsic spin of the dyon.

   Here you can just use Newton's equation. More interesting is the interpretation of the extra term – where did it come from? It is the contribution from the angular momentum in the EM fields, $\vec{L}_{EM} = \int \vec{r}\times\left(\vec{E}\times\vec{B}\right)$.

   (b) To confirm that the dyon has fermionic statistics, consider the wavefunction of two such dyons, $\psi(x_1, x_2)$. The exchange of the two dyons can be accomplished by a π-rotation about $\vec{x}_1$, followed by a translation by $\vec{x}_1 - \vec{x}_2$. By analyzing the Aharonov-Bohm phases, show that this process produces a phase Φ

   $$\psi(x_2, x_1) = e^{\mathbf{i}\Phi}\psi(x_1, x_2)$$

   with $\Phi = \pi$ in the case where q, g saturate the Dirac quantization condition.

   The charges must satisfy Dirac quantization $4\pi gq \in 2\pi\mathbb{Z}$, so the minimal charges have $gq = \frac{1}{2}$. Moving one electron all the way around the other monopole produces the phase

   $$q\int_C \vec{A}\cdot d\vec{\ell} \overset{\text{Stokes}}{=} \int_{\text{hemisphere}} \vec{B}\cdot d\vec{a} = 2\pi gq.$$

   A π rotation only acquires half this phase. But now we mustn't forget that *each* monopole makes a field that the other electron moves through. So $\Phi = 2\pi gq = \pi$.

2. **Jordan-Wigner.**

Solve the following spin chains using the mapping to Majorana fermions.

In all these problems we will use the following Jordan-Wigner transformation:

$$\chi_1(j) = \mathbf{Z}_j \prod_{k>j} \mathbf{X}_k, \quad \chi_2(j) = \mathbf{Y}_j \prod_{k>j} \mathbf{X}_k \quad .$$

$$\mathbf{X}_j = \mathbf{i}\chi_1(j)\chi_2(j), \qquad \mathbf{Z}_j\mathbf{Z}_{j+1} = -\mathbf{i}\chi_1(j)\chi_2(j+1) \tag{1}$$

So the majoranas satisfy $\{\chi_\alpha(j), \chi_\beta(l)\} = 2\delta_{\alpha\beta}\delta_{jl}, \alpha, \beta = 1, 2$.

For comparison: With these conventions the transverse-field ising model hamiltonian is

$$\mathbf{H}_{TFIM} = -J\sum_j (g_x\mathbf{X}_j + g_z\mathbf{Z}_j\mathbf{Z}_{j+1}) = -\mathbf{i}J\sum_j (g_x\chi_1(j)\chi_2(j) - g_z\chi_1(j)\chi_2(j+1))$$

so that the heisenberg eom are

$$\partial_t\chi_2(j) = -J\left(g_x\chi_1(j) - g_z\chi_1(j-1)\right), \partial_t\chi_1(j) = J\left(g_x\chi_2(j) - g_z\chi_2(j+1)\right),$$

and from this we see that in the continuum $\chi_\pm \equiv \frac{1}{2\sqrt{a}}(\chi_1 \pm \chi_2)$ are chiral majorana fermions:

$$\left(\partial_0 \mp \partial_x\right)\chi_\pm = m\chi_\mp,$$

with $m \propto g_z - g_x$.

(a) **XY-model.**

$$\mathbf{H} = -J\sum_j (\mathbf{Z}_j\mathbf{Z}_{j+1} + \mathbf{Y}_j\mathbf{Y}_{j+1})$$

I find

$$\mathbf{H} = -J\sum_j (\mathbf{i}\boldsymbol{\chi}_2(j)\boldsymbol{\chi}_1(j+1) - \mathbf{i}\boldsymbol{\chi}_1(j)\boldsymbol{\chi}_2(j+1))$$

$$\mathbf{H} = -J\sum_j \mathbf{i}\left(\boldsymbol{\chi}_2(j)\boldsymbol{\chi}_1(j+1) + \boldsymbol{\chi}_2(j+1)\boldsymbol{\chi}_1(j)\right)$$

In fourier space,

$$\boldsymbol{\chi}_\alpha(j) = \frac{1}{\sqrt{N}}e^{-\mathbf{i}kaj}\boldsymbol{\chi}_\alpha(k)$$

we get

$$\mathbf{H} = +\mathbf{i}J\sum_k \boldsymbol{\chi}_1(k)\boldsymbol{\chi}_2(-k)2\cos ka = +\sum_k (2J\cos ka)\mathbf{c}_k^\dagger\mathbf{c}_k$$

where $\mathbf{c}_k \equiv \frac{1}{2}(\boldsymbol{\chi}_1(k) + \mathbf{i}\chi_2(-k))$. (Beware my factors of two here.)

This model has a U(1) symmetry which rotates $\mathbf{Z}$ into $\mathbf{Y}$, *i.e.* acting by $U = e^{\mathbf{i}\alpha\mathbf{X}}$. How does it act on the fermions?

The $\mathsf{U}(1)$ acts by

$$\mathbf{c} \to e^{\mathbf{i}\theta}\mathbf{c}.$$

(b) **Solve an interacting fermion system.**

$$\mathbf{H}_{\text{int}} = -J \sum_j (\mathbf{X}_j \mathbf{X}_{j+1} + \mathbf{Y}_j \mathbf{Y}_{j+1}) \tag{2}$$

This model is in fact related by a basis rotation ($\mathbf{U} = \prod_j e^{\mathbf{i}\frac{\pi}{4}\mathbf{Y}_j}$) to the one in part 2a.

But if you directly use the mapping we introduced in class in these variables, you'll find quartic terms in the fermions.

The basis transformation above therefore solves this interacting fermion system.

$$\mathbf{H}_{\text{int}} = -J \sum_j (\mathbf{i}\boldsymbol{\chi}_1(j)\boldsymbol{\chi}_2(j)\mathbf{i}\boldsymbol{\chi}_1(j+1)\boldsymbol{\chi}_2(j+1) + \mathbf{i}\boldsymbol{\chi}_2(j)\boldsymbol{\chi}_1(j+1))$$

In terms of the complex fermions, $\mathbf{i}\boldsymbol{\chi}_1(j)\boldsymbol{\chi}_2(j) = 1 - 2\mathbf{c}_j^\dagger \mathbf{c}_j = 1 - 2\mathbf{n}_j$ this first term is a near-neighbor density-density interaction, $\propto \mathbf{n}_j \mathbf{n}_{j+1}$.

How does the $\mathsf{U}(1)$ symmetry of (2) act on these fermion variables?

It mixes particles and holes.

(c) **A spin chain with a non-onsite Ising symmetry.**

Consider the Hamiltonian

$$\mathbf{H} = -J \sum_j (\mathbf{X}_j + \lambda \mathbf{Z}_{j-1}\mathbf{X}_j\mathbf{Z}_{j+1})$$

i. [Slightly more optional] Show that when $\lambda = -1$ this model is invariant under the action of

$$\mathbf{S}_1 \equiv \prod_j \mathbf{X}_j \prod_j e^{\mathbf{i}\frac{\pi}{4}\mathbf{Z}_j\mathbf{Z}_{j+1}}. \tag{3}$$

This symmetry is "not-onsite" in that its action on the spin at site j depends on the state of the neighboring sites.

ii. Solve this model by Jordan-Wigner. Show that the spectrum is gapless and that each momentum state is doubly-degenerate.
The chain falls apart into two decoupled pieces, since the ZXZ term only couples odd sites to odd sites and even sites to even sites. Hence the doubling of the spectrum.

iii. [Challenge problem] The previous part shows that this model produces *two* massless majorana fermions of each chirality. Find the action of the $\mathbb{Z}_2$ symmetry (3) on these fermions.

iv. [Challenge problem] Consider the effect of adding the ferromagnetic term $\sum_j \mathbf{Z}_j \mathbf{Z}_{j+1}$ on this system. Is it invariant under the symmetry?

Majorana fermion solution of edge Hamiltonian with non-onsite $\mathbb{Z}_2$ symmetry.

In this problem we consider adding an extra term:

$$\mathbf{H}_2 = -J \sum_j \left(g_x \mathbf{X}_j + g_z \mathbf{Z}_j \mathbf{Z}_{j+1} + \tilde{g}_x \mathbf{Z}_{j-1} \mathbf{X}_j \mathbf{Z}_{j+1} \right) \ .$$

When $\tilde{g}_x = -g_x$, this hamiltonian has the symmetry

$$\mathbf{S}_1 = \left(\prod_j \mathbf{X}_j \right) \left(\prod_l e^{\mathbf{i} Q_{l,l+1}} \right)$$

where $e^{\mathbf{i}Q_{l,l+1}} = \sqrt{\mathbf{Z}_l \mathbf{Z}_{l+1}} = e^{\frac{\mathbf{i}\pi}{4}(1 - \mathbf{Z}_l \mathbf{Z}_{l+1})}$.

I claim that

$$\mathbf{Z}_{j-1} \mathbf{X}_j \mathbf{Z}_{j+1} = -\mathbf{i} \chi_1(j-1) \chi_2(j+1) \ .$$

So this hamiltonian is

$$\mathbf{H}_2 = -\mathbf{i} J \sum_j \left(g_x \chi_1(j) \chi_2(j) - \left(g_z \chi_1(j) - \tilde{g}_x \chi_1(j-1) \right) \chi_2(j+1) \right) \ .$$

Near $k = 0$, this has the same continuum eom as the usual TFIM model with the replacements: $g_z \to g_z + \tilde{g}_x$ and double the velocity. So the critical point is now at $0 = g_z - \tilde{g}_x - g_x$. But focusing e.g. on $g_z = 0$, the dispersion is that of the TFIM with $k \to 2k$. So for example at $\tilde{g}_x = 1$,

$$\epsilon_k = 2J\sqrt{2 - 2\cos 2ka}$$

has zeros at both $k = 0$ *and* at $k = \pi$.

Let's understand the symmetry action on the fermions. The trivial symmetry action is

$$\mathbf{S}_0 = \prod_j \mathbf{X}_j = i^N \prod_j \chi_1(j)\chi_2(j).$$

This acts as

$$\mathbf{S}_0 \chi_\alpha \mathbf{S}_0^\dagger = -\chi_\alpha,$$

which is indeed a symmetry of $\mathbf{H}_{\text{TFIM}}$.

Using (1), the additional factors in $\mathbf{S}_1$ can be written as:

$$e^{\mathbf{i}Q_{l,l+1}} = e^{\frac{\mathbf{i}\pi}{4}(1+\mathbf{i}\chi_1(j)\chi_2(j+1))} = e^{\frac{\mathbf{i}\pi}{4}} e^{\mathbf{i}\frac{\pi}{4}\mathbf{i}\chi_1(j)\chi_2(j+1)}$$

$$= e^{\frac{\mathbf{i}\pi}{4}} \left(\cos\frac{\pi}{4} + \mathbf{i}\sin\frac{\pi}{4}\mathbf{i}\chi_1(j)\chi_2(j+1) \right) = e^{\frac{\mathbf{i}\pi}{4}} \frac{1-\chi_1(j)\chi_2(j+1)}{\sqrt{2}}.$$

Notice that this object is indeed unitary:

$$\frac{1-ab}{\sqrt{2}} \left(\frac{1-ab}{\sqrt{2}} \right)^\dagger = \frac{1-ab}{\sqrt{2}} \frac{1+ab}{\sqrt{2}} = \frac{1}{2}(1-ab+ab-abab) = 1 \ .$$

Acting this on $\chi_\alpha(j)$ gives

$$\mathbf{S}_1 \chi_1(j) \mathbf{S}_1^\dagger \stackrel{?}{=} -\chi_2(j+1) \ . \tag{4}$$

$$\mathbf{S}_1 \chi_2(j) \mathbf{S}_1^\dagger \stackrel{?}{=} \chi_1(j-1) \ .$$

The key step (the only factor in $\prod_j e^{\mathbf{i}Q}$ that matters) comes from

$$\frac{1 \pm \chi_1(j)\chi_2(j+1)}{\sqrt{2}} \chi_2(j+1) \frac{1 \mp \chi_1(j)\chi_2(j+1)}{\sqrt{2}} = \pm\chi_1(j).$$

The essential fact here is the identity:

$$\frac{1 \pm ab}{\sqrt{2}} b \frac{1 \mp ab}{\sqrt{2}} = \pm a, \quad \frac{1 \pm ab}{\sqrt{2}} a \frac{1 \mp ab}{\sqrt{2}} = \mp b,$$

for any two distinct majorana modes a, b. So $\mathbf{S}_1$ seems to act like

$$\begin{pmatrix} 0 & -T \\ T^\dagger & 0 \end{pmatrix}$$

where T is the shift-by-1 operator and the matrix is acting on the α, β space. Notice that this does not seem to square to $\mathbb{1}$! Rather it squares to the operation which reverses the sign of all the majoranas. In the spin chain, this is a gauge redundancy: the sign of the fermion operators is not observable. So there is no contradiction.

Since

$$\mathbf{Z}_{j-1}\mathbf{X}_j\mathbf{Z}_{j+1} = -\mathbf{i}\boldsymbol{\chi}_1(j-1)\boldsymbol{\chi}_2(j+1) \overset{(4)}{\mapsto} -\mathbf{i}\left(-\boldsymbol{\chi}_2(j)\right)\left(\boldsymbol{\chi}_1(j)\right) = -\mathbf{i}\boldsymbol{\chi}_1(j)\boldsymbol{\chi}_2(j) = -\mathbf{X}_j$$

this $\mathbb{Z}_2$ action indeed preserves the form of $\mathbf{H}_2$ if $g_x = -\tilde{g}_x$ (set $g_z = 0$ for a moment):

$$\mathbf{H}_2^\star = -\mathbf{i}J\sum_j \left(\chi_1(j)\chi_2(j) - \chi_1(j-1)\chi_2(j+1)\right) \text{ has } 0 = [\mathbf{S}_1, \mathbf{H}_2^\star] \ .$$

In the continuum limit, if we ignore the shift, the transformation (4) is basically $\chi_1 \to \chi_2, \chi_2 \to \chi_1$, which acts on $\chi_\pm$

$$\mathbf{S}_1 : \chi_+ \to \chi_+, \quad \chi_- \to -\chi_- \ ,$$

which indeed acts nontrivially only on χ_-. This is a chiral symmetry.

More microscopically, it seems that we should define the chiral majoranas to be

$$\chi_\pm(j) \sim \chi_1(j) \pm \chi_2(j+1).$$

Notice that this regrouping is very much like the dual jordan-wigner fermions.

That is: if I relabel my degrees of freedom as living on the links as follows:

$$\gamma_1(j+\frac{1}{2}) \equiv \chi_1(j), \gamma_2(j+\frac{1}{2}) = -\chi_2(j+1)$$

then in terms of the gammas, the TFIM hamiltonian has the roles of the X and Z terms reversed:

$$\mathbf{H}_{TFIM} = -\mathbf{i}J\sum_j \left(g_x\gamma_1(j)\gamma_2(j+1) - g_z\gamma_1(j)\gamma_2(j)\right)$$

which if I then rewrite in terms of new spin variables amounts to a duality transformation, i.e. produces the original hamiltonian with the replaxement $g_x/g_z \to g_z/g_x$.

Addendum: The preceding discussion is correct, *except*: a 1d chain with *only* next-nearest-neighbor hopping falls apart into two decoupled chains: odd sites only couple to odd sites and even sites only couple to even sites. This means we actually get two copies of the majorana with this chiral realization of the symmetry.

In fact, the ferromagnetic term is also invariant under $\mathbf{S}_1$. This is clear in terms of the spins, since the extra phase factors commute with $\mathbf{Z}$s. And in fact the extra phase factors are *made* from the combination $\mathbf{Z}_j\mathbf{Z}_{j+1}$. In terms of the majoranas,

$$\mathbf{Z}_j\mathbf{Z}_{j+1} = -\mathbf{i}\boldsymbol{\chi}_1(j)\boldsymbol{\chi}_2(j+1) \mapsto -\mathbf{i}\left(-\boldsymbol{\chi}_2(j+1)\right)\left(+\boldsymbol{\chi}_1(j)\right) = -\mathbf{i}\boldsymbol{\chi}_1(j)\boldsymbol{\chi}_j(j+1).$$

Adding this term will gap out the majorana fields (*i.e.* it adds a term in the eom which is not proportional to k). *However*: there will still be two degenerate groundstates. With an open chain, these are the dangling majorana modes at the ends. This is because with both $g_z \neq 0$ and $\tilde{g}_x \neq 0$ (both of which couple j to $j+\frac{1}{2}$), we will always be in the regime (the ferromagnetic phase) where the pairing is between site j and site $j+\frac{1}{2}$, leaving out the sites $\frac{1}{2}$ and $N+\frac{1}{2}$.

This model is discussed in this paper by Xie Chen et al.

(d) **Kitaev-honeycomb-model-like chain** [optional]

Consider

$$\mathbf{H}_\mathrm{K} = \sum_j \left(\mathbf{X}_{2j}\mathbf{X}_{2j+1} + \mathbf{Y}_{2j}\mathbf{Y}_{2j-1}\right)$$

where the bonds alternate between XX interactions and YY interactions. There are now two sites per unit cell, which means that the solution in terms of momentum-space fermion operators will involve two bands. Find their dispersion.

$$\mathbf{H}_\mathrm{K} = \sum_j \mathbf{i}\left(-\boldsymbol{\chi}_1(2j)\boldsymbol{\chi}_2(2j+1) + \boldsymbol{\chi}_2(2j)\boldsymbol{\chi}_1(2j-1)\right)$$

Introduce fourier modes for the bravais lattice with two sites (hence four majoranas) per unit cell

$$\begin{pmatrix} \boldsymbol{\chi}_1(2j) \\ \boldsymbol{\chi}_2(2j+1) \\ \boldsymbol{\chi}_1(2j+1) \\ \boldsymbol{\chi}_2(2j) \end{pmatrix} = \frac{1}{\sqrt{N}} e^{\mathbf{i}k(2ja)}\boldsymbol{\chi}_A(k)$$

whose hamiltonian is

$$\mathbf{H}_\mathrm{K} = \sum_k \mathbf{i}\boldsymbol{\chi}_A(k)\boldsymbol{\chi}_B(-k)t_{AB}(k)$$

with

$$t_{AB}(k) = \begin{pmatrix} 0 & e^{\mathbf{i}ka} & 0 & 0 \\ e^{-\mathbf{i}ka} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Since the bottom block of this matrix doesn't depend on k, one of the bands is flat.

3. **Homage to Onsager.** [optional]

   Show that the groundstate energy of Ising chan with $N \gg 1$ sites may be written as

   $$E_0(g) = -NJ \int_0^\pi \frac{dk}{2\pi} \epsilon_k$$

   where ϵ_k is the dispersion we derived for the fermions.

   Show that this can be written as

   $$\frac{1}{NJ} E_0(g) = -\frac{2}{\pi}(1+g) E(\pi/2, \sqrt{1-\gamma^2}), \quad \gamma = \left| \frac{1-g}{1+g} \right|$$

   (notice that this expression is manifestly self-dual) where $E(\pi/2, x)$ is the elliptic integral

   $$E(\pi/2, x) \equiv \int_0^{\pi/2} d\theta \sqrt{1 - x^2 \sin^2 \theta}.$$

   Expand this result in $g - g_c$.

   Use the quantum-to-classical mapping to infer the critical behavior of the 2d (classical) Ising model.

   This calculation is done pretty explicitly in Fradkin's 2d edition, page 125.

4. **Heisenberg chain**

   Consider the Heisenberg hamiltonian

   $$\mathbf{H} = -J \sum_j \left(\mathbf{X}_j \mathbf{X}_{j+1} + \mathbf{Y}_j \mathbf{Y}_{j+1} + v \mathbf{Z}_j \mathbf{Z}_{j+1} \right) .$$

   When $v = 1$ there is $\mathsf{SU}(2)$ symmetry. What are the generators?

   The generators are just the Pauli operators $\sum_j \vec{\boldsymbol{\sigma}}_j$.

   On the previous problem set we successfully fermionized the model with $v = 0$. Fermionize the v term.

   We saw (in a different basis) that this is a 4-fermion term. It is

   $$\left(2\mathbf{c}_j^\dagger \mathbf{c}_j - 1 \right) \left(2\mathbf{c}_{j+1}^\dagger \mathbf{c}_{j+1} - 1 \right) .$$

   Take the continuum limit.

Plugging in $\Psi(x_j) = \frac{1}{\sqrt{a}}\mathbf{c}_j$ gives

$$\begin{aligned}\mathbf{\Delta H} &\simeq -J\int dx\,\big(2\Psi^\dagger(x)\Psi(x)-1\big)\,\big(2\Psi^\dagger(x+a)\Psi(x+a)-1\big)\\ &= -Ja^2\int dx\,\big(2\Psi^\dagger(x)\Psi(x)\big)\,\big(2\partial\Psi^\dagger(x)\partial\Psi(x)\big) + \text{quadratic terms} \qquad (5)\end{aligned}$$

Fermi statistics get rid of the term with fewer derivatives.