University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 6

Due 12:30pm Monday, May 13, 2019

1. **Brain-warmer.**

   Compute the expectation values of $\mathbf{X}$ and $\mathbf{Z}$ in the spin-coherent state $|\check{n}\rangle$.

2. **Mean field theory is product states.**

   Consider the transverse field Ising model on an arbitrary lattice:

$$\mathbf{H} = -J \left(\sum_{\langle x,y \rangle} Z_x Z_y + g \sum_x X_x \right).$$

   We will study the mean field state:

$$|\psi_{\rm MF}\rangle \equiv \otimes_x \left(\sum_{s_x \pm} \psi_{s_x} |s_x\rangle \right). \tag{1}$$

   Restrict to the case where the state of each spin is the same.

   (a) Write the variational energy for the mean field state, $E(\hat{n}) \equiv \langle \psi_{\rm MF} | \mathbf{H} | \psi_{\rm MF} \rangle$.

   (b) Assuming s_x is independent of x, minimize it for each value of the dimensionless parameter g. Find the groundstate magnetization $\langle \psi | Z_x | \psi \rangle$ in this approximation, as a function of g. Draw the mean-field phase diagram.

3. **Potentials for matrix-valued fields.**

   (a) Convince yourself that by a symmetry transformation $\Sigma \to g_L \Sigma g_R^\dagger$ we can put the complex matrix Σ in the form $\Sigma = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}$.

   (b) Consider the $\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R$-symmetric potential

$$V(\Sigma) = -m^2 \mathrm{tr} \Sigma \Sigma^\dagger + \frac{\lambda}{4} \left(\mathrm{tr} \Sigma \Sigma^\dagger \right)^2 + g \mathrm{tr} \Sigma \Sigma^\dagger \Sigma \Sigma^\dagger. \tag{2}$$

   Show that for any $g > 0$ this potential has a minimum at $\langle \Sigma \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Find v. Show that if $g = 0$ there are other minima which are not related by rotations $\Sigma \to g_L \Sigma g_R^\dagger$.

(c) [bonus problem] Now consider a hermitian-matrix-valued field $\Phi = \Phi^a T^a$. Suppose T^a are generators of the adjoint of $\mathsf{SU}(5)$, so there are 24 components of Φ^a. In order for grand unification to work, there must be a potential for such a Higgs field Φ which has a minimum of the form

$$\langle \Phi \rangle = v \text{diag}(2, 2, 2, -3, -3) \equiv \Phi_{3,2}$$

which breaks $\mathsf{SU}(5)$ down to $\mathsf{SU}(3)_{\text{color}} \times \mathsf{SU}(2)_{\text{weak}}$. Consider the most general quartic potential for Φ which is invariant under $\mathsf{SU}(5)$:

$$V = -m^2 \text{tr}\Phi^2 + a \text{tr}\Phi^4 + b \left(\text{tr}\Phi^2\right)^2 .$$

Choose a basis where $\Phi = v \text{diag}(a_1, a_2, a_3, a_4, a_5)$, with $\sum_{i=1}^5 a_i = 0$. (Impose this last condition with a Lagrange multiplier.)

For what values of m, a, b is $\Phi_{3,2}$ an extremum?

Show that $\Phi_{3,2}$ is a minimum.

Find all possible minima of this potential.

For the minimum of the form $\langle \Phi \rangle = v \text{diag}(1, 1, 1, 1, -4)$, what are the masses of the massive gauge bosons, and what is the unbroken gauge group?

I may add another problem here.