University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 3 – Solutions

Due 12:30pm Monday, April 22, 2019

1. **Anomaly cancellation in the Standard Model.** If we try to gauge a chiral symmetry (such as hypercharge in the Standard Model (SM)), it is important that it is actually a symmetry, *i.e.* is not anomalous. In $D = 3 + 1$, a possible anomaly is associated with a choice of three currents, out of which to make a triangle diagram. We'll call a "$\mathsf{G}_1\mathsf{G}_2\mathsf{G}_3$ anomaly" the diagram with insertions of currents for $\mathsf{G}_1, \mathsf{G}_2$ and G_3. Generalizing a little, we showed that the divergence of the current for G_1 is

$$\partial_\mu j_1^{A\mu} = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} F^{2B}_{\mu\nu} F^{3C}_{\rho\sigma} \sum_f (-1)^f \mathrm{tr}_{R(f)} \{T_1^A, T_2^B\} T_3^C .$$

The sum is over each Weyl fermion, $R(f)$ is its representation under the combined group $\mathsf{G}_1 \times \mathsf{G}_2 \times \mathsf{G}_3$, and T_1^A are a basis of generators of the Lie algebra of G_1 etc. in the representation of the field f. By $(-1)^f$ I mean $\pm$ for left- and right-handed fermions respectively.

We consider the possibilities in turn.

Schwartz §30.4 does most of this pretty explicitly.

(a) Convince yourself that the divergence of the $\mathsf{U}(1)_Y$ hypercharge current gets a contribution of the form

$$\partial_\mu J^\mu_Y = \left(\sum_{\text{left}} Y_l^3 - \sum_{\text{right}} Y_r^3\right) \frac{g'^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} B_{\rho\sigma}$$

from the triangle with three insertions of the current itself (here B is the hypercharge gauge field strength). The sum on the RHS is over all left- and right-handed Weyl spinors weighted by the cube of their hypercharge. Check that this sum evaluates to zero in the SM.

(b) Show that any anomaly of the form $\mathsf{SU}(N)\mathsf{U}(1)^2$ or $\mathsf{SU}(N)G_1G_2$ is zero.

(c) (Easy) Convince yourself that there is no $\mathsf{SU}(3)^3$ anomaly for QCD.

The charges of the fields under $\mathsf{SU}(3)$ are symmetric under $L \leftrightarrow R$ – *i.e.* QCD is non-chiral – so there is a cancellation between the contributions of left- and right-handed fields.

(d) Check that there is never an $\mathsf{SU}(2)^3$ anomaly. (Hint: the generators satisfy $\{\tau^a, \tau^b\} = 2\delta^{ab}$.)

(e) Show that the $\mathsf{SU}(3)^2\mathsf{U}(1)_Y$ anomaly demands that $2Y_Q - Y_u - Y_d = 0$. Check that this is true in the SM.

(f) Show that a necessary condition for hypercharge to not have an anomaly with the Electroweak gauge bosons on the RHS is $Y_L + 3Y_Q = 0$, where Y_L and Y_Q are the hypercharges of the left-handed leptons and quarks. Check that this works out in the SM.

It gets contributions only from left-handed fields (those charged under $\mathsf{SU}(2)_{EW}$): $\mathrm{tr}\{\tau^a, \tau^b\}Y = \delta^{ab} \sum_{\text{left}} Y_l = Y_L + 3Y_Q$ because the quarks carry 3 colors.

(g) There is another kind of anomaly called a *gravitational anomaly*. This is a violation of current conservation in response to coupling to curved space. An example is of the form

$$\partial_\mu j^\mu_Y = a \mathrm{tr} \mathcal{R} \wedge \mathcal{R}$$

where $\mathcal{R}$ is a two-form related to the curvature of spacetime (analogous to the field strength F). The coefficient a is proportional to $\sum_{\text{left}} \mathrm{tr} Y_l - \sum_{\text{right}} \mathrm{tr} Y_r$. Check that this too vanishes for hypercharge in the Standard Model.

These conditions, plus the assumption that the right-handed neutrino is neutral, actually determine all the hypercharge assignments.

2. **Right-handed neutrinos.**

[from Iain Stewart, and hep-ph/0210271]

Consider adding a right-handed singlet (under all gauge groups) neutrino N_R to the Standard Model. It may have a majorana mass M; and it may have a coupling g_ν to leptons, so that all the dimension ≤ 4 operators are

$$\mathcal{L}_N = \bar{N}_R \mathbf{i} \partial\!\!\!/ N_R - \frac{M}{2} \bar{N}^c_R N_R - \frac{M}{2} \bar{N}_R N^c_R + \left(g_\nu \bar{N}_R H^T_i L_j \epsilon^{ij} + h.c. \right)$$

where $N^c_R = C \left(\bar{N}_R \right)^T$ is the the charge conjugate field, $C = \mathbf{i}\gamma_2\gamma_0$ (in the Dirac representation), H is the Higgs doublet, L is the left-handed lepton doublet, containing ν_L and e_L. Take the mass M to be large compared to the electroweak scale. Integrate out the right-handed neutrinos at tree level. [Hint: you may find it useful to work in terms of the Majorana field

$$N \equiv N_R + N^c_R$$

which satisfies $N = N^c$.]

Show that the leading term in the expansion in $1/M$ is a dimension-5 operator made of Standard Model fields. Explain the consequences of this operator for neutrino physics, assuming a vacuum expectation value for the Higgs field.

In terms of N, the lagrangian is

$$\mathcal{L}_N = \frac{1}{2}\bar{N}\left(\mathbf{i}\partial\!\!\!/ - M\right)N + g_\nu \bar{N} H_i L_j \epsilon^{ij} + g_\nu \bar{N} H_i^\star L_j^c \epsilon^{ij}.$$

The equation of motion for N (from varying $\bar{N}$) is

$$\left(\mathbf{i}\partial\!\!\!/ - M\right)N = -g_\nu \left(H_i L_j + H_i^\star L_j^c\right)\epsilon^{ij}$$

which gives

$$\mathcal{L}_N| = -\frac{1}{2} g_\nu \left(\bar{L}_j^c H_i + \bar{L}_j H_i^\star\right)\epsilon^{ij} \frac{1}{\mathbf{i}\partial\!\!\!/ - M} g_\nu \left(H_k L_\ell + H_k^\star L_\ell^c\right)\epsilon^{k\ell}.$$

As for our discussion of W-bosons, we expand this in powers of $1/M$ to get a local effective field theory. The leading term is

$$\mathcal{O}^{(5)} = \frac{g_\nu^2}{M} \bar{L}_j^c H_i \epsilon^{ij} L_\ell H_k \epsilon^{k\ell} + h.c.$$

Plugging in $\langle H \rangle \neq 0$, this is a neutrino mass.

Place a bound on M assuming that the observed neutrinos have masses $m_\nu < 0.5$ eV.

In terms of the parameterization from lecture, $m_\nu = \frac{c_5 v^2}{2\Lambda_{\text{new}}}$. This gives $\Lambda_{\text{new}} \geq 10^{14}$GeV for $c_5 \sim 1$. We find $\Lambda_{\text{new}}/c_5 \sim M$, so $M \geq 10^{14}$GeV.

3. **Gross-Neveu model.**

Here's an example which illustrates the manipulations we did in describing the BCS phenomenon. Now that we've learned about fermionic path integrals, consider the partition function for an N-vector of fermionic spinor fields in D dimensions:

$$Z = \int [d\psi d\bar{\psi}] e^{\mathbf{i}S[\psi]}, \quad S[\vec{\psi}] = \int \mathrm{d}^D x \left(\bar{\psi}^a \mathbf{i}\partial\!\!\!/ \psi^a - \frac{g}{N}\left(\bar{\psi}^a\psi^a\right)^2\right).$$

(a) At the free fixed point, what is the dimension of the coupling g as a function of the number of spacetime dimensions D? Show that it is classically marginal in $D = 2$, so that this action is (classically) scale invariant.

(b) We will show that this model in $D=2$ exhibits dimensional transmutation in the form of a dynamically generated mass gap. Here are the steps: first use the Hubbard-Stratonovich trick to replace ψ^4 by $\sigma\psi^2+\sigma^2$ in the action, where σ is a scalar field. Then integrate out the ψ fields. Find the saddle point equation for σ; argue that the saddle point dominates the integral for large N. Find a translation invariant saddle point. Plug the saddle point configuration of σ back into the action for ψ and describe the resulting dynamics.

We can decouple the quartic term by writing

$$Z=\int[D\psi\bar{\psi}]e^{\mathbf{i}S[\psi]}=\int[D\psi D\bar{\psi}D\sigma]e^{\mathbf{i}S_2[\psi]+\mathbf{i}\int d^Dx\left(\sigma\bar{\psi}^a\psi^a+h.c.\right)+\mathbf{i}\int \mathrm{d}^Dx\frac{N\sigma^2(x)}{2g}}\ . \tag{1}$$

Now the integral over ψ is gaussian:

$$\int[D\psi D\bar{\psi}D\sigma]e^{\int d^Dx\bar{\psi}^a\left(\mathbf{i}\partial\!\!\!/+\sigma\right)\psi^a}=\left(\det\left(\mathbf{i}\partial\!\!\!/+\sigma\right)\right)^N=e^{N\mathrm{tr}\log\left(\mathbf{i}\partial\!\!\!/+\sigma\right)}.$$

The resulting path integral is

$$Z=\int[D\sigma]e^{\mathbf{i}NS_{\text{eff}}[\sigma]}$$

with $S_{\text{eff}}[\sigma]=\int d^Dx\frac{\sigma^2}{2g}+\delta S[\sigma]$ where the term generated by the fermionic fluctuations is

$$\delta S[\sigma]=\mathrm{tr}\log\left(\partial\!\!\!/+\sigma\right).$$

We can take care of the spin indices by noticing that

$$\begin{aligned}\mathrm{tr}_{\text{spin}}\log\left(\partial\!\!\!/+\sigma\right)&=\frac{1}{2}\left(\mathrm{tr}_{\text{spin}}\log\left(\partial\!\!\!/+\sigma\right)+\mathrm{tr}_{\text{spin}}\log\left(-\partial\!\!\!/+\sigma\right)\right) &(2)\\ &=\frac{1}{2}\mathrm{tr}_{\text{spin}}\log\left(-\partial^2+\sigma^2\right)\overset{D=2}{=}\log\left(-\partial^2+\sigma^2\right) &(3)\end{aligned}$$

where at the last step we used the fact that the Dirac spinor in 2D has two components.

If we assume that σ is constant in spacetime, we can do the trace in momentum space (V is the volume of spacetime):

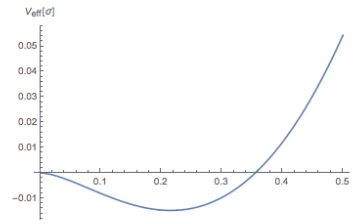
$$\begin{aligned}\mathrm{tr}\log\left(-\partial^2+\sigma^2\right)&=V\int đ^Dp\log\left(p^2+\sigma^2\right) &(4)\\ &\overset{\text{Wick rotate}}{=}\mathbf{i}V\frac{1}{2\pi}\mathbf{i}\int_0^\Lambda pdp\log\left(p^2+\sigma^2\right) &(5)\\ &=\mathbf{i}\frac{V}{\pi}\left(-\sigma^2\log\frac{\sigma^2}{\Lambda^2}+\text{UV divergent terms}\right). &(6)\end{aligned}$$

I introduced a hard UV cutoff, since we have no gauge invariance to preserve. At the last step I've assumed $\sigma \ll \Lambda$. We ignore the divergent constants.

Because of the big honking factor of N in front of S_{eff}, the σ integral is dominated by its saddle point configuration, where

$$0 = \frac{\delta S_{\text{eff}}}{\delta \sigma} = V\left(\frac{\sigma}{g} + \frac{2\sigma}{\pi}\left(1 + \log \sigma/\Lambda\right)\right)$$

from which we conclude that there is a minimum for σ at $\sigma = \Lambda e^{-\frac{\pi}{g}}/\sqrt{e}$. (The figure at right is for $g = .3, \Lambda = 1000$.)



Thus, the fermions get a mass of order $\Lambda e^{-\pi/g}$, non-perturbative in g, and parametrically smaller than the cutoff.

4. **Polyacetylene returns.**

On HW01 problem 4, you may have wondered what is the connection between the field theory we were studying (a scalar coupled to fermions in $D = 2$) and polyacetylene. I'd like to explain that connection a bit.

Consider an extension of the model above to include also *phonon* modes, *i.e.* degrees of freedom encoding the positions of the ions in the solid. (Again we ignore the spins of the electrons for simplicity.)

$$H = -t\sum_n (1 + u_n)c_n^\dagger c_{n+1} + h.c. + \sum_n K(u_n - u_{n+1})^2 \equiv H_F + H_E.$$

Here u_n is the deviation of the nth ion from its equilibrium position (in the $+x$ direction), so the second term represents an elastic energy.

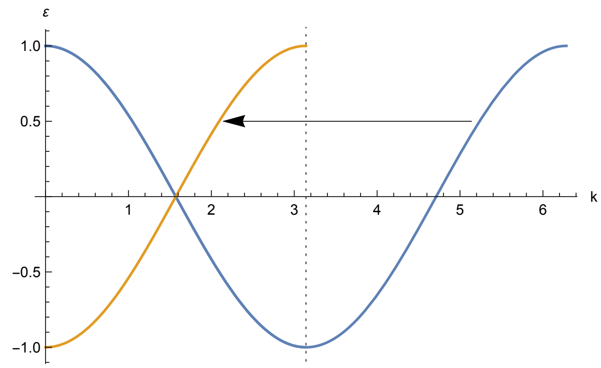
(a) (The part with the free massless Dirac field you had time to do on HW2.)

(b) Consider a configuration

$$u_n = \phi(-1)^n \tag{7}$$

where the ions move closer in pairs. Compute the electronic spectrum. (Hint: this enlarges the unit cell. Define $c_{2n} \equiv a_n, c_{2n+1} \equiv b_n$, and solve in Fourier space, $a_n \equiv \oint đk e^{2\mathbf{i}kn} a_k$ *etc.*) You should find that when $\phi \neq 0$ there is a gap in the electron spectrum (unlike $\phi = 0$). Expand the spectrum near the minimum gap and include the effects of the field ϕ in the continuum theory.

When doubling the unit cell, we halve the Brillouin zone. So even when $\phi = 0$, the spectrum gets folded on itself, like this:

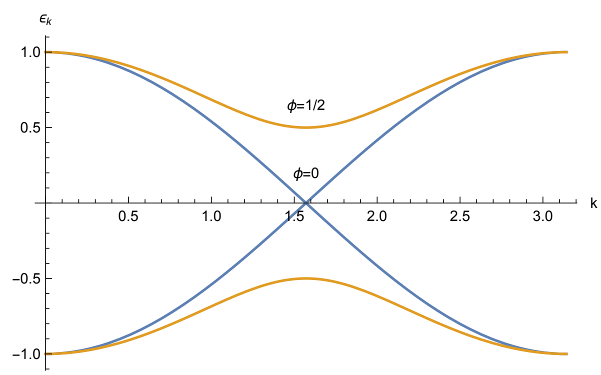


This means that at half-filling, with $\phi = 0$, it looks like there is a Dirac point at $k = \pi/2$.

Now, including ϕ, it allows the two branches of the Dirac point to mix with each other and produces a gap:

$$\epsilon(k) = \pm\sqrt{\cos^2 k + \phi^2 \sin^2 k}$$

which looks like this:



Near the minimum gap at $k = \pi/2$, we can expand to find

$$\epsilon(k = \frac{\pi}{2} + \delta k) = \pm\sqrt{\cos^2 k(1 - \phi^2) + \phi^2} = \pm\sqrt{\delta k^2(1 - \phi^2) + \phi^2}. \qquad (8)$$

Comparing to the spectrum of a Dirac fermion with action

$$S[\psi, \phi] = \int d^2x \left(\bar{\psi}\mathbf{i}\partial\!\!\!/\psi - \phi\bar{\psi}\psi\right)$$

which has

$$H = \gamma^0(\mathbf{i}\gamma^1\partial_x - \phi) = \begin{pmatrix} \phi & k \\ k & -\phi \end{pmatrix}$$

and therefore

$$\epsilon_k = \pm\sqrt{k^2 + \phi^2}$$

which agrees with (**??**) at small k (which is really the deviation from $k = \pi/2$) and small ϕ.

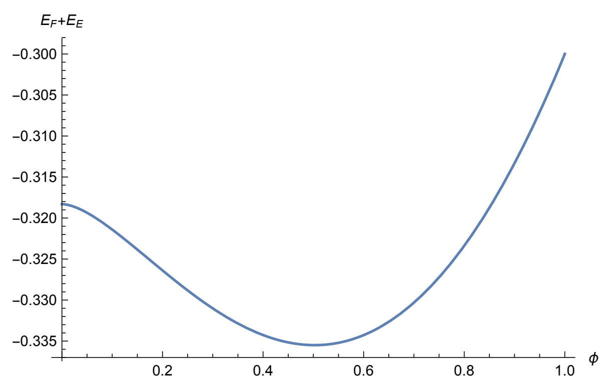
(c) **Peierls' instability.** Compute the groundstate energy of the electrons H_F in the configuration (1), at half-filling (*i.e.* the number of electrons is half the number of available states). Check that you recover the previous answer when $\phi = 0$. Interpret the answer when $\phi = 1$.

Compute H_E in this configuration, and minimize the sum of the two as a function of ϕ.

At half-filling, in the groundstate the lower band is filled. The energy is

$$E_F(\phi) = -\oint dk\sqrt{\cos^2 k + \phi^2 \sin^2 k} = -\frac{1}{\pi}\text{EllipticE}(1 - \phi^2).$$

For $8K^2 = .2$ the total energy looks like this:



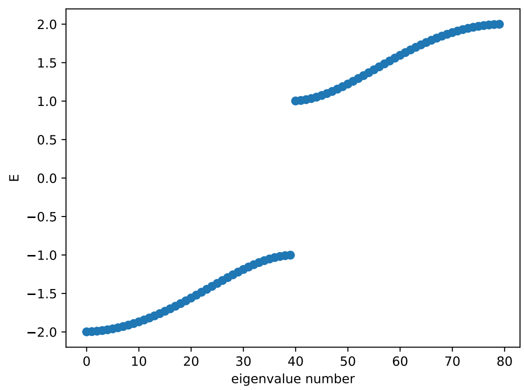
There is a minimum at $\phi^2 \neq 0$, *i.e.* two minima at $\phi = \pm\phi_0$. Increasing ϕ lowers the total energy because it lowers the energy of the filled states.

(d) You should find that the energy is independent of the *sign* of ϕ. This means that there are two groundstates. We can consider a domain wall between a region of $+$ and a region of $-$. Show that this domain wall carries a fermion mode whose energy lies in the bandgap and has fermion number $\pm\frac{1}{2}$.

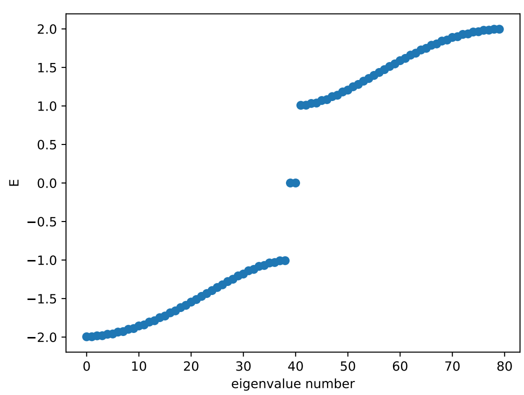
The basic idea is that ϕ must go through zero in between. We showed on HW01 that this is the case using the field theory we derived in the earlier parts of the problem. In particular, the two states (zero-mode occupied and zero-mode unoccupied) must have a fermion number which differ by 1, but they are related to each other by particle-hole symmetry, so they must have fermion number $\pm\frac{1}{2}$ (as we found on HW01).

(e) Verify the result of the previous part by diagonalizing the relevant tight-binding matrix.

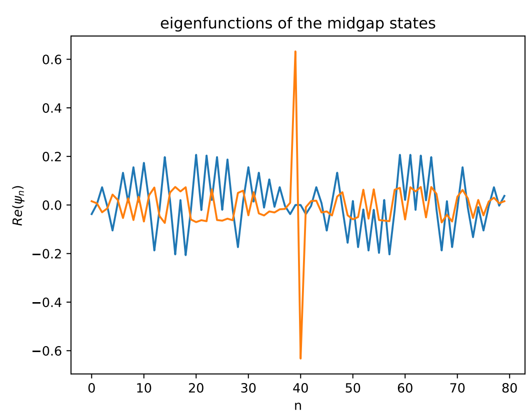
Here is the spectrum of a chain (of length 40) with $\phi = +0.5$ everywhere:



And here is the result when ϕ switches to -0.5 in the middle:



The wavefunctions of the states in the middle look like



(f) Time-reversal played an important role here. If we allow complex hopping amplitudes, we can make a domain wall without midgap modes. Explain this from field theory. Bonus: explain this from the lattice hamiltonian.

If the mass is allowed to be complex, then we can interpolate between $-m$ and $+m$ without going through $m = 0$.