University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 2 – Solutions

Due 12:30pm Monday, April 15, 2019

1. **Emergence of the Dirac equation.** Consider a chain of free fermions with

$$H = -t\sum_n c_n^\dagger c_{n+1} + h.c.$$

(a) Show that the low-energy excitations at a generic value of the filling are described the the massless Dirac lagrangian in 1+1 dimensions. Find an explicit choice of 1+1-d gamma matrices which matches the answer from the lattice model. Show that the right-movers are right-handed $\gamma^5 \equiv \gamma^0\gamma^1 = 1$ and the left-movers are left-handed.

This system has a conserved charge $N \equiv \sum_n c_n^\dagger c_n$ counting the number of fermions, which we get to pick. The easiest way to do this is to add a chemical potential $H \to H - \mu N$ and choose μ to get the desired number of particles on average. (This is the same as fixing the number of particles in the thermodynamic limit.) In that case we have

$$H = -t\sum_n c_n^\dagger c_{n+1} + h.c. - \mu\sum_n c_n^\dagger c_n = \oint_{\mathrm{BZ}} đk c_k^\dagger c_k \epsilon_k$$

with $\epsilon_k = -2t\cos ka - \mu$, and the integral is over the Brillouin zone. $a = 1$ is the lattice spacing. By 'generic filling' I mean choose the number of particles per site to be between 0 and 1. The former and latter correspond to choosing $\mu = \pm 2t$ at the bottom or top of the band, where the dispersion is quadratic, rather than linear.

We can focus on the physics at the two Fermi points $k = \pm k_F$ (where k_F solves $\epsilon_{k_F} = 0$) by plugging in

$$\psi(x) \simeq \int_R đk e^{(k_F+k)x}\psi_R + \int_R đk e^{(-k_F+k)x}\psi_L$$

where R is a small-enough region in momentum space that the two domains don't overlap. This gives

$$H = \int_R đk \left(v_F k\psi_R^\dagger\psi_R - v_F k\psi_L^\dagger\psi_L\right)$$

where $v_F \equiv \partial_k \epsilon_k|_{k=k_F}$. Translating into an action, setting $v_F = 1$, and pretending R goes on forever (this is how we can fool ourselves that the chiral current is conserved), this is

$$S = \int dxdt \left(\psi_R^\dagger \left(\partial_t - \partial_x \right) \psi_R + \psi_L^\dagger \left(\partial_t + \partial_x \right) \psi_L \right) = \int d^2x \left(\bar{\Psi} \gamma^\mu \partial_\mu \Psi \right)$$

with

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

and

$$\gamma^0 = \sigma^1, \gamma^1 = \mathbf{i}\sigma^2, \gamma^5 \equiv \gamma^0 \gamma^1 = -\sigma^3.$$

This gives

$$\gamma^5 \Psi = -\sigma^3 \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} -\psi_L \\ \psi_R \end{pmatrix}$$

so indeed the left-moving particle has left-handed chirality.

[I am adding the rest of this problem on Friday April 12, so it is all bonus material. It will carry over to HW3.]

On the previous problem set problem 4, you may have wondered what is the connection between the field theory we were studying (a scalar coupled to fermions in $D = 2$) and polyacetylene. I'd like to explain that connection a bit.

Consider an extension of the model above to include also *phonon* modes, *i.e.* degrees of freedom encoding the positions of the ions in the solid. (Again we ignore the spins of the electrons for simplicity.)

$$H = -t \sum_n (1 + u_n) c_n^\dagger c_{n+1} + h.c. + \sum_n K (u_n - u_{n+1})^2 \equiv H_F + H_E.$$

Here u_n is the deviation of the nth ion from its equilibrium position (in the $+x$ direction), so the second term represents an elastic energy.

(b) Consider a configuration

$$u_n = \phi(-1)^n \tag{1}$$

where the ions move closer in pairs. Compute the electronic spectrum. (Hint: this enlarges the unit cell. Define $c_{2n} \equiv a_n, c_{2n+1} \equiv b_n$, and solve in Fourier space, $a_n \equiv \oint \mathrm{d}k e^{2\mathbf{i}kn} a_k$ *etc.*) You should find that when $\phi \neq 0$ there is a gap in the electron spectrum (unlike $\phi = 0$). Expand the spectrum near the minimum gap and include the effects of the field ϕ in the continuum theory.

(c) **Peierls' instability.** Compute the groundstate energy of the electrons H_F in the configuration (1), at half-filling (*i.e.* the number of electrons is half the number of available states). Check that you recover the previous answer when $\phi = 0$. Interpret the answer when $\phi = 1$.

Compute H_E in this configuration, and minimize the sum of the two as a function of ϕ.

(d) You should find that the energy is independent of the *sign* of ϕ. This means that there are two groundstates. We can consider a domain wall between a region of $+$ and a region of $-$. Show that this domain wall carries a fermion mode whose energy lies in the bandgap and which carries fermion number $\frac{1}{2}$.

(e) Verify the result of the previous part by diagonalizing the relevant tight-binding matrix.

(f) Time-reversal played an important role here. If we allow complex hopping amplitudes, we can make a domain wall without midgap modes. Explain this from field theory. Bonus: explain this from the lattice hamiltonian.

2. **An application of effective field theory in quantum mechanics.**

[I learned this example from Z. Komargodski.]

Consider a model of two canonical quantum variables ($[\mathbf{x}, \mathbf{p}_x] = \mathbf{i} = [\mathbf{y}, \mathbf{p}_y], 0 = [\mathbf{x}, \mathbf{p}_y] = [\mathbf{x}, \mathbf{y}]$, etc) with Hamiltonian

$$\mathbf{H} = \mathbf{p}_x^2 + \mathbf{p}_y^2 + \lambda \mathbf{x}^2 \mathbf{y}^2.$$

(This is similar to the degenerate limit of the model studied in lecture with two QM variables where both natural frequencies are taken to zero.)

(a) Based on a semiclassical analysis, would you think that the spectrum is discrete?

The potential has flat directions along the coordinate axes, $\{x = 0\} \cup \{y = 0\}$. This means there are unbounded classical orbits, which suggests that the spectrum should be continuous. This conclusion is in fact wrong. (An excuse for discounting it is that the set of initial conditions which follow unbounded orbits have measure zero.)

(b) Study large, fixed x near $y = 0$. We will treat x as the slow (= low-energy) variable, while y gets a large restoring force from the background x value. Solve the y dynamics, and find the groundstate energy as a function of x:

$$V_{\text{eff}}(x) = E_{\text{g.s. of y}}(x).$$

If we treat x as a constant, the hamiltonian for y is a harmonic oscillator problem. The groundstate energy is

$$V_{\text{eff}}(x) = E_{\text{g.s. of y}}(x) = \sqrt{\lambda}|x|$$

(c) The result is not analytic in x at $x = 0$. Why?

At $x = 0$, y becomes massless (i.e. it is a spring whose natural frequency goes to zero there). Integrating out massless degrees of freedom produces singularities in the effective action.

(d) Is the spectrum of the resulting 1d model with

$$\mathbf{H}_{\text{eff}} = \mathbf{p}_x^2 + V_{\text{eff}}(\mathbf{x})$$

discrete? Is this description valid in the regime which matters for the semi-classical analysis?

[Bonus: determine the spectrum of $\mathbf{H}_{\text{eff}}$.]

The potential $V \sim |x|$ bounds the trajectories and has a discrete spectrum. Integrating out y is a better approximation at larger $|x|$, which is where the dangerous flat directions occur. That is: this approximation is valid outside of a compact region of field space near $x = y = 0$ in which the potential is bounded below. Such a region cannot produce a continuum in the spectrum.

The actual spectrum of the absolute value potential is fun. The solutions of the Schrödinger problem (we can rescale x to get rid of the constant prefactor in the potential) $\psi(x) = \psi_>(x)\theta(x) + \psi_<(x)\theta(-x)$ satisfy

$$\begin{cases} (-\partial_x^2 + (x - E))\,\psi_> = 0, & x > 0 \\ (-\partial_x^2 + (-x - E))\,\psi_< = 0, & x > 0 \end{cases}.$$

The solutions for $x > 0$ are the two Airy functions

$$\psi_>(x) = a_> \text{Ai}(x - E) + b_> \text{Bi}(x - E)$$

of which the second blows up at large argument and hence cannot be normalized so we must set $b_> = 0$. Similarly, for $x < 0$, we have

$$\psi_<(x) = a_< \text{Ai}(-x - E) + b_< \text{Bi}(-x - E)$$

and again we must set $b_< = 0$. Since the potential has finite measure near $x = 0$ (i.e. no delta function) the wavefunction and its first derivative must be continuous at $x = 0$ and we have

$$\begin{aligned} \psi_>(0) = \psi_<(0) &\implies a_> \mathrm{Ai}(-E) = a_< \mathrm{Ai}(-E) \\ \psi'_>(0) = \psi'_<(0) &\implies a_> \mathrm{Ai}'(-E) = -a_< \mathrm{Ai}'(-E) \end{aligned} \tag{2}$$

which means either $a_> = a_< = 0$ OR $\mathrm{Ai}(-E) = 0$ OR $\mathrm{Ai}'(-E) = 0$. This means that the boundstates occur at zeros of the airy function or its derivative:

$$\{\text{boundstate energies }\} \propto \{E | \mathrm{Ai}(-E) = 0 \text{ or } \mathrm{Ai}'(-E) = 0\}.$$

3. **Matching with massive electrons.** [I got this problem from Iain Stewart.]

Consider QED in the regime where the photon momenta q^μ are much smaller than the electron mass m_e. In this regime, we can integrate out the electron and write an effective field theory involving only the photon.

(a) Calculate the QED one-loop vacuum polarization diagram using dimensional regularization, in the $\overline{\mathrm{MS}}$ scheme. (You've done this before. Here's the new ingredient:) Expand $\Pi(q^2)$ *through* first order in $\frac{q^2}{m_e^2}$.

We did this calculation in lecture and found

$$\delta\Pi_2^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\mu\right) \delta\Pi_2(q^2)$$

with

$$\delta\Pi_2(q^2)^{\overline{\mathrm{MS}}} = \frac{2\alpha}{\pi} \int_0^1 \mathrm{d}x \; x(1-x) \log\left(\frac{m^2 - x(1-x)q^2}{\mu^2}\right).$$

To facilitate the expansion in q^2, write

$$\log \frac{m^2 - x(1-x)q^2}{\mu^2} = \log \frac{m^2}{\mu^2} + \log\left(1 - x(1-x)\frac{q^2}{\mu^2}\right) \; .$$

Taylor expanding the $\log \log(1-u) = -\sum_{n=1}^{\infty} \frac{u^n}{n}$ gives

$$\delta\Pi_2(q^2)^{\overline{\mathrm{MS}}} = \frac{\alpha}{3\pi} \log \frac{m^2}{\mu^2} - \frac{\alpha}{15\pi}\frac{q^2}{m^2} + \mathcal{O}\left(\frac{q^4}{m^4}\right)$$

since

$$\int_0^1 \mathrm{d}x \; x(1-x) = \frac{1}{6}, \quad \int_0^1 \mathrm{d}x \; x^2(1-x)^2 = \frac{1}{30}.$$

(b) Write down a Lagrangian involving only the photon field operator that reproduces the first two terms in the expansion. (Hint: it should be gauge invariant and Lorentz invariant. There is essentially (up to integration by parts) only one addition to the Maxwell term). Use the calculation above to match between the photon-only EFT and QED at $\mu = m_e$, at this order in the fine structure constant α.

The leading term is the correction to the Maxwell term. The order-q^2 bit is produced by an operator involving two gauge fields and two extra derivatives.

$$\delta L = c_1 \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{c_2}{m^2} \frac{1}{4} F_{\mu\nu} \partial_\rho \partial^\rho F^{\mu\nu} + ...$$

Other gauge-invariant dimension six operators made of two As either are related by integration by parts ($\partial_\rho F_{\mu\nu} \partial^\rho F^{\mu\nu}$) or break parity ($(\star F)_{\mu\nu} \partial_\rho \partial^\rho F^{\mu\nu}$) and are therefore not produced by QED, which is parity invariant. Matching to our expression above gives:

$$c_1 = \delta\Pi_2(0), c_2 = m^2 \partial_{q^2} \delta\Pi_2(0).$$

(c) What symmetry of QED forbids dimension-6 operators involving three field strengths?

Charge conjugation invariance acts by $F_{\mu\nu} \to -F_{\mu\nu}$, and $j^\mu \to -j^\mu$, where j is the charge current. In the absence of any external charges, it acts like $(-1)^n$ where n is the number of external photons in the amplitude. This is called 'Furry's theorem'.

(d) At dimension 8, there are operators in the photons-only EFT which describe light-by-light scattering. Write them down (there are two). Draw the QED Feynman diagram which matches to these terms, and determine the number of factors of α in their coefficients. (Don't do the integrals unless you find it enjoyable.)

They are going to involve four Fs and therefore go like $1/m^4$. A single loop of an electron with four photon lines coming off (a box diagram) has four vertices, which therefore goes like $e^4 \sim \alpha^2$. We must decide what to do with the indices on the Fs. Parity demands that an even number of the Fs must be $\star F$s (can be zero). This leaves two terms, which are $\left(F_{\mu\nu} (\star F)^{\mu\nu}\right)^2 \sim \left(\vec{E} \cdot \vec{B}\right)^2$ and $\left(F_{\mu\nu} F^{\mu\nu}\right)^2 \sim \left(E^2 - B^2\right)^2$. For the actual matching coefficients, see Itzykson-Zuber. (What about $F_{\mu\nu} F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu}$?)

(e) [bonus] Use dimensional analysis in the low-energy EFT to estimate the size of the $\gamma\gamma \to \gamma\gamma$ cross section.

The amplitude goes like

$$\mathcal{A} \sim \alpha^2 \frac{p^4}{m^4}$$

with one factor of p from each F in the interaction. So the cross section is

$$\sigma(p) \sim \frac{1}{p^2}|\mathcal{A}|^2 \sim \frac{\alpha^4 p^6}{m^8}.$$

It's pretty small.