University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 1

Due 12:30pm Monday, April 8, 2019

Some of you will have seen some of these problems in Winter 2018 215B. Please do the parts you didn't do then.

1. **Brain-warmer: chiral anomaly in two dimensions.**

   Consider a massive relativistic Dirac fermion in 1+1 dimensions, with

   $$S = \int \mathrm{d}x\mathrm{d}t\bar{\psi}\left(\mathbf{i}\gamma^\mu\left(\partial_\mu + eA_\mu\right) - m\right)\psi.$$

   By heat-kernel regularization of its expectation value, show that the divergence of the axial current $j^5_\mu \equiv \mathbf{i}\bar{\psi}\gamma_\mu\gamma^5\psi$ is

   $$\partial_\mu j^5_\mu = 2\mathbf{i}m\bar{\psi}\gamma^5\psi + \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}.$$

2. **Where to find a Chern-Simons term.**

   Consider a field theory in $D = 2+1$ of a massive Dirac fermion, coupled to a background $\mathsf{U}(1)$ gauge field A:

   $$S[\psi, A] = \int d^3x\bar{\psi}\left(\mathbf{i}\not{D} - m\right)\psi$$

   where $D_\mu = \partial_\mu - \mathbf{i}A_\mu$.

   (a) Convince yourself that the mass term for the Dirac fermion in $D = 2+1$ breaks parity symmetry. That is, parity takes $m \to -m$. (Note that the definition of a parity transformation in d spatial dimensions is an element of $\mathsf{O}(d, 1)$ that's not in $\mathsf{SO}(d, 1)$, *i.e.* one with $\det(g) = -1$.)

   (b) We would like to study the effective action for the gauge field that results from integrating out the fermion field

   $$e^{-S_{eff}[A]} = \int [D\psi]e^{-S[\psi,A]}.$$

   Focus on the term quadratic in A:

   $$S_{eff}[A] = \int d^Dq A_\mu(q)\Pi^{\mu\nu}(q)A_\nu(-q) + ...$$

   We can compute $\Pi^{\mu\nu}$ by Feynman diagrams. Convince yourself that Π comes from a single loop of ψ with two A insertions.

(c) Evaluate this diagram using dim reg near $D = 3$. Show that, in the low-energy limit $q \ll m$ (where we can't make on-shell fermions),

$$\Pi^{\mu\nu} = a\frac{m}{|m|}\epsilon^{\mu\nu\rho}q_\rho + ...$$

for some constant a. Find a. Convince yourself that in position space this is a Chern-Simons term with level $k = \frac{1}{2}\frac{m}{|m|}$.

(d) [bonus] Redo this calculation by doing the Gaussian path integral over ψ.

3. **A bit more about Chern-Simons theory.**

Consider again $\mathsf{U}(1)$ gauge theory in $D = 2{+}1$ dimensions with the Chern-Simons action

$$S[a] = \frac{k}{4\pi}\int_\Sigma a \wedge da.$$

(Here I've changed the name of the dynamical gauge field to a lowercase a to distinguish it from the electromagnetic field A which will appear anon.)

(a) Show that the Chern-Simons action is gauge invariant under $a \to a + d\lambda$, as long as there is no boundary of spacetime Σ. Compute the variation of the action in the presence of a boundary of Σ.

(b) [bonus] Actually, the situation is a bit more subtle than the previous part suggests. The actual gauge transformation is

$$a \to g^{-1}ag + \frac{1}{\mathbf{i}}g^{-1}dg$$

which reduces to the previous if we set $g = e^{\mathbf{i}\lambda}$. That expression, however, ignores the global structure of the gauge group (*e.g.* in the abelian case, the fact that g is a periodic function). Consider the case where spacetime is $\Sigma = S^1 \times S^2$, and consider a *large gauge transformation*:

$$g = e^{\mathbf{i}n\theta}$$

where θ is the coordinate on the circle. Show that the variation of the CS term is $\frac{k}{4\pi}\int g^{-1}\partial g \wedge f$ (where $f = da$). Since the action appears in the path integral in the form $e^{\mathbf{i}S}$, convince yourself that the path integrand is gauge invariant if

(1) $\int_\Gamma f \in 2\pi\mathbb{Z}$ for all closed 2-surfaces Γ in spacetime, and

(2) $k \in \mathbb{Z}$.

The first condition is called flux quantization, and is closely related to Dirac's condition.

(c) [bonus] In the case where G is a non-abelian lie group, the argument for quantization of the level (k) is more straightforward. Show that the variation of the CS Lagrangian

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\mathrm{tr}\left(a \wedge da + \frac{2}{3} a \wedge a \wedge a\right)$$

under $a \to gag^{-1} - \partial g g^{-1}$ is

$$\mathcal{L}_{CS} \to \mathcal{L}_{CS} + \frac{k}{4\pi} d\mathrm{tr} dg g^{-1} \wedge a + \frac{k}{12\pi}\mathrm{tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right).$$

The integral of the second term over any closed surface is an integer. Conclude that $e^{\mathbf{i}S_{CS}}$ is gauge invariant if $k \in \mathbb{Z}$.

(d) If there is a boundary of spacetime, something must be done to fix up this problem. Consider the case where $\Sigma = \mathbb{R} \times \mathrm{UHP}$ where $\mathbb{R}$ is the time direction, and UHP is the upper half-plane $y > 0$. One way to fix the problem is simply to declare that the would-be gauge transformations which do not vanish at $y = 0$ are not redundancies. This means that they represent physical degrees of freedom.

The exterior derivative on this spacetime decomposes into $d = \partial_t dt + \tilde{d}$ where $\tilde{d}$ is just the spatial part, and similarly the gauge field is $a = a_0 dt + \tilde{a}$.

Let us choose the gauge $a_0 = 0$. We must still impose the equations of motion for a_0 (in the path integral it is a Lagrange multiplier). Solve this equation, and evaluate the action for the resulting solution.

We can also add local terms at the boundary to the action. Consider adding $\Delta S = g\int_{\partial\Sigma} \tilde{a}_x^2$ (for some coupling constant g). In the presence of such a boundary term, find the equations of motion for the boundary degrees of freedom.

Interpretation: the Chern-Simons theory on a space with boundary necessarily produces a chiral edge mode.

(e) Suppose we had a system in $2+1$ dimensions with a gap to all excitations, which breaks parity symmetry and time-reversal invariance, and involves a conserved current J^μ, with

$$0 = \partial^\mu J_\mu. \tag{1}$$

Solve this equation by writing $J^\mu = \epsilon^{\mu\nu\rho}\partial_\nu a_\rho$ in terms of a one-form $a = a_\mu dx^\mu$. Guess the leading terms in the action for a_μ in a derivative expansion.

(f) Now suppose the current J^μ is coupled to an external electromagnetic field A_μ by $S \ni \int J^\mu A_\mu$. Ignoring the Maxwell term for a, compute the Hall conductivity, σ^{xy}, which is defined by Ohm's law $J^x = \sigma^{xy}E^y$.

4. **An application of the anomaly to a theory without gauge fields.**

Consider a 1+1d theory of Dirac fermions coupled to a background scalar field θ as follows:

$$\mathcal{L} = \bar{\Psi}\left(\mathbf{i}\partial\!\!\!/ + me^{\mathbf{i}\theta\gamma^5}\right)\Psi.$$

We wish to ask: if we subject the fermion to various configurations of $\theta(x)$ (such as a domain wall where $\theta(x) = \pi + \theta(x)$) what does the fermion number do in the groundstate?

(a) Convince yourself that when θ is constant

$$\langle j^\mu \rangle = 0$$

where $j^\mu = \bar{\Psi}\gamma^\mu\Psi$ is the fermion number current.

(b) Minimally couple the fermion to a *background* gauge field A_μ. Let $e^{\mathbf{i}\Gamma[A,\theta]} = \int [d\Psi] e^{\mathbf{i}S}$. Convince yourself that the term linear in A in $\Gamma[A,\theta] = \text{const} + \int A_\mu J^\mu + \mathcal{O}(A^2)$ is the vacuum expectation value of the current $\langle j^\mu \rangle = J^\mu$.

(c) Show that by a local chiral transformation $\Psi \to e^{\mathbf{i}\theta(x)\gamma^5/2}\Psi$ we can remove the dependence on θ from the mass term.

(d) Where does the theta-dependence go? Use the 2d chiral anomaly to relate $\langle j^\mu \rangle$ to $\partial\theta$. Notice that the result is independent of m. [This relation was found by Goldstone and Wilczek. The associated physics is realized in Polyacetylene.]

(e) Show that a domain wall where θ jumps from 0 to π localizes *fractional* fermion number.

(f) [bonus problem] Consider the Dirac hamiltonian in the presence of such a soliton. Show that there is a localized mode of zero energy.