University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239/139 Spring 2018
Assignment 5

Due 12:30pm Monday, May 7, 2018

1. **Chain rule for mutual information.** [optional]

   Show from the definitions that the mutual information satisfies the following chain rule:

   $$I(X:YZ) = I(X:Y) + I(X:Z|Y) = I(X:Z) + I(X:Y|Z).$$

   More generally,

   $$I(X_1 \cdots X_n : Y) = \sum_{i=1}^{n} I(X_i Y | X_{i-1} \cdots X_i). \tag{1}$$

2. **Control-X brainwarmer.**

   Show that the operator control-X can be written variously as

   $$\mathsf{CX}_{BA} = |0\rangle\langle 0|_B \otimes 1\!\!1_A + |1\rangle\langle 1|_B \otimes \mathbf{X}_A = \mathbf{X}_A^{\frac{1}{2}(1-\mathbf{Z}_B)} = e^{\frac{\mathbf{i}\pi}{4}(1-\mathbf{Z}_B)(1+\mathbf{X}_A)}.$$

3. **Density matrix exercises.**

   (a) Show that the most general density matrix for a single qbit lies in the Bloch ball, *i.e.* is of the form

   $$\boldsymbol{\rho}_v = \frac{1}{2}\left(1\!\!1 + \vec{v}\cdot\vec{\boldsymbol{\sigma}}\right), \quad \sum_i v_i^2 \leq 1.$$

   Find the determinant, trace, and von Neumann entropy of $\boldsymbol{\rho}_v$.

   (b) [from Barnett] A single qbit state has $\langle \mathbf{X} \rangle = s$. Find the most general forms for the corresponding density operator with the minimum and maximum von Neumann entropy. (Hint: the Bloch ball is your friend.)

   (c) Show that the *purity* of a density matrix $\pi[\boldsymbol{\rho}] \equiv \mathrm{tr}\boldsymbol{\rho}^2$ satisfies $\pi[\boldsymbol{\rho}] \leq 1$ with saturation only if $\boldsymbol{\rho}$ is pure.

(d) [from Barnett] Show that the quantum relative entropy satisfies the following

$$D(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B || \boldsymbol{\sigma}_A \otimes \boldsymbol{\sigma}_B) = D(\boldsymbol{\rho}_A || \boldsymbol{\sigma}_A) + D(\boldsymbol{\rho}_B || \boldsymbol{\sigma}_B). \tag{2}$$

$$\sum_i p_i D\left(\boldsymbol{\sigma}_i || \boldsymbol{\rho}\right) = \sum_i p_i D(\boldsymbol{\sigma}_i || \boldsymbol{\sigma}_{\text{av}}) + D(\boldsymbol{\sigma}_{\text{av}} || \boldsymbol{\rho}) \tag{3}$$

$$D\left(\boldsymbol{\sigma}_{\text{av}} || \boldsymbol{\rho}\right) \leq \sum_i p_i D(\boldsymbol{\sigma}_i || \boldsymbol{\rho}) \tag{4}$$

for any probability distribution $\{p_i\}$ and density matrices $\boldsymbol{\rho}, \boldsymbol{\sigma}_i$, and where $\boldsymbol{\sigma}_{\text{av}} \equiv \sum_i p_i \boldsymbol{\sigma}_i$.

4. **Teleportation for qdits.** [optional, from Christiandl]

   Show that it is possible to teleport a state $|\xi\rangle_A \in \mathcal{H}_A, |A| \equiv d$ from A to B using the maximally-entangled state

   $$|\Phi\rangle_{AB} \equiv \frac{1}{\sqrt{d}} \sum_{n=1}^{d} |nn\rangle_{AB} .$$

   Hint: Consider the clock and shift operators

   $$\mathbf{Z} \equiv \sum_{n=1}^{d} |n\rangle \langle n| \omega^n, \ \omega \equiv e^{\frac{2\pi \mathbf{i}}{d}}, \quad \mathbf{X} \equiv \sum_{n=1}^{d} |n+1\rangle \langle n|$$

   where the argument of the ket is to be understood mod d. Show that these generalize some of the properties of the Pauli $\mathbf{X}$ and $\mathbf{Z}$ in that they are unitary and that they satisfy the (discrete) Heisenberg algebra

   $$\mathbf{XZ} = a\mathbf{ZX}$$

   for some c-number a which you should determine.

5. **Thermal density matrix.** Suppose given a Hamiltonian H. In lecture we showed that the thermal density matrix $\boldsymbol{\rho}_T \equiv \frac{e^{-\frac{H}{k_B T}}}{Z}$ has the maximum von Neumann entropy for any state with the same expected energy. Show that if instead we are given a fixed temperature T, the thermal density matrix s minimizes the free energy functional

   $$F_T[\boldsymbol{\rho}] \equiv \text{tr}\boldsymbol{\rho} H - T S_{vN}[\boldsymbol{\rho}].$$

6. **Amplitude-damping channel.** [from Preskill 3.4.3, Le Bellac §15.2.4]

   This is a very simple model for a two-level atom, coupled to an environment in the form of a (crude rendering of a) radiation field.

The atom has a groundstate $|0\rangle_A$; if it starts in this state, it stays in this state, and the radiation field stays in its groundstate $|0\rangle_E$ (zero photons). If the atom starts in the excited state $|1\rangle_A$, it has some probability p per time dt to return to the groundstate and emit a photon, exciting the environment into the state $|1\rangle_E$ (one photon). This is described by the time evolution

$$\mathbf{U}_{AE} |0\rangle_A \otimes |0\rangle_E = |0\rangle_A \otimes |0\rangle_E$$

$$\mathbf{U}_{AE} |1\rangle_A \otimes |0\rangle_E = \sqrt{1-p}\, |1\rangle_A \otimes |0\rangle_E + \sqrt{p}\, |0\rangle_A \otimes |1\rangle_E .$$

(a) Show that the evolution of the atom's density matrix can be written in terms of two Kraus operators $\mathcal{K}_i$, find those operators and show that they satisfy $\sum_i \mathcal{K}_i^\dagger \mathcal{K}_i = \mathbb{1}_{\text{atom}}$.

(b) Assuming that the environment is forgetful and resets to $|0\rangle_E$ after each time step dt, find the fate of the density matrix after time $t = ndt$ for late times $n \gg 1$, *i.e.* upon repeated application of the channel.

(c) Evaluate the *purity* $\text{tr}\boldsymbol{\rho}_n^2$ of the nth iterate. (Recall that the purity is 1 IFF the state is pure.)

7. **Phase-flipping decoherence channel.** [from Schumacher]

Consider the following model of decoherence on an N-state Hilbert space, with basis $\{|k\rangle, k = 1..N\}$.

Define the unitary operator

$$\mathbf{U}_\alpha \equiv \sum_k \alpha_k |k\rangle \langle k|$$

where α_k is an N-component vector of signs, ± 1 – it flips the signs of some of the basis states. There are 2^N distinct such operators.

Imagine that interactions with the environment act on any state of the system with the operator $\mathbf{U}_\alpha$, for some α, chosen randomly (with uniform probability from the 2^N choices).

[Hint: If you wish, set $N = 2$.]

(a) Warmup question: If the initial state is $|\psi\rangle$, what is the probability that the resulting output state is $\mathbf{U}_\alpha |\psi\rangle$?

(b) Write an expression for the resulting density matrix, $\mathcal{D}(\boldsymbol{\rho})$, in terms of $\boldsymbol{\rho}$.

(c) Think of $\mathcal{D}$ as a superoperator, an operator on density matrices. How does $\mathcal{D}$ act on a density matrix which is diagonal in the given basis,

$$\boldsymbol{\rho}_{\text{diagonal}} = \sum_k p_k \left|k\right\rangle \left\langle k\right| \quad ?$$

(d) The most general initial density matrix is not diagonal in the k-basis:

$$\boldsymbol{\rho}_{\text{general}} = \sum_{kl} \rho_{kl} \left|k\right\rangle \left\langle l\right| \quad .$$

what does $\mathcal{D}$ do to the off-diagonal elements of the density matrix?