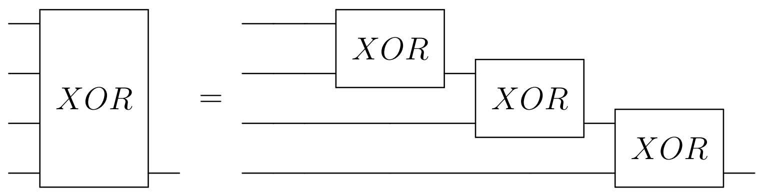
University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239/139 Spring 2018
Assignment 2

Due 12:30pm Monday, April 16, 2018

1. **Classical circuits brain-warmer.**

   (a) Show that this circuit adds the input bits (at left) mod two:

   (b) [Optional] Construct a circuit with n input bits and one output bit which gives zero unless exactly one of the bits is one. The ingredients available are any gates that take two bits to at most two bits.[1]

2. **Entanglement entropy in a quantum not-so-many-body system made from spins.**

   Consider the transverse-field Ising model on a lattice with only two ($L = 2$) sites, $i = 1, 2$, so that the Hilbert space is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where each of $\mathcal{H}_{1,2}$ is a two-state system, and the Hamiltonian is

   $$\mathbf{H} = -J\left(2Z_1Z_2 + gX_1 + gX_2\right).$$

   (a) Find the matrix elements of the Hamiltonian in the eigenbasis of Z_1, Z_2

   $$h_{ab} = \langle s_a | \mathbf{H} | s_b \rangle$$

   where $a, b = 1..N$. What is N in terms of the system size L? Check that your matrix is hermitian.

   (b) Find the eigenvalues of h and plot them as a function of g. (You may wish to use a computer for this and other parts of this problem.)

[1] Thanks to Hans Singh and Brian Shotwell for pointing out that this function cannot be computed with only two-to-one gates! I had an error in my solution, and the problem is a little trickier than I thought.

(c) Find the eigenvector (the groundstate) and eigenvalue of the matrix h with the lowest eigenvalue. Write the groundstate as

$$|\Psi\rangle = \sum_{a=1}^{N} \alpha_a |\phi_a\rangle .$$

(d) The Hilbert space is of the form $\mathcal{H}_1 \otimes \mathcal{H}_2$ where $\mathcal{H}_{1,2}$ are the Hilbert spaces of a single spin. Construct the reduced density matrix for the first site in the groundstate

$$\boldsymbol{\rho}_1 \equiv \mathrm{tr}_{\mathcal{H}_2} |\Psi\rangle\langle\Psi|.$$

(e) Find the eigenvalues λ_α of $\boldsymbol{\rho}_1$. Calculate the von Neumann entropy of $\boldsymbol{\rho}_1$, $S(\boldsymbol{\rho}_1) = -\sum_\alpha \lambda_\alpha \log \lambda_\alpha$ as a function of g. What is the numerical value when $g \to \infty$? What about $g \to 0$? Do they agree with your expectations?

(f) [Bonus] Redo this problem with $L = 3$ sites (or more):

$$\mathbf{H} = -J\left(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1 + gX_1 + gX_2 + gX_2\right).$$

3. **Entanglement entropy in a quantum not-so-many-body system made from electrons.** [This problem is optional for students in physics 139.][2]

Consider a system consisting of two electrons, each with spin one-half, and each of which can occupy either of two sites labelled $i = 1, 2$. The dynamics is governed by the following (Hubbard) Hamiltonian:

$$\mathbf{H} = -t \sum_{\sigma=\uparrow,\downarrow} \left(\mathbf{c}^\dagger_{1\sigma}\mathbf{c}_{2\sigma} + \mathbf{c}^\dagger_{2\sigma}\mathbf{c}_{1\sigma}\right) + U \sum_i \mathbf{n}_{i\uparrow}\mathbf{n}_{i\downarrow}.$$

$\sigma = \uparrow, \downarrow$ labels the electron spin. $\mathbf{c}$ and $\mathbf{c}^\dagger$ are fermion creation and annihilation operators,

$$\{\mathbf{c}_{i\sigma}, \mathbf{c}^\dagger_{i'\sigma'}\} = \delta_{ii'}\delta_{\sigma\sigma'}$$

and $\mathbf{n}_{i\sigma} \equiv \mathbf{c}^\dagger_{i\sigma}\mathbf{c}_{i\sigma}$ is the number operator. The condition that there is a total of two electrons means we only consider states $|\psi\rangle$ with

$$\left(\sum_{i,\sigma} \mathbf{n}_{i\sigma} - 2\right)|\psi\rangle = 0.$$

The first term is a kinetic energy which allows the electrons to hop between the two sites. The second term is a potential energy which penalizes the states where two electrons sit at the same site, by an energy $U > 0$.

[2] I got this problem from Tarun Grover.

(a) Enumerate a basis of two-electron states (make sure they satisfy the Pauli exclusion principle).

(b) The Hamiltonian above has some symmetries. In particular, the total electron spin in the $\hat{z}$ direction is conserved. For simplicity, let's focus on the states where it is zero, such as $\mathbf{c}_{1\uparrow}^\dagger \mathbf{c}_{2\downarrow}^\dagger |0\rangle$ where $|0\rangle$ is the state with no electrons, $\mathbf{c}_{i\sigma} |0\rangle = 0$. Find a basis for this subspace, $\{\phi_a\}, a = 1..N$.

(c) Find the matrix elements of the Hamiltonian in this basis,

$$h_{ab} \equiv \langle \phi_a | \mathbf{H} | \phi_b \rangle , \quad a, b = 1..N.$$

(d) Find the eigenstate and eigenvalue of the matrix h with the lowest eigenvalue. Write the groundstate as

$$|\Psi\rangle = \sum_{a=1}^{N} \alpha_a |\phi_a\rangle .$$

(e) Before imposing the global constraints on particle number and S^z, the Hilbert space can be factored (up to some signs because fermions are weird) by site: $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_i = \text{span}\{|0\rangle , \mathbf{c}_{i\uparrow}^\dagger |0\rangle , \mathbf{c}_{i\downarrow}^\dagger |0\rangle , \mathbf{c}_{i\uparrow}^\dagger \mathbf{c}_{i\downarrow}^\dagger |0\rangle\}$. Using this bipartition, construct the reduced density matrix for the first site in the groundstate:

$$\boldsymbol{\rho}_1 \equiv \text{tr}_{\mathcal{H}_2} |\Psi\rangle \langle\Psi| .$$

(f) Find the eigenvalues λ_α of $\boldsymbol{\rho}_1$. Calculate the von Neumann entropy of $\boldsymbol{\rho}_1$, $S(\boldsymbol{\rho}_1) = -\sum_\alpha \lambda_\alpha \log \lambda_\alpha$ as a function of U/t. What is the numerical value when $U/t \to \infty$?

(g) **Super-Exchange.** Go back to the beginning and consider the limit $U \gg t$. What are the groundstates when $U/t \to \infty$, so that we may completely ignore the hopping term?

At second order in degenerate perturbation theory, find the effective Hamiltonian which splits the degeneracy for small but nonzero t/U. Write the answer in terms of the spin operator

$$\vec{\mathbf{S}}_i \equiv \mathbf{c}_{i\sigma}^\dagger \vec{\sigma}_{\sigma\sigma'} \mathbf{c}_{i\sigma'}.$$

The sign is important!

(h) Redo all the previous parts for the case where the two particles are spin-half bosons,

$$\mathbf{c}_{i\sigma} \rightsquigarrow \mathbf{b}_{i\sigma}, \quad [\mathbf{b}_{i\sigma}, \mathbf{b}_{i'\sigma'}^\dagger] = \delta_{ii'}\delta_{\sigma\sigma'}.$$

4. **Chain rules.**

   Show that for a joint distribution of n random variables $p(X_1 \cdots X_n)$, the joint and conditional entropies satisfy the following chain rule:

$$H(X_1 \cdots X_n) = \sum_{i=1}^{n} H(X_i | X_{i-1} \cdots X_1).$$

   Show that the $n = 2$ case is the expectation of the log of the BHS of Bayes rule. Then repeatedly apply the $n = 2$ case to increasing values of n.

5. **Learning decreases ignorance only on average.**

   Consider the joint distribution $p_{yx} = \begin{pmatrix} 0 & a \\ b & b \end{pmatrix}_{yx}$, where $y = \uparrow, \downarrow$ is the row index and $x = \uparrow, \downarrow$ is the column index (so yx are like the indices on a matrix). Normalization implies $\sum_{xy} p_{xy} = a + 2b = 1$, so we have a one-parameter family of distributions, labelled by b.

   What is the allowed range of b?

   Find the marginals for x and y. Find the conditional probabilities $p(x|y)$ and $p(y|x)$.

   Check that $H(X|Y) \leq H(X)$ and $H(Y|X) \leq H(Y)$ for any choice of b.

   Show, however, that $H(X|Y = \downarrow) > H(X)$ for any $b < \frac{1}{2}$.