University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239/139 Spring 2018
Assignment 1.5

Due 12:30pm Wednesday, April 11, 2018

Here are some bonus problems for the benefit of those of you with limited prior experience with quantum mechanics. These problems are strictly optional, unless you find them difficult, in which case they are compulsory. If any of the notation is not clear please ask. You may find these notes helpful.

1. **Pauli spin matrix gymnastics.**

   (This problem is long but each part is pretty simple.) Recall the definition of the Pauli spin matrices:

$$\boldsymbol{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boldsymbol{\sigma}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \boldsymbol{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   (Occasionally I will write $\boldsymbol{\sigma}^x \equiv \boldsymbol{\sigma}^1 \equiv \boldsymbol{X}, \boldsymbol{\sigma}^y \equiv \boldsymbol{\sigma}^2 \equiv \boldsymbol{Y}, \boldsymbol{\sigma}^z \equiv \boldsymbol{\sigma}^3 \equiv \boldsymbol{Z}$.)

   (a) Show that the $\boldsymbol{\sigma}^i$ are Hermitian.

   (b) Check that the $\boldsymbol{\sigma}^i$ all square to the identity operator, $(\boldsymbol{\sigma}^i)^2 = \mathbb{1}, \forall i$.

   (c) Check that the $\boldsymbol{\sigma}$s are all traceless, $\text{tr}\boldsymbol{\sigma}^i = 0, \forall i$, where the trace operation is defined as $\text{tr}\,(M) \equiv \sum_a M_{aa}$.

   (d) Find their eigenvalues and eigenvectors.

   (e) There are only so many two-by-two matrices. A product of sigmas can be written in terms of sigmas. Show that

$$\boldsymbol{\sigma}^i \boldsymbol{\sigma}^j = i\epsilon^{ijk}\boldsymbol{\sigma}^k + \delta^{ij}\mathbb{1} \tag{1}$$

   where ϵ^{ijk} is the completely antisymmetric object with $\epsilon^{123} = 1$ (that is: $\epsilon^{ijk} = 0$ if any of ijk are the same, $= 1$ if ijk is a cyclic permutation of 123 and $= -1$ if ijk is an odd permutation of 123, like 132). You may prefer to do parts 1g and 1h of the problem first.

   (f) There are only so many two-by-two hermitian matrices. Convince yourself that an arbitrary hermitian operator $\mathbf{A}$ acting on a two-dimensional Hilbert space can be decomposed as

$$\mathbf{A} = a_0\mathbb{1} + a_1\mathbf{X} + a_2\mathbf{Y} + a_3\mathbf{Z} \equiv a_\mu \boldsymbol{\sigma}^\mu$$

where $\boldsymbol{\sigma}^0 \equiv \mathbb{1}$ is the identity operator (which does nothing to everyone). Furthermore, show that the coefficients a_μ can be extracted by taking traces:

$$a^\mu = c\text{tr}\left(\mathbf{A}\sigma^\mu\right)$$

for some constant c. Find c.

(g) Convince yourself that (1) implies

$$[\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j] = 2i\epsilon^{ijk}\boldsymbol{\sigma}^k,$$

(where $[A, B] \equiv AB - BA$ is the *commutator*) and that therefore $\mathbf{J}^i_{\frac{1}{2}} \equiv \frac{1}{2}\boldsymbol{\sigma}^i$ satisfy

$$[\mathbf{J}^i_{\frac{1}{2}}, \mathbf{J}^j_{\frac{1}{2}}] = i\epsilon^{ijk}\mathbf{J}^k_{\frac{1}{2}},$$

the same algebra as the rotation generators on the 3-state system in §1.5 of these lecture notes. [Cultural note: this is the Lie algebra called **su**(2).]

(h) Convince yourself that (1) implies

$$\{\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j\} = 2\delta^{ij}$$

where $\{A, B\} \equiv AB + BA$ is called the *anti-commutator*.

[Cultural note: this is called the Dirac algebra or Clifford algebra.]

It may be useful to note that $\{A, B\} + [A, B] = 2AB$.

(i) Convince yourself that (1) is the same as

$$(\vec{\boldsymbol{\sigma}} \cdot \vec{a})(\vec{\boldsymbol{\sigma}} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\vec{\boldsymbol{\sigma}} \cdot \left(\vec{a} \times \vec{b}\right).$$

In particular, check that $(\vec{\boldsymbol{\sigma}} \cdot \hat{n})^2 = \mathbb{1}$ if $\hat{n}$ is a unit vector.

(j) Show that

$$e^{i\frac{\theta}{2}\vec{\boldsymbol{\sigma}}\cdot\hat{n}} = \mathbb{1}\cos\frac{\theta}{2} + i\vec{\boldsymbol{\sigma}} \cdot \hat{n}\sin\frac{\theta}{2}$$

where $\hat{n}$ is a unit vector.

[Hint: use the Taylor expansion of the LHS $e^A = 1 + A + A^2/2! + ...$ and the previous results for $(\vec{\boldsymbol{\sigma}} \cdot \hat{n})^2$.]

2. **Dirac notation exercises.**

Dirac's notation for state vectors is extremely useful and we will use it all the time. The following problems are intended to test your understanding of the discussion on pages 1-6 – 1-10 of these notes.

(a) Consider some operators acting on a Hilbert space with a resolution of the identity of the form

$$\mathbb{1} = \sum_n |n\rangle \langle n| \ .$$

Recall that the matrix representation of an operator in this basis is $A_{nm} = \langle n| \hat{A} |m\rangle$. Using Dirac notation, show that the matrix representation of a product of operators $\left(\hat{A}\hat{B}\right)_{nr}$ is given by the matrix product of the associated matrices $\sum_m A_{nm} B_{mr}$.

(b) For a normalized state $|a\rangle$ (normalized means $\langle a|a\rangle = 1$), show that the operator

$$P_a \equiv |a\rangle\langle a|$$

is a *projector*, in the sense that $P_a^2 = P_a$ (doing it twice is the same as doing it once), and $P_a = P_a^\dagger$.

3. **Normal matrices.**

An operator (or matrix) $\hat{A}$ is *normal* if it satisfies the condition $[\hat{A}, \hat{A}^\dagger] = 0$.

(a) Show that real symmetric, hermitian, real orthogonal and unitary operators are normal.

(b) Show that any operator can be written as $\hat{A} = \hat{H} + \mathbf{i}\hat{G}$ where $\hat{H}, \hat{G}$ are Hermitian. [Hint: consider the combinations $\hat{A} + \hat{A}^\dagger, \hat{A} - \hat{A}^\dagger$.] Show that $\hat{A}$ is normal if and only if $[\hat{H}, \hat{G}] = 0$.

(c) Show that a normal operator $\hat{A}$ admits a spectral representation

$$\hat{A} = \sum_{i=1}^{N} \lambda_i \hat{P}_i$$

for a set of projectors $\hat{P}_i$, and complex numbers λ_i.

4. **Clock and shift operators.**

Consider an N-dimensional Hilbert space, with orthonormal basis $\{|n\rangle, n = 0, \ldots, N-1\}$. Consider operators $\mathbf{T}$ and $\mathbf{U}$ which act on this N-state system by

$$\mathbf{T}|n\rangle = |n+1\rangle, \qquad \mathbf{U}|n\rangle = e^{\frac{2\pi \mathbf{i} n}{N}} |n\rangle \ .$$

In the definition of $\mathbf{T}$, the label on the ket should be understood as its value modulo N, so $N + n \equiv n$ (like a clock).

(a) Find the matrix representations of $\mathbf{T}$ and $\mathbf{U}$ in the basis $\{|n\rangle\}$.

(b) What are the eigenvalues of $\mathbf{U}$? What are the eigenvalues of its adjoint, $\mathbf{U}^\dagger$?

(c) Show (using Dirac notation, not matrices) that

$$\mathbf{UT} = e^{\frac{2\pi\mathbf{i}}{N}}\mathbf{TU}.$$

(d) From the definition of adjoint, how does $\mathbf{T}^\dagger$ act, *i.e.*

$$\mathbf{T}^\dagger |n\rangle = ?$$

(e) Show that the 'clock operator' $\mathbf{T}$ is normal – that is, commutes with its adjoint – and therefore can be diagonalized by a unitary basis rotation.

(f) Find the eigenvalues and eigenvectors of $\mathbf{T}$.
[Hint: consider states of the form $|\theta\rangle \equiv \sum_n e^{\mathbf{i}n\theta} |n\rangle$.]