University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2017
Assignment 10

Due 12:30pm Wednesday, June 14, 2017

1. **Kosterlitz-Thouless transition.**

   (a) **Kosterlitz-Thouless estimate.** Recall that the energy of an n-fold vortex in a global $\mathsf{U}(1)$-symmetry-breaking order-parameter field is of the form

   $$E_v(L) = \frac{1}{2\pi} K \int_a^L \left(\frac{n}{r}\right)^2 d^2r \sim Kn^2 \ln \frac{L}{a}$$

   where L is an IR cutoff, a is a UV cutoff, and an n-fold vortex is of the form $\phi = n\varphi$ (far from the core). To make contact with our $D = 1 + 1$ analysis, let's replace the coupling with $\rho_s \equiv K/T$ where more precisely ρ_s is defined by

   $$\beta H[\phi] = \int d^2x \frac{\rho_s}{2\pi} (\partial\phi)^2 .$$

   Convince yourself that the free energy density (per vortex) of a collection of n-fold vortices and antivortices with spacing L_0 is

   $$F = E_v(L_0) - T \ln(L_0/a)^2$$

   (where $(L_0/a)^2$ is the number of possible locations). Show that this free energy is lowered by creating more $n = \pm 1$ vortices if $\rho_s < \frac{2}{\pi}$, while for $\rho_s > \frac{2}{\pi}$ the vortices want to stay home. Note the sameness of this critical value with the value found in lecture.

   (b) **Another derivation of the T-dual description.** [Chaikin-Lubensky, problem 9.10] Consider a Coulomb-gas description of the sum over vortices:

   $$Z(y) = \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} \frac{1}{N_+!N_-!} y^{N_++N_-} \prod_{\alpha=1}^{N_+} \prod_{\beta=1}^{N_-} \int d^2x_+^\alpha d^2x_-^\beta e^{-\beta \int d^2x \int d^2y n_v(x) G(x-y) n_v(y)}$$

   where $n_v(x) = \sum_\alpha \delta(x - x_+^\alpha) - \sum_\beta \delta(x - x_-^\beta)$ is the total density of vortices, and $G(x) = -\frac{\rho_s}{2\pi} \ln(|x|/a)$ is the potential by which the vortices interact (the free massless 2d scalar Green's function), y is the vortex fugacity, and ρ_s is the spin stiffness. Use a Hubbard-Stratonovich transformation and do the sums over vortices to write this as

$$Z(y) = c \int [d\Theta] e^{-\beta S_{SG}[\Theta]}$$

where

$$S_{SG}[\phi] = \int d^2x \left(\frac{1}{2(2\pi)^2 \rho_s} (\partial\Theta)^2 - y \cos\Theta \right)$$

is the Sine-Gordon action.

(c) So consider the sine-Gordon model in two euclidean dimensions. In lecture we studied only infinitesimal y (in the Hamiltonian description). Using either OPEs or ordinary perturbation theory, find the form of the beta functions for the couplings ρ_s and y near the KT point (small $x \equiv 2 - \rho_s$) to quadratic order in the deviations. (Don't worry about the values of the coefficients.) Draw the phase diagram in the xy plane.

(d) **This is an infinite-order phase transition.** [bonus problem] The correlation length "exponent" near this critical point is very strange. Integrate the RG equation $\frac{dy}{dx} = \frac{x}{Ay}$ with initial conditions $y(\ell = 0) = y(0), x(\ell = 0) = -\sqrt{A}y(0)(1+t)$ where $t = (T - T_c)/T_c$ is small. Find the value of the RG parameter ℓ when $x(\ell) = 1$, and find the resulting dependence of the correlation length $\xi = \xi_0 e^{\ell}$ on the t, the deviation from the critical point.

2. **The shape of the Mott lobes.**

Consider again the Bose-Hubbard model

$$H_{BH} = \sum_i (-\mu n_i + U n_i(n_i - 1)) + \sum_{ij} b_i^\dagger w_{ij} b_j$$

on a lattice with uniform coordination number z. The hopping matrix is $w_{ij} \equiv w$ if ij share a link, and zero otherwise.

(a) Show that the coherent state path integral representation of the euclidean partition function is the one I wrote in lecture, namely

$$Z = \int [d^2b] e^{-\int d\tau \sum_i \left(b_i^\star \partial_\tau b_i - \mu b_i^\star b_i + U b_i^\star b_i^\star b_i b_i + b_i^\star w_{ij} b_j \right)}.$$

(b) In lecture we derived a condition for the boundary of the Mott lobes in the phase diagram in mean field theory: it is when the coefficient r in the variational energy

$$\langle \Psi_{\text{var}} | H_{BH} | \Psi_{\text{var}} \rangle = E_0^0 + r|\psi|^2 + \mathcal{O}(|\psi|^4|)$$

changes sign. Using second order perturbation theory (in the $\psi b^\dagger + \psi^\star b$ terms of the mean field hamiltonian) or otherwise, derive the form of r as a function of μ/U. The answer is

$$r = \chi_0(n_0)\left(1 - Zw\chi(n_0)\right)$$

where

$$\chi_0(n_0) \equiv \frac{n_0+1}{Un_0 - \mu} + \frac{n_0}{\mu - U(n_0-1)},$$

and n_0 is the integer which minimizes $Un(n-1) - \mu n$.

(c) Draw the Mott lobes using this formula.

(d) [bonus problem] Define $\tilde{r}$ as the coefficient of $|\psi|^2$ in $W[\psi] = \int d^2x\,(... + \tilde{r}|\psi|^2)$ where

$$e^{-W[\psi]} = \int [d^2b] e^{-\int d\tau \mathcal{L}'_b},$$

with $\mathcal{L}'_b = \sum_i \left(b_i^\dagger \partial_\tau b_i - \mu b_i^\dagger b_i + U b_i^\dagger b_i^\dagger b_i b_i - \Psi b_i^\dagger - \Psi^\star b_i\right) + \sum_{ij} \Psi_i J_{ij}^{-1} \Psi_j.$

Show that $\tilde{r} \propto r$ as computed in mean field theory.

3. **Path integral duality for gauge theory in $D = 3+1$.**

Consider Maxwell theory in $D = 3+1$,

$$S[A] = -\int d^4x \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{16\pi^2}\int F \wedge F.$$

Following a similar strategy to the T-duality problem, derive the electric-magnetic dual description, where $\tau \equiv \frac{2\pi}{g^2} + \mathbf{i}\theta$ is replaced by $\tau_{\text{dual}} \equiv -\frac{1}{\tau}$.

For help, see here.

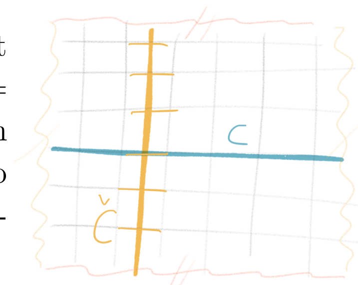
4. **Continuum $\mathsf{U}(1)$ version of canonical loop operators.**

In the toric code on T^2, there are four degenerate groundstates which are not distinguishable by local operators. They are made from a given groundstate (say the one where all the electric flux lines are contractible) by the action of the algebra of loop operators, defined (in the basis where the plaquette operator is $\prod_{\ell \in \partial p} \mathbf{X}_\ell$) by

$$W(C) = \prod_{\ell \in C} \mathbf{X}_\ell, \quad V(\check{C}) = \prod_{\ell \perp \check{C}} \mathbf{Z}_\ell$$

where C is a path through the lattice , and $\check{C}$ is a path through the *dual* lattice (the lattice whose sites are the faces of the original lattice).

The groundstate degeneracy on the torus arises from the fact that the groundstates must represent the algebra $W_x V_{\check{y}} = -V_{\check{y}} W_x$, where α is a loop wrapping the α-direction, as in the figure at right. (This relation follows because these two curves share a single link, on which the operators anticommute.)



Here we would like to understand the analogs of these operators in (the deconfined phase of) $\mathsf{U}(1)$ gauge theory and in $D = 3+1$. The analogy is: $\mathbf{X}_\ell \sim e^{\mathbf{i}\int_\ell A}$, $\mathbf{Z}_\ell \sim e^{\mathbf{i}\int_\ell d\ell \times E}/g^2$. The analog of $W(C)$ is the Wilson loop $W(C) = e^{\mathbf{i}\oint_C A}$, the phase produced by adiabatically moving a charge along the path C. The analog of $V(\hat{C})$ depends on the number of dimensions. In $D = 2+1$, $j_\mu \equiv \epsilon_{\mu\nu\rho} F_{\nu\rho}/g^2$ is the vortex current. Consider

$$V(\hat{C}) = e^{\mathbf{i}\oint_{\hat{C}} dx^\nu \epsilon_{\mu\nu\rho} F_{\mu\nu}}.$$

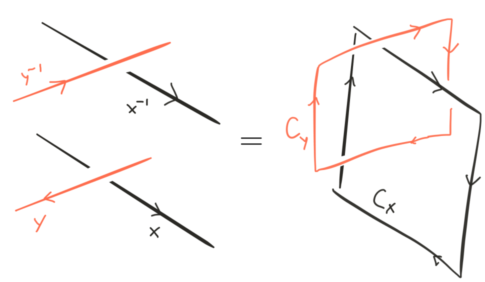
Check that the exponent is dimensionless. Interpret $V(\hat{C})$ as the phase produced by adiabatically moving a vortex along the curve $\hat{C}$,

We would like to show that $W_x V_y W_x^{-1} V_y^{-1}$ is a nontrivial phase, just like in the toric code. We'll do it using the path integral.

Convince yourself that

$$\left\langle T\left(W_x V_y W_x^{-1} V_y^{-1}\right)\right\rangle = \langle W(C_x) V(C_y)\rangle$$

where C_x and C_y are indicated in the right figure. Time goes vertically. In the left figure, the curves are closed by the periodic identifications of the spatial torus.



Write a path integral expression for the RHS.

Do the gaussian integral.

Show that the answer is $e^{\mathbf{i}a\ell(C_x, C_y)}$ where $\ell(C, C')$ is the linking number of the two curves, and a is a constant. Find a.

Reproduce this result using canonical methods, $[A_i(x,t), E_j(y,t)] = \mathbf{i}g^2\delta^2(x-y)$ and the Baker-Campbell-Hausdorff formula.

In $D = 3+1$, the analog of j_μ is a two-form: $j_{\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}/g^2$, the vortex *string* current. So we may consider *surface* operators

$$V_\Sigma \equiv e^{\mathbf{i}\int_\Sigma j}$$

which are the phase produced by moving a vortex string along this worldsheet trajectory. Using canonical methods or path integrals, study the commutation of W_x with

$$V_{yz} \equiv e^{\mathbf{i}\int_{yz} j}$$

where the integral is over the yz plane at $x = 0$.

Bonus: using the path integral, consider the effects of charged matter on this calculation.

5. **Right-handed neutrinos.** [from Iain Stewart, and hep-ph/0210271]

Consider adding a right-handed singlet (under all gauge groups) neutrino N_R to the Standard Model. It may have a majorana mass M; and it may have a coupling g_ν to leptons, so that all the dimension ≤ 4 operators are

$$\mathcal{L}_N = \bar{N}_R\mathbf{i}\partial\!\!\!/ N_R - \frac{M}{2}\bar{N}_R^c N_R - \frac{M}{2}\bar{N}_R N_R^c + \left(g_\nu \bar{N}_R H_i^T L_j \epsilon^{ij} + h.c.\right)$$

where $N_R^c = C\left(\bar{N}_R\right)^T$ is the the charge conjugate field, $C = \mathbf{i}\gamma_2\gamma_0$ (in the Dirac representation), H is the Higgs doublet, L is the left-handed lepton doublet, containing ν_L and e_L. Take the mass M to be large compared to the electroweak scale. Integrate out the right-handed neutrinos at tree level. [Hint: you may find it useful to work in terms of the Majorana field

$$N \equiv N_R + N_R^c$$

which satisfies $N = N^c$.]

Show that the leading term in the expansion in $1/M$ is a dimension-5 operator made of Standard Model fields. Explain the consequences of this operator for neutrino physics, assuming a vacuum expectation value for the Higgs field.

Place a bound on M assuming that the observed neutrinos have masses $m_\nu < 0.5$ eV.