University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2017
Assignment 8

Due 12:30pm Wednesday, May 31, 2017

1. **2PI generating functional.**

   In this problem, we will consider $0+0$-dimensional ϕ^4 theory for definiteness, but the discussion also works with labels. Parametrize the integral as

   $$Z[K] \equiv \int_{-\infty}^{\infty} dq\ e^{-\frac{1}{2}m^2q^2 - \frac{g}{4!}q^4 - \frac{1}{2}Kq^2} \ \equiv e^{-W[K]}\ ;$$

   we will regard K as a source for the two-point function.

   The 2PI generating functional is defined by the Legendre transform

   $$\Gamma[G] \equiv W[K] - K\partial_K W[K]$$

   with $K = K(G)$ determined by $G = \partial_K W[K]$. Here G is the 2-point Green's function $G \equiv 2\left\langle q^2 \right\rangle_K$.

   (a) Show from the definition that Γ satisfies the Dyson equation

   $$\partial_G \Gamma[G] = -\frac{1}{2}K.$$

   (b) Show that the perturbative expansion of Γ can be written as

   $$\Gamma[G] \overset{\text{A}}{=} \frac{1}{2}\log G^{-1} + \frac{1}{2}m^2 G + \gamma_{2PI} + \text{const.}$$

   (The A over the equal sign means that this is a perturbative statement.) Here γ_{2PI} is defined as everything beyond one-loop order in the perturbation expansion.

   (c) Show that $-\gamma_{2PI}$ is the sum of connected two-particle-irreducible (2PI) vacuum graphs, where 2PI means that the diagram cannot be disconnected by cutting any *two* propagators.

   (d) Define the 1PI self-energy in this Euclidean setting as

   $$-\Sigma \overset{\text{A}}{=} \sum \left(\text{1PI diagrams with two nubbins}\right).$$

   Show that

   $$\Sigma \overset{\text{A}}{=} \frac{\partial \gamma_{2PI}}{\partial K}$$

   by examining the diagrams on the BHS.

(e) Show that the expression for G that results from the Dyson equation

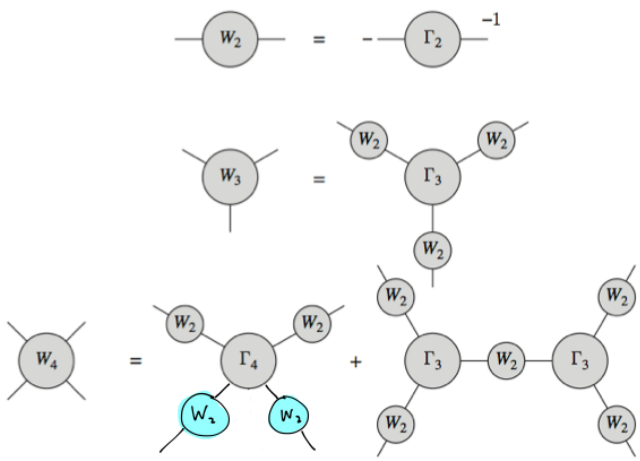
$$G^{-1} = K + 2\partial_G \gamma_{2PI}$$

agrees with our summation of 1PI diagrams.

(f) I would like add another part to this problem which shows more clearly why 2PI effective actions might be useful. (For example, they are used in Luttinger's proof of the Luttinger theorem relating the number of particles to the volume of the Fermi surface in a Fermi liquid.)

2. **Making connected functions out of 1PI functions.**

In this problem we prove the relations illustrated in Fig. 1. More precisely, we prove that the ϕ^n term in the expansion of $\Gamma[\phi]$ is the 1PI n-point function.



Legendre transform $W[J] = \Gamma[\phi] + \int \phi J$ makes trees.

Figure 1: [From Banks, *Modern Quantum Field Theory*, with some improvement] W_n denotes the connected n-point function, $\left(\frac{\partial}{\partial J}\right)^n W[J] = \langle \phi^n \rangle$. Γ_n denotes the 1PI n-point function, as encoded in the 1PI effective action.

(a) Show using the definitions of ϕ_c and the 1PI effective action that

$$\frac{\delta^2 W}{\delta J(x)\delta J(y)} = \frac{\delta\phi(y)}{\delta J(x)} = \left(\frac{\delta J(x)}{\delta\phi(y)}\right)^{-1} = -\left(\frac{\delta^2 \Gamma}{\delta\phi(x)\delta\phi(y)}\right)^{-1}. \quad (1)$$

(where $\phi \equiv \phi_c$ here). Inverse here means in the sense of integral operators: $\int d^D z K(x,z) K^{-1}(z,y) = \delta^D(x-y)$. Convince yourself that (1) can be written more compactly as:

$$W_2 = -\Gamma_2^{-1}.$$

(b) Show that

$$W_3(x,y,z) = \int dw_1 \int dw_2 \int dw_3 W_2(x,w_1) W_2(y,w_2) W_2(z,w_3) \Gamma_3(w_1,w_2,w_3)\,.$$

Do this by differentiating (1) again with respect to J and using the matrix differentiation formula $dK^{-1} = -K^{-1} dK K^{-1}$ and the chain rule.

(c) Show that W_n can be constructed from Γ and W_2 as indicated in the figure. The idea is to compute W_n by taking more derivatives with respect to J. To get the rest of the W_n requires an induction step.

3. **Coleman-Weinberg potential.**

(a) Fill in the details of the Coleman-Weinberg calculation for ϕ^4 theory in $D = 4$ which went by quickly in lecture.

(b) [Zee problem IV.3.4] What set of Feynman diagrams are summed by the Coleman-Weinberg calculation?

[Hint: expand the logarithm as a series in V''/k^2 and associate a Feynman diagram with each term.]

(c) [Zee problem IV.3.3] Consider a massless fermion field coupled to a scalar field ϕ by a coupling $g\phi\bar{\psi}\psi$ in $D = 1+1$. Show that the one loop effective potential that results from integrating out the fluctuations of the fermion has the form

$$V_F = \frac{1}{2\pi} (g\phi)^2 \log\left(\frac{\phi^2}{M^2}\right)$$

after adding an appropriate counterterm.