University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2017
Assignment 3

Due 12:30pm Wednesday, April 26, 2017

1. **Brain-warmer on spin coherent states.**

   Show that
   $$\langle \check{n}| \vec{h} \cdot \vec{\mathbf{S}} |\check{n}\rangle = s\vec{h} \cdot \check{n}$$
   where $|\check{n}\rangle = \mathcal{R}|s, s\rangle$ is a coherent state of spin s (where $|s, s\rangle$ is the eigenvector of $\mathbf{S}^z$ with maximal eigenvalue, and $\mathcal{R}$ is the rotation operator which takes $\check{z}$ to $\check{n}$).

   Show that for several spins and $i \neq j$
   $$\langle \check{n}| \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j |\check{n}\rangle = s^2 \check{n}_i \cdot \check{n}_j,$$
   where now $|\check{n}\rangle \equiv \otimes_j (\mathcal{R}_i |s_i\rangle)$ is a product of coherent states of each of the spins individually.

2. **Brain-warmer on Schwinger bosons.**

   Recall the Schwinger-boson representation of the $\mathsf{SU}(2)$ algebra:
   $$\mathbf{S}^+ = a^\dagger b, \ \mathbf{S}^- = b^\dagger a, \ \mathbf{S}^z = a^\dagger a - b^\dagger b,$$
   where the modes a, b satisfy $[a, a^\dagger] = 1 = [b, b^\dagger], [a, b] = [a, b^\dagger] = 0$. This is the algebra of a simple harmonic oscillator in two dimensions,
   $$H = \frac{1}{2}\left(p_x^2 + p_y^2 + x^2 + y^2\right).$$
   Is the $\mathsf{SU}(2)$ a symmetry of this Hamiltonian? How does it act on the oscillator coordinates? Check that the oscillator algebra does indeed imply that $\vec{\mathbf{S}}$ defined this way satisfy the $\mathsf{SU}(2)$ algebra.

3. **Simplicial homology and the toric code.**

   In lecture we discussed the (de Rham) cohomology of the exterior derivative d acting on vector spaces (over $\mathbb{R}$) of differential forms on some smooth manifold X. The dimensions $b^p(X, \mathbb{R})$ of the cohomology groups are topological properties of X. This same data is manifested in many other ways; in this problem we study another one, along with an important and familiar physical realization of it.

(a) The toric code we've discussed so far has qbits on the links $\ell \in \Delta_1(\mathbf{\Delta})$ of a graph $\mathbf{\Delta}$. But the definition of the Hamiltonian involves more information than just the links of the graph: we have to know which vertices v lie at the boundaries of each link ℓ, and we have to know which links are boundaries of which faces. The Hamiltonian has two kinds of terms: a 'plaquette' operator $B_p = \prod_{\ell \in \partial p} X_\ell$ associated with each 2-cell (plaquette) $p \in \Delta_2(\mathbf{\Delta})$. and 'star' operators, $A_s = \prod_{\ell \in \partial^{-1}(s)} Z_\ell$, associated with each 0-cell (site) $s \in \Delta_0(\mathbf{\Delta})$. Here I've introduced some notation that will be useful, please be patient: Δ_k denotes a collection of k-dimensional polyhedra which I'll call k-simplices or more accurately k-cells – k-dimensional objects making up the space. (It is important that each of these objects is topologically a k-ball.) This information constitutes (part of) a *simplicial complex*, which says how these parts are glued together:

$$\Delta_d \overset{\partial}{\to} \Delta_{d-1} \overset{\partial}{\to} \cdots \Delta_1 \overset{\partial}{\to} \Delta_0 \tag{1}$$

where ∂ is the (signed) boundary operator. For example, the boundary of a link is $\partial \ell = s_1 - s_0$, the difference of the vertices at its ends. The boundary of a face $\partial p = \sum_{\ell \in \partial p} \ell$ is the (oriented) sum of the edges bounding it. By $\partial^{-1}(s)$ I mean the set of links which contain the site s in their boundary (with sign).

Think of this collection of objects as a triangulation (or more generally some chopping-up) of a smooth manifold X. Convince yourself that the sequence of maps (1) is a complex in the sense that $\partial^2 = 0$.

(b) [not actually a question] This means that the simplicial complex defines a set of homology groups, which are topological invariants of X, in the following way. (It is homology and not cohomology because ∂ decreases the degree k). To define these groups, we should introduce one more gadget, which is a collection of vector spaces over some ring R (for the ordinary toric code, $R = \mathbb{Z}_2$)

$$\Omega_p(\mathbf{\Delta}, R), \ p = 0...d \equiv \dim(X)$$

basis vectors for which are p-simplices:

$$\Omega_p(\mathbf{\Delta}, R) = \text{span}_R\{\sigma \in \Delta_p\}$$

– that is, we associate a(n orthonormal) basis vector to each p-simplex (which I've just called σ), and these vector spaces are made by taking linear combinations of these spaces, with coefficients in R. Such a linear combination of p-simplices is called a *p-chain*. It's important that we can *add* (and subtract) p-chains, $C + C' \in \Omega_p$. A p-chain with a negative coefficient can be regarded as having the opposite orientation. We'll see below how better to interpret the coefficients.

The boundary operation on Δ_p induces one on Ω_p. A chain C satisfying $\partial C = 0$ is called a *cycle*, and is said to be *closed*.

So the pth homology is the group of equivalence classes of p-cycles, modulo boundaries of $p+1$ cycles:

$$H_p(X, R) \equiv \frac{\ker(\partial : \Omega_p \to \Omega_{p-1}) \subset \Omega_p}{\mathrm{Im}(\partial : \Omega_{p+1} \to \Omega_p) \subset \Omega_p}$$

This makes sense because $\partial^2 = 0$ – the image of $\partial : \Omega_{p+1} \to \Omega_p$ is a subset of $\ker(\partial : \Omega_p \to \Omega_{p-1})$. It's a theorem that the dimensions of these groups are the same for different (faithful-enough) discretizations $\mathbf{\Delta}$ of X. Furthermore, their dimensions (as vector spaces over R) $b_p(X)$ contain (much of[1]) the same information as the Betti numbers defined by de Rham cohomology. For more information and proofs, see the great book by Bott and Tu, *Differential forms in algebraic topology.*

(c) A state of the toric code on a cell-complex $\mathbf{\Delta}$ can be written (for the hamiltonian described above, this is in the basis where Z_ℓ is diagonal) as an element of $\Omega_1(X, \mathbb{Z}_2)$,

$$|\Psi\rangle = \sum_C \Psi(C) |C\rangle$$

where C is an assignment of an element of $\mathbb{Z}_2$ in X (the eigenvalue of Z_ℓ). For the case of $\mathbb{Z}_2$ coefficients, $1 = -1$ mod 2 and we don't care about the orientations of the cells. Show that the conditions for a state $\Psi(C)$ to be a groundstate of the toric code ($A_s |\Psi\rangle = |\Psi\rangle \, \forall s$ and $B_p |\Psi\rangle = |\Psi\rangle \, \forall p$) are exactly those defining an element of $H_1(X, \mathbb{Z}_2)$.

(d) Consider putting a spin variable on the p-simplices of $\mathbf{\Delta}$. More generally, let's put an N-dimensional hilbert space $\mathcal{H}_N \equiv \mathrm{span}\{|n\rangle, n = 1..N\}$ on each p-simplex, on which act the operators

$$\mathbf{X} \equiv \sum_{n=1}^{N} |n\rangle \langle n| \omega^n = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & \omega & 0 & \dots \\ 0 & 0 & \omega^2 & \dots \\ 0 & 0 & 0 & \ddots \end{pmatrix}, \quad \mathbf{Z} \equiv \sum_{n=1}^{N} |n\rangle \langle n+1| = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots \\ 1 & 0 & 0 & \dots \end{pmatrix}$$

where $\omega^N = 1$ is an nth root of unity. If you haven't already, check that satisfy the clock-shift algebra: $\mathbf{XZ} = \omega \mathbf{ZX}$. For $N = 2$ these are Pauli matrices and $\omega = -1$.

[1] I don't want to talk about torsion homology.

Consider the Hamiltonian

$$\mathbf{H} = -J_{p-1} \sum_{s \in \Delta_{p-1}} A_s - J_{p+1} \sum_{\mu \in \Delta_{p+1}} B_\mu - g_p \sum_{\sigma \in \Delta_p} \mathbf{Z}_\sigma$$

with

$$A_s \equiv \prod_{\sigma \in \partial^{-1}(s) \subset \Delta_p} \mathbf{Z}_\sigma$$

$$B_\mu \equiv \prod_{\sigma \in \partial \mu} \mathbf{X}_\sigma \ .$$

This is a lattice version of p-form $\mathbb{Z}_N$ gauge theory, at a particular, special point in its phase diagram.

Show that

$$0 = [A_s, A_{s'}] = [B_\mu, B_{\mu'}] = [A_s, B_\mu], \quad \forall s, s', \mu, \mu'$$

so that for $g_p = 0$ this is solvable.

(e) Show that the groundstates of $\mathbf{H}_p$ (with $g_p = 0$) are in one-to-one correspondence with elements of $H_p(\mathbf{\Delta}, \mathbb{Z}_N)$.

4. **Non-linear sigma models on more general spaces.** [Warning: some knowledge of general relativity is helpful here.]

In lecture we considered the 2d non-linear sigma model whose target space was a round 2-sphere, motivated by the low-energy physics of antiferromagnets. At weak coupling (large radius of sphere, which means large spin), we saw that the sphere wants to shrink in the IR.

Consider now a 2d non-linear sigma model (NLSM) whose target space is a more general manifold X with Riemannian metric $ds^2 = L^2 g_{ij}(x) dx^i dx^j$. Assume that the space is *big*, in the sense that we will treat the parameter L^{-1} as a small parameter, and *smooth* in the sense that we can Taylor expand around any point.

The NLSM is a field theory whose fields $x^i(\sigma)$ are maps from spacetime (here 2d flat space) to the *target space* X. The simplest action is

$$S[x(\sigma)] = \int d^2\sigma L^2 g_{ij}(x) \partial_{\sigma^\mu} x^i \partial_{\sigma^\nu} x^j \eta^{\mu\nu}$$

where $\eta^{\mu\nu}$ is the flat metric on the 2d spacetime ‘worldsheet’.

$D = 2$ is special because the free scalar field $x(\sigma)$ is dimensionless. As long as g_{ij} is nonsingular, in the limit $L \to \infty$, the local coordinate field becomes free.

Regard $g_{ij}(x)$ as a coupling *function*. What is the leading beta function (actually beta functional) for this set of couplings?

Hint: use the fact that the answer must be covariant under changes of coordinates on X plus dimensional analysis.

5. **Haldane phase.** [bonus problem]

   Consider the $D = 1 + 1$ nonlinear sigma model with target space S^2 at $\theta = 2\pi$. The θ term is a total derivative in the action, so it can manifest itself when we study the path integral on a spacetime with boundary.

   (a) Put this field theory on the half-line $x > 0$. Suppose that the boundary conditions respect the $\mathsf{SO}(3)$ symmetry, so that the boundary values $\vec{n}(\tau, x = 0)$ are free to fluctuate. By remembering that the θ-term is a total derivative, and considering the strong-coupling (IR) limit, $g \to \infty$, show that there is a spin-$\frac{1}{2}$ at the boundary. (Hint: Recall the coherent state path integral for a spin-$\frac{1}{2}$.)

   (b) Now cut the path integral open at some fixed euclidean *time* $\tau = 0$. (Consider periodic boundary conditions in space.) Such a path integral computes the groundstate wavefunction, as a function of the boundary values of the fields, $\vec{S}(x, \tau = 0)$. Find the groundstate wavefunctional is $\Psi[\vec{n}(x, \tau = 0)]$ in the strong coupling limit $g \to \infty$ (where the gap is big).