University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2017
Assignment 2

Due 12:30pm Wednesday, April 19, 2017

1. **Topology brain warmer.**

   Compute the de Rham cohomology of the d-torus. Hint: you can choose p-chains of the form

   $$A_{i_1 \cdots i_p}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

   where $x^i \simeq x^i + 1$ are periodically identified coordinates, and $A_{i_1 \cdots i_p}$ is a *single-valued* (*i.e.* periodic) function. Notice that x^i is not a single-valued function.

2. **Coherent state quantization brain-warmers.**

   (a) Start with first order action $S = \int \mathrm{d}t \; z_\alpha^\dagger \dot{z}_\alpha$. Show that the Hamiltonian is $\mathbf{H} = 0$.

   (b) Check the completeness relation in the spin 1/2 coherent state basis.

   (c) Show that different spinor representations, *i.e.* different choices of ψ in

   $$z = \begin{pmatrix} e^{\mathbf{i}(\psi + \varphi/2)} \cos\theta/2 \\ e^{\mathbf{i}(\psi - \varphi/2)} \sin\theta/2 \end{pmatrix}$$

   shift the coefficient of the total derivative $\dot{\varphi}$ part of the WZW functional.

3. **Topological terms in QM.** [from Abanov]

   The euclidean path integral for a particle on a ring with magnetic flux $\theta = \int \vec{B} \cdot \mathrm{d}\vec{a}$ through the ring is given by

   $$Z = \int [D\phi] e^{-\int_0^\beta \mathrm{d}\tau \left(\frac{m}{2} \dot{\phi}^2 - \mathbf{i} \frac{\theta}{2\pi} \dot{\phi} \right)} .$$

   Here

   $$\phi \equiv \phi + 2\pi \tag{1}$$

   is a coordinate on the ring. Because of the identification (1), ϕ need not be a single-valued function of τ – it can wind around the ring. On the other hand, $\dot{\phi}$ is single-valued and periodic and hence has an ordinary Fourier decomposition. This means that we can expand the field as

   $$\phi(\tau) = \frac{2\pi}{\beta} Q \tau + \sum_{\ell \in \mathbb{Z}} \phi_\ell e^{\mathbf{i} \frac{2\pi}{\beta} \ell \tau} . \tag{2}$$

(a) Show that the $\dot{\phi}$ term in the action does not affect the classical equations of motion. In this sense, it is a topological term.

(b) Using the decomposition (2), write the partition function as a sum over topological sectors labelled by the *winding number* $Q \in \mathbb{Z}$ and calculate it explicitly.

[Hint: use the Poisson resummation formula

$$\sum_{n\in\mathbb{Z}} e^{-\frac{1}{2}tn^2+\mathbf{i}zn} = \sqrt{\frac{2\pi}{t}}\sum_{\ell\in\mathbb{Z}} e^{-\frac{1}{2t}(z-2\pi\ell)^2}. \quad]$$

(c) Use the result from the previous part to determine the energy spectrum as a function of θ.

(d) [more optional] Derive the canonical momentum and Hamiltonian from the action above and verify the spectrum.

(e) Consider what happens in the limit $m \to 0, \theta \to \pi$ with $X \equiv \frac{\theta-\pi}{m} \sim \beta^{-1}$ fixed. Interpret the result as the partition function for a spin 1/2 particle. What is the meaning of the ratio X in this interpretation?

An important lesson here is that total derivative terms in the action do affect the physics.

4. **Geometric Quantization of the 2-torus.**

Redo the analysis that we did in lecture for the two-sphere for the case of the two-torus, $S^1 \times S^1$. The coordinates on the torus are $(x, y) \simeq (x + 2\pi, y + 2\pi)$; use $N\mathrm{d}x \wedge \mathrm{d}y$ as the symplectic form. Show that the resulting Hilbert space represents the Heisenberg algebra

$$e^{\mathbf{i}\mathbf{x}}e^{\mathbf{i}\mathbf{y}} = e^{\mathbf{i}\mathbf{y}}e^{\mathbf{i}\mathbf{x}}e^{\frac{2\pi\mathbf{i}}{N}}.$$

(I am using boldface letters for operators.) The irreducible representation of this algebra is the same Hilbert space as a particle on a periodic one-dimensional lattice with N sites.

5. **Particle on a sphere with a monopole inside.**

Consider a particle of mass m and electric charge e with action

$$S[\vec{x}] = \int \mathrm{d}t \left(\frac{1}{2}m\dot{\vec{x}}^2 + e\dot{\vec{x}}\cdot\vec{A}(\vec{x})\right)$$

constrained to move on a two sphere of radius r in three-space, $\vec{x}^2 = r^2$. Suppose further that there is a *magnetic monopole* inside this sphere: this means that $4\pi g = \int_{S^2}\vec{B}\cdot \mathrm{d}\vec{a} = \int_{S^2} F$, where $F = \mathrm{d}A$. (Since the particle lives only at $\vec{x}^2 = r^2$, the form of the field in the core of the monopole is not relevant here.)

(a) Find an expression for $A = A_i \mathrm{d}x^i = A_\theta \mathrm{d}\theta + A_\varphi \mathrm{d}\varphi$ such that $F = \mathrm{d}A$ has flux $4\pi g$ through the sphere.

(b) Show that the Witten argument gives the Dirac quantization condition $2eg \in \mathbb{Z}$.

(c) Take the limit $m \to 0$. Count the states in the lowest Landau level. Compare with the calculation in lecture for coherent state quantization of a spin-s.