University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Spring 2016
Assignment 8

Due 11am Thursday, June 9, 2016

1. **Direct application of Lieb's theorem.**

   We only used a very special case of Lieb's theorem to prove monotonicity of the relative entropy. Surely there is more to learn from it.

   Consider an ensemble of states $\boldsymbol{\rho} = \sum_i p_i \boldsymbol{\rho}_i$, and a unitary operator $\mathbf{U}$ (for example, it may be closed-system time evolution).

   Show that the relative entropy between $\boldsymbol{\rho}(t) \equiv \mathbf{U}\boldsymbol{\rho}\mathbf{U}^\dagger$ and $\boldsymbol{\rho}$ is convex in $\boldsymbol{\rho}$:

$$D(\boldsymbol{\rho}(t)||\boldsymbol{\rho}) \leq \sum_i p_i D(\boldsymbol{\rho}_i(t)||\boldsymbol{\rho}_i).$$

   Open ended bonus problem: see if you can find a better result by directly applying Lieb's joint concavity theorem to a problem in many body physics.

2. **Majorization questions.**

   (a) Show that if a doubly stochastic map is reversible (invertible and the inverse is also doubly stochastic) then it is a permutation.

   (b) Show that the set of doubly stochastic maps is convex. (This is the easier direction of the Birkhoff theorem.)

   (c) Show that a pure state and uniform state satisfy $(1, 0, 0 \cdots) \succ p \succ (1/L, 1/L \cdots)$ for any p on an L-item space.

   (d) A useful visualization of majorization relations is called the 'Lorenz curve': this is just a plot of the cumulative probability $P_p(K) = \sum_{k=1}^{K} p_k$ as a function of K. What does $p \succ q$ mean for the Lorenz curves of p and q? Draw the Lorenz curves for the uniform distribution and for a pure state.

   (e) Show that the set of probability vectors majorized by a fixed vector x is convex. That is: if $x \succ y$ and $x \succ z$ then $x \succ ty + (1-t)z, t \in [0, 1]$. Hints: (1) the analogous relation is true if we replace x, y, z with real numbers and $\succ$ with $\geq$. (2) Show that $P_{p^\downarrow}(K) \geq P_{\pi p^\downarrow}(K)$ (where $\pi p^\downarrow$ indicates any other ordering of the distribution).

(f) For the case of a 3-item sample space we can draw some useful pictures of the whole space of distributions. The space of probability distributions on three elements is the triangle $x_1 + x_2 + x_3 = 1, x_i \geq 0$, which can be drawn in the plane. We can simplify the picture further by ordering the elements $x_1 \geq x_2 \geq x_3$, since majorization does not care about the order. Pick some distribution x with $x_1 \neq x_2 \neq x_3$ and draw the set of distributions which x majorizes, the set of distributions majorized by x, and the set of distributions with which x does not participate in a majorization relation ('not comparable to x').

3. **Majorization and catalytic majorization fractions.** [open-ended]

   How often do two randomly-chosen probability distributions participate in a majorization relation? Assume for simplicity that they are distributions on the same sample space, with L elements, and think about the fraction of p and q which satisfy $p \succ q$ or $q \succ p$ as a function of L.

   When they fail to majorize each other, how often is it possible to find a catalyst?

4. Compute the trace distance between the two single-qbit states

$$\boldsymbol{\rho}_v = \frac{1}{2}\left(1 + \vec{v} \cdot \boldsymbol{\sigma}\right) \quad \text{and} \quad \boldsymbol{\rho}_w = \frac{1}{2}\left(1 + \vec{w} \cdot \boldsymbol{\sigma}\right).$$

5. **Random singlets.**

   [from PRL 111, 170501 (2013)]

   Consider qbits arranged on a chain. Suppose that the groundstate is made of random singlets, where the probability for spins at i and j to be paired is $f(|i-j|a)$ (a is the lattice spacing). Consider in turn the case of short-range singlets $f(x) \propto e^{-x/\xi}$, and long-range singlets $f(x) \propto \frac{1}{x^2+\delta^2}$.

   Consider a A which is an interval $\left[-\frac{R-\epsilon}{2}, \frac{R-\epsilon}{2}\right]$ ($\epsilon \ll R$) and B is what we called $\bar{A}^-$ (nearly the complement), more precisely: $B \equiv [-\infty, \frac{R}{2}] \cup [\frac{R}{2}, \infty]$. Let $I_\epsilon(R) \equiv I(A:B) = S(A) + S(B) - S(AB)$ be their mutual information.

   Find $\overline{\left\langle \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j \right\rangle}$ and $\overline{I_\epsilon(R)}$. In both cases assume the regions are big enough that you can average over regions and use a continuum approximation ($\xi, \delta \gg$ lattice spacing).

   Check that the answer is consistent with the mutual information bound on correlations.

6. **Checking the operational interpretation of trace distance.** [optional bonus problem, thanks to S.M.Kravec for input]

   (a) **Warmup.** Show that for two pure states $|1\rangle, |2\rangle$, their trace distance T and their fidelity F satisfy
   $$F^2 + T^2 = 1.$$

   (b) In lecture the possibility was raised that by considering POVMs which are not projective measurements it might be possible to evade the theorem we proved on the probability of success at distinguishing two states by a single measurement in terms of the trace distance.

   Consider two non-orthogonal pure states $|1\rangle, |2\rangle$. with overlap $\delta = |\langle 1|2\rangle|^2$ and consider the POVM made of :
   $$E_1 = \chi |1\rangle \langle 1|, E_2 = \alpha |2\rangle \langle 2|, E_3 = 1 - E_1 - E_2.$$

   For which χ, α is this a POVM?

   Find the probability of success of the strategy: if outcome is 1 guess 1, if outcome is 2 guess 2, if outcome is 3 do a little dance then guess randomly. Show that the bound we proved is not violated.

   (c) Nevertheless, POVMs (which are not projective measurements) are indeed useful for state discrimination. Find a POVM with the property that distinguishes between two non-orthogonal pure states $|1,2\rangle$ in such a way that for one outcome we are *certain* that the state is $|1\rangle$ and for another we are *certain* that the state is $|2\rangle$. (There is a third outcome where we learn nothing from the measurement.)