University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Spring 2016
Assignment 5

Due 11am Thursday, May 12, 2016

1. **Warmup problems.**

   (a) Show that the most general density matrix for a single qbit lies in the Bloch ball, *i.e.* is of the form

   $$\boldsymbol{\rho}_v = \frac{1}{2}\left(\mathbb{1} + \vec{v}\cdot\vec{\boldsymbol{\sigma}}\right), \quad \sum_i v_i^2 \leq 1.$$

   Find the determinant, trace, and von Neumann entropy of $\boldsymbol{\rho}_v$.

   (b) [from Barnett] A single qbit state has $\langle \mathbf{X} \rangle = s$. Find the most general forms for the corresponding density operator with the minimum and maximum von Neumann entropy. (Hint: the Bloch ball is your friend.)

   (c) Show that the *purity* of a density matrix $\pi[\boldsymbol{\rho}] \equiv \text{tr}\boldsymbol{\rho}^2$ satisfies $\pi[\boldsymbol{\rho}] \leq 1$ with saturation only if $\boldsymbol{\rho}$ is pure.

   (d) [from Barnett] Show that the quantum relative entropy satisfies the following

   $$D(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B || \boldsymbol{\sigma}_A \otimes \boldsymbol{\sigma}_B) = D(\boldsymbol{\rho}_A||\boldsymbol{\sigma}_A) + D(\boldsymbol{\rho}_B||\boldsymbol{\sigma}_B). \tag{1}$$

   $$\sum_i p_i D\left(\boldsymbol{\sigma}_i||\boldsymbol{\rho}\right) = \sum_i p_i D(\boldsymbol{\sigma}_i||\boldsymbol{\sigma}_{\text{av}}) + D(\boldsymbol{\sigma}_{\text{av}}||\boldsymbol{\rho}) \tag{2}$$

   $$D\left(\boldsymbol{\sigma}_{\text{av}}||\boldsymbol{\rho}\right) \leq \sum_i p_i D(\boldsymbol{\sigma}_i||\boldsymbol{\rho}) \tag{3}$$

   for any probability distribution $\{p_i\}$ and density matrices $\boldsymbol{\rho}, \boldsymbol{\sigma}_i$, and where $\boldsymbol{\sigma}_{\text{av}} \equiv \sum_i p_i \boldsymbol{\sigma}_i$.

2. **Teleportation for qdits.** [optional, from Christiandl]

   Show that it is possible to teleport a state $|\xi\rangle_A \in \mathcal{H}_A, |A| \equiv d$ from A to B using the maximally-entangled state

   $$|\Phi\rangle_{AB} \equiv \frac{1}{\sqrt{d}} \sum_{n=1}^{d} |nn\rangle_{AB}.$$

Hint: Consider the clock and shift operators

$$\mathbf{Z} \equiv \sum_{n=1}^{d} |n\rangle \langle n| \, \omega^n, \ \omega \equiv e^{\frac{2\pi \mathbf{i}}{d}}, \quad \mathbf{X} \equiv \sum_{n=1}^{d} |n+1\rangle \langle n|$$

where the argument of the ket is to be understood mod d. Show that these generalize some of the properties of the Pauli $\mathbf{X}$ and $\mathbf{Z}$ in that they are unitary and that they satisfy the (discrete) Heisenberg algebra

$$\mathbf{XZ} = a\mathbf{ZX}$$

for some c-number a which you should determine.

The following is a collection of examples of quantum channels, wherein it is fun and profitable to determine the long-term behavior on repeated action of the channel. Do as many of them as you find instructive.

3. **Amplitude-damping channel.** [from Preskill 3.4.3, Le Bellac §15.2.4]

   This is a very simple model for a two-level atom, coupled to an environment in the form of a (crude rendering of a) radiation field.

   The atom has a groundstate $|0\rangle_A$; if it starts in this state, it stays in this state, and the radiation field stays in its groundstate $|0\rangle_E$ (zero photons). If the atom starts in the excited state $|1\rangle_A$, it has some probability p per time dt to return to the groundstate and emit a photon, exciting the environment into the state $|1\rangle_E$ (one photon). This is described by the time evolution

$$\mathbf{U}_{AE} |0\rangle_A \otimes |0\rangle_E = |0\rangle_A \otimes |0\rangle_E$$

$$\mathbf{U}_{AE} |1\rangle_A \otimes |0\rangle_E = \sqrt{1-p}\, |1\rangle_A \otimes |0\rangle_E + \sqrt{p}\, |0\rangle_A \otimes |1\rangle_E .$$

   (a) Show that the evolution of the atom's density matrix can be written in terms of two Kraus operators $\mathcal{K}_i$, find those operators and show that they satisfy $\sum_i \mathcal{K}_i^\dagger \mathcal{K}_i = \mathbb{1}_{\text{atom}}$.

   (b) Assuming that the environment is forgetful and resets to $|0\rangle_E$ after each time step dt, find the fate of the density matrix after time $t = ndt$ for late times $n \gg 1$, *i.e.* upon repeated application of the channel.

   (c) Evaluate the *purity* $\text{tr}\boldsymbol{\rho}_n^2$ of the nth iterate. (Recall that the purity is 1 IFF the state is pure.)

4. **Phase-flipping decoherence channel.** [from Schumacher]

   Consider the following model of decoherence on an N-state Hilbert space, with basis $\{|k\rangle, k = 1..N\}$.

   Define the unitary operator

$$\mathbf{U}_\alpha \equiv \sum_k \alpha_k |k\rangle \langle k|$$

   where α_k is an N-component vector of signs, ± 1 – it flips the signs of some of the basis states. There are 2^N distinct such operators.

   Imagine that interactions with the environment act on any state of the system with the operator $\mathbf{U}_\alpha$, for some α, chosen randomly (with uniform probability from the 2^N choices).

   [Hint: If you wish, set $N = 2$.]

   (a) Warmup question: If the initial state is $|\psi\rangle$, what is the probability that the resulting output state is $\mathbf{U}_\alpha |\psi\rangle$?

   (b) Write an expression for the resulting density matrix, $\mathcal{D}(\boldsymbol{\rho})$, in terms of $\boldsymbol{\rho}$.

   (c) Think of $\mathcal{D}$ as a superoperator, an operator on density matrices. How does $\mathcal{D}$ act on a density matrix which is diagonal in the given basis,

$$\boldsymbol{\rho}_{\text{diagonal}} = \sum_k p_k |k\rangle \langle k| \quad ?$$

   (d) The most general initial density matrix is not diagonal in the k-basis:

$$\boldsymbol{\rho}_{\text{general}} = \sum_{kl} \rho_{kl} |k\rangle \langle l| \quad .$$

   what does $\mathcal{D}$ do to the off-diagonal elements of the density matrix?

5. **Decoherence by phase damping with non-orthogonal states** [from Preskill]

   Suppose that a heavy particle A begins its life in outer space in a superposition of two positions

$$|\psi_0\rangle_A = a |x_0\rangle + b |x_1\rangle .$$

These positions are not too far apart. The particle interacts with the electromagnetic field, and in time dt, the whole system evolves according to

$$\mathbf{U}_{AE} |x_0\rangle_A \otimes |0\rangle_E = \sqrt{1-p}\, |x_0\rangle_A \otimes |0\rangle_E + \sqrt{p}\, |x_0\rangle_A \otimes |\gamma_0\rangle_E$$

$$\mathbf{U}_{AE} |x_1\rangle_A \otimes |0\rangle_E = \sqrt{1-p}\, |x_1\rangle_A \otimes |0\rangle_E + \sqrt{p}\, |x_1\rangle_A \otimes |\gamma_1\rangle_E$$

But because x_0 and x_1 are close, the (normalized) photon states $|\gamma_0\rangle, |\gamma_1\rangle$ have a large overlap:

$$\langle\gamma_0|\gamma_1\rangle_E = 1 - \epsilon, \quad \text{with } 0 < \epsilon \ll 1.$$

(a) Find the Kraus operators describing the time evolution of the reduced density matrix $\boldsymbol{\rho}_A$.

(b) How long does it take the superposition to decohere? More precisely, at what time t is $(\rho_A)_{01}(t) = \frac{1}{e}(\rho_A)_{01}(t=0)$?

6. **Decoherence on the Bloch sphere** [from Preskill]

Parametrize the density matrix of a single qubit as

$$\boldsymbol{\rho}_A = \frac{1}{2}\left(\mathbb{1} + \vec{P}\cdot\vec{\boldsymbol{\sigma}}\right).$$

(a) **Polarization-damping channel.**

Consider the (unitary) evolution of a qbit A coupled to a 4-state environment via

$$\mathbf{U}_{AE} |\phi\rangle_A \otimes |0\rangle_E = \sqrt{1-p}\, |\phi\rangle_A \otimes |0\rangle_E + \sqrt{p/3} \sum_{i=1}^{3} \boldsymbol{\sigma}^i_A \otimes \mathbb{1}_E |\phi\rangle_A \otimes |i\rangle_E$$

Show that this evolution can be accomplished with the Kraus operators

$$\mathbf{M}_0 = \sqrt{1-p}\mathbb{1}, \qquad \mathbf{M}_i = \sqrt{p/3}\boldsymbol{\sigma}^i,$$

and show that they obey the completeness relation requred by unitarity of $\mathbf{U}_{AE}$.

Show that the polarization P_i of the qbit evolves according to

$$\vec{P} \to \left(1 - \frac{4p}{3}\right)\vec{P}.$$

Describe this evolution in terms of what happens to the Bloch ball.

What happens if $p > 3/4$?

(b) **Two-Pauli channel.**

Consider the (unitary) evolution of a qbit A coupled to a *three*-state environment via

$$\mathbf{U}_{AE} |\phi\rangle_A \otimes |0\rangle_E = \sqrt{1-p}\, |\phi\rangle_A \otimes |0\rangle_E + \sqrt{p/2} \sum_{i=1}^{2} \boldsymbol{\sigma}^i_A \otimes \mathbb{1}_E |\phi\rangle_A \otimes |i\rangle_E$$

Show that this evolution can be accomplished with the Kraus operators

$$\mathbf{M}_0 = \sqrt{1-p}\mathbb{1}, \qquad \mathbf{M}_i = \sqrt{p/2}\boldsymbol{\sigma}^i, i = 1, 2$$

and show that they obey the completeness relation requred by unitarity of $\mathbf{U}_{AE}$.

Describe this evolution in terms of what happens on the Bloch ball, and evaluate the purity.

(c) **Phase-damping channel.**

For the evolution of problem 5,

$$\mathbf{U}_{AE} |0\rangle_A \otimes |0\rangle_E = \sqrt{1-p}\, |0\rangle_A \otimes |0\rangle_E + \sqrt{p}\, |0\rangle_A \otimes |\gamma_0\rangle_E$$

$$\mathbf{U}_{AE} |1\rangle_A \otimes |0\rangle_E = \sqrt{1-p}\, |1\rangle_A \otimes |0\rangle_E + \sqrt{p}\, |1\rangle_A \otimes |\gamma_1\rangle_E$$

now thinking of A as a qbit, describe the evolution of its polarization vector on the Bloch ball.