The Fractional Quantum Hall Effect and DMRG

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The Fractional Quantum Hall effect is a topological phase of matter meaning that it is characterized by certain properties relating to topology. Of notable importance is the existence of quasiparticle excitations known as anyons with unusual exchange statistics. Methods using quantum information and entanglement have been used to characterize the system. Since it is the low energy, long range physics that is of interest and it is mappable to a 1-D problem, the Density Matrix Renormalization Group method is viable.

INTRODUCTION

The general Quantum Hall Effect (QHE) involves the study of 2+1 D electrons in strong, constant magnetic fields, where experimentally it is observed that the conductance gets quantized, among other things.[1] Under certain conditions, of magnetic field strength and number density, we get the Fractional Quantum Hall effect (FQHE).[2] The FQHE ground states have quasi-particle excitations that can be anyons, which means that the many-body wave function doesn't transform under exchange like with bosons or fermions (+1 or -1), but instead it acquires a "statistical" complex phase, as in the Ahronov-Bohm effect or the Berry phase.[3] Some of these quasi-particles can be non-abelian anyons or "non-abelions", meaning that exchanges not only change the phase of the wave function, but can take the wavefunction to another degenerate state.[4] This degenerate subspace of states (and operations upon it) can be described by non-commutative matrices, hence non-abelian. Kitaev showed that non-abelian particles can be used for universal quantum computing.[5]

Normally, one solves the system using the classical method of exact diagonalization (ED), which just means find the eigenvalues and eigenfunctions. In the past 5 years, these studies have been accelerated by the use of the Density Matrix Renormalization Group (DMRG) method, as have been many systems that are 1D or that can be mapped to 1D.[6] DMRG is a way of solving the system by throwing away degrees of freedom using entanglement and density matrices. It is a variational method that gets you the ground state energy of a system. The ground sate and the operators (Hamiltonian) can be put into a form known as Matrix Product States or Operators, that is, MPS or MPO, respectively.[7]

THE FRACTIONAL QUANTUM HALL EFFECT

To study the entanglement, specifically the bi-partite entanglement of a system, or self-entanglement, one divides the system, in the state ψ, into two parts A and B, in either position (real) space or some spectral (e.g. momentum) space. Specifically, from the density matrix, $\rho = |\psi\rangle\langle\psi|$, one calculates the reduced density matrix of one part of the system, say ρ^A, (eq. 1) then this matrix is diagonalized to give the eigenvalues, called the entanglement spectrum. Chronologically around 2006, this spectrum was not paid much attention in regards to the study of topological phases of matter, instead one would calculate some specific function of the spectrum called the entanglement entropy. It has been shown that for many systems the entanglement entropy follows an area (boundary) law. Specifically, in 2+1 for the FQHE, the entanglement entropy grows linearly in the boundary, L, for large system size but has $\frac{1}{L^n}$ terms for finite size (eq. 3).[8] The entanglement entopy is calculated using the Von Neumann entropy, S in eq. 2 borrowed from stat mech. for quantum information theory. For part A of the system:

$$\rho^A = \mathrm{tr}_B[\rho] = \sum_B \langle B|\rho|B\rangle \tag{1}$$

$$S = \mathrm{tr}[\rho^A \log \rho^A] = \sum_i \rho_i^A \log \rho_i^A \tag{2}$$

$$S = \gamma + cL + O(\frac{1}{L}) \tag{3}$$

Kitaev and Preskill gave one of the first descriptions of the significance of a term in the entanglement entropy called the topological entanglement entropy (TEE), γ.[8] It is toplogical because it does not depend on the geometry, that is, shape of the system, but only on the specific quantum phase we are in. The TEE is related to another quantity called the quantum dimension of particles in a conformal field theory (CFT) for the boundary (edge). Later, Li and Haldane related the real space entanglement spectrum to the edge states of the Quantum Hall system, which are described by the aforementioned CFT.[9] A heuristic explanation is that cutting the system in two is like introducing a false edge.

To understand how one uses DMRG to help with the quantum hall effect, first we discuss the structure of its free single particle spectrum, as if the particles did not interact with each other. This is like in eq. 4 below, but with $V(|r_i - r_j|) = 0$, where we have electrons with charge, e, and mass, m, and a constant magnetic field with vector potential A. Then, the states of the systems would just be tensor products of all the individual particles states. Basically, it is your garden-variety quantum qualifying exam problem, solve in 2D a single spin-less particle in a constant perpendicular magnetic field to get the famous Landau Levels where each level has a bunch of degenerate states (see Figure 1). Solving the problem using algebraic methods gives two types of creation and annihilation operators, one that jumps to other Landau levels (increasing cyclotron energy eigenvalue) and another that jumps between degenerate states (increasing momentum eigenvalue), so this is a 2-D spectral space, as one might expect.

Depending on the geometry of the surface, which will suggest a certain gauge potential choice, one may use orbital or transverse momentum (sphere/plane or torus/cylinder) as a good quantum number for the degenerate orbitals (states) in each Landau level. One must keep in mind that putting the system on a different geometry may be experimentally difficult, for example a magnetic monopole is needed to get a nice radial magnetic field on a sphere. So, geometry is a theoretical device in many cases. Also, in a real system, the level degeneracies are huge, approximately infinite, but for a finite model system they can be manageable. Each Landau level is separated by the cyclotron energy for the problem. In Figure 1, we see that FQHE states can be described by a filling fraction, ν, which is the number of electrons divided by the number of orbitals in all Landau levels up to the lowest filled one.

$$H = \sum_i \frac{(p_i + eA/c)^2}{2m} + \sum_{i<j} V(|r_i - r_j|) \tag{4}$$

So, we have a practical basis of states to work with that gives us nice mental pictures. One expects the true eigenfunctions to be a sum, possibly infinite, of tensor products of single particle Landau states, whereas in spin 1/2 chains we instead have tensor products of spin up or down states. But is it realistic to perturb around free electrons for the Coulomb interaction? There is an interesting paper by Ye and Sachdev, that shows the Coulomb potential to be a dangerously irrelevant operator with a nonzero fixed point that is sensitive against disordered (random) perturbations.[10] Dangerously irrelevant means that it can contribute to long range physics, such as the topological properties.

Now, by rotation invariance of the Hamiltonian, a group theoretical analysis of angular momentum says that the ground state is unique on an infinite plane or the sphere and of total angular momentum zero. However, it has been shown by Wen et. al. that on surface with higher number of holes (genus, $g > 0$), like the torus, there is topological degeneracy (different from Landau level degeneracy) for the Fractional Quantum Hall effect. This topological degeneracy grows according to a power related to the genus, which is a topology dependent quantity.[11] We can use the topological methods that have been discussed in this paper to classify each of the phases of the FQHE and look for experimentally measurable quantities.

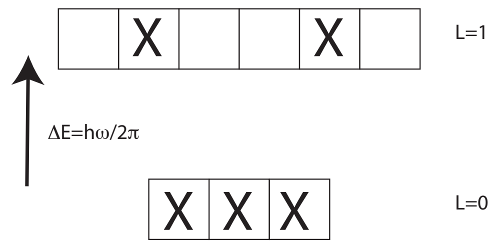

FIG. 1: Each Landau level is separated by the cyclotron energy with $\omega = \frac{qB}{mc}$. An X marks the location of an electron. This filling of one and a third levels gives filling fraction $\nu = 1 + \frac{1}{3} = \frac{4}{3}$

Different QHE states are distinguished by filling fraction. If the fraction of an experimental sample is an integer, then we might get the Integer Quantum Hall effect (IQHE). Basically, the electrons are stuck in a filled Landau level, so they cant jump to the next Landau or Zeeman level, so the conductance goes to zero. However, the Hall conductance (off-diagonal part of the conductivity tensor) flattens to some plateau. It can also happen even when the Landaul level filling is fractional (FQHE), but the explanation is not so concrete. The system is thought to be in a collective (entangled) ground state, brought upon by interactions between particles. Supposedly, these are a type of quantum phase of matter that is an incompressible fluid, such that adding pressure or energy does not change the density or filling fraction, hence you also get zero conductance and plateaus for the Hall conductance.[12]

So, remember from stat mech, that low energy states for electrons will be favored if the experimental temperatures are very small compared to the relevant energy scales I will mention. That is the Fermi distribution has a sharp looking drop. Second, imagine the magnetic field is strong enough, so as to make the Landau Level energy separation large. Then, there will be only a single Landau Level that is partially filled. Third, we can make all the spins line up with the magnetic field if the Zeeman energy is also large, this is called spin polarization, which is why we can ignore spin.

Therefore, now we now come to the primary reason one can use DMRG. Since we are stuck on a single Landau Level, this is a 1-D system, where the momentum states in this lowest-filled Landau level (LLL) can be hopped to and fro using momentum annihilation and creation operators. In Figure 1, this corresponds to the $L = 1$ level. If we work with a finite number of particles and a finite number of Landau Levels, then we have a finite basis, which means we can solve a finite matrix in a finite amount of time. Since this a condensed matter problem, where ideally we have an infinite number or realistically have Avogadros number of particles, then we hope that our finite size solutions scale to this thermodynamic limit. In real systems, we dont have pure quantum mechanics (zero-temperature), so we expect some Landau Level mixing or spin flipping depending on energy (Fermi distribution for electrons), but the ideal theory has done very well to explain the many experiments.

DENSITY MATRIX RENORMALIZATION GROUP

Now what is DMRG? It is a way to approximately solve the many-body system at low energies by: 1. Solving the system exactly (ED) at a reasonably small size. 2. Taking the new ground state and making a partition into two blocks to get a reduced density matrix. 3. Saving only states that have high non-zero entanglement in the reduced density matrix up to a limit of one's choosing. 4. Projecting the ground state and operators to this new smaller basis, which is called truncation. 5. Growing the system by adding sites. 5. Rinse and repeat. So technically, it should be labeled RDMRG. Also, this process just described is called "infinite DMRG" which allows you to scale to higher particle number (towards the thermodynamic limit). There is another process of sweeping called "finite DMRG", which just refines your ground state by making block A bigger, while block B gets smaller, and then reverse. In Figure 2, we see a sweeping to the right.[7] DMRG helps computation-

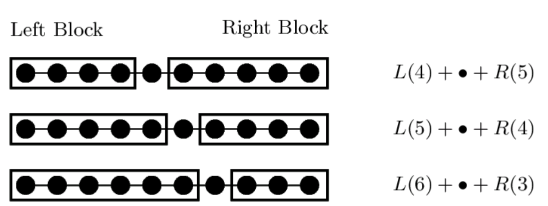

FIG. 2: Finite DMRG sweep to the right. Infinite DMRG would add extra sites in the middle at each step.

ally because we are throwing away basis states, which means smaller matrices! It is a refinement of the numerical renormalization group (RG) method by Wilson, which uses space-time or momentum-energy re-scaling to make the problem simpler i.e. less degrees of freedom.[13] Depending on how the entanglement spectrum decays though, we have to keep more states for convergence to happen. Also, some Hilbert spaces grow faster than others. In the FQHE, growing the system means adding an electron, so in order to maintain the filling fraction one increases the orbital degeneracy, N_L, in the Landau level. The number of possible states is N choose N_L, where N is the number of electrons. In a spin system the Hilbert space grows by (2s+1) states per spin or however many degrees of freedom per site for an O(N) or SU(N) or Blah(N) system, so its $(2s + 1)^N$.

Basically more basis states (bigger Hilbert space) means larger matrices means, so eventually impossible in your lifetime. However, these bases can be cut down using the symmetries of the problem. For example, if we have a conserved quantity like total angular momentum, we can reduce our basis to only those states of the conserved eigenvalue. This is the idea that density matrices and other operators like the total angular momentum become block diagonal, where each block acts on a subspace of states. Also, testing the limits of DMRG for the FQHE, we can add Landau level mixing and spin mixing. We just attach the extra Landau Levels of higher cyclotron energy or opposite spin to the end of the line to get back to a 1-D system. This makes operations more complicated and the matrices bigger, but people have done it.

CONCLUSION

There are also recent developments that touch more deeply upon why DMRG gives a right answer, specifically, casting the problem in terms of Matrix Product States or Operators, which are special forms of the DMRG wavefunction and Hamiltonian, respectively. A MPS encodes the state of the system as a chain of matrices, which is traced like so:

$$|\psi\rangle = \sum_{\{\sigma\}} \mathrm{tr}[A_1^{\sigma_1} A_2^{\sigma_2} ... A_n^{\sigma_n}] |\sigma_1 \sigma_2 ... \sigma_n\rangle$$

Each matrix corresponds to a site on the chane, the index σ represents spin states. If there is translation invariance, then all the matrices will end up being the same. It turns out that DMRG works well if the system should be well approximated by a MPS. This is easy to believe for spin chains given how the MPS itself looks like a chain of transfer matrices. How about for the FQHE? In fact Zalatel and Mong, found an exact representation for the Laughlin state as an MPS.[14] The Laughlin state is the parent of the Moore-Read or Read-Rezayi states, which are model wave functions that are thought to match very well the ground state properties of the FQHE for specific filling fractions.[12][15][16] The Laughlin state, which has $\nu = \frac{1}{m}$, has only abelian anyonic excitations, while the other states have non-abelian ones. DMRG has recently been used to study the FQHE 12/5 and 13/5 filling fraction that has so called Fibonacci anyons, which according to them is well described by the Read-Rezayi state.[17][18] They have gotten up to a system size of 30+ electrons. There TEE is related to the golden ratio. If one were interested in numerology, then one might think that maybe this wasn't an accident and that perhaps there is some underlying order in the universe. Thus, while Hilbert space sizes are limited by computer memory and processing power, there has been success and it seems that using DMRG for the FQHE is treading on solid ground.

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