

Can Spacetime Geometries Emerge From a Many-Body Quantum System If We Let Entanglement Define Distance?

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Following work of Brian Swingle [1], we summarize the basics of applying Entanglement Renormalization repeatedly to a quantum many-body state to create a MERA network. Then we discuss a correspondence between the entropy of subsystems and a discrete geometry.

INTRODUCTION

Recent progress in real-space renormalization techniques have shown that the entanglement structure of a many-body quantum system can be used to simplify the description of the system's ground state. Much like knowledge of oscillatory solutions in a classical Hamiltonian allows a significant simplification by considering the problem under a change to a Fourier decomposition in the canonical variables.

If a many body lattice quantum system has entanglement on every length scale, then a MERA network description of that state is computationally advantageous.[2] Such a structure is common amongst the ground states of Hamiltonians which can be written as the sum of local operators.[5] So the study of ground states of these local Hamiltonians is often done with the help of the MERA technique.

A MERA network is defined by a quantum state, and essentially represents the information in the state in a layered structure, where each layer corresponds to information about correlations or bulk properties on some length scale. There is a proposed connection between certain quantum field theories in n dimensions and geometries in $n + 1$ dimensions. MERA networks may provide the link to study this connection by taking the length scale as degree of freedom in the geometry and using entanglement entropy to define notions of distance.

DENSITY MATRIX RENORMALIZATION

The Multiscale Entanglement Renormalization Ansatz, which seems to have an ambiguous part of speech, is a name given to the procedure of applying Entanglement Renormalization to a system repeatedly. Entanglement Renormalization is technique that grew out of Density Matrix Renormalization[3]. Where Density Matrix Renormalization is a technique for truncating the density matrix of a quantum state in a many body system with controlled error in calculating expectation values.[4] Entanglement Renormalization is essentially Density Matrix Renormalization with an extra "disentanglement step." We will start by first understanding Density Matrix Renormalization (DMRG)[6].

In DMRG, we consider a state in an N -body quantum system. So we have a vector, $|\psi\rangle$, in a Hilbert space, $\mathcal{H} \cong \mathcal{V}^{\otimes N}$, where \mathcal{V} can be interpreted as the Hilbert space for a single particle in this large identical body system. The dimension of \mathcal{H} is then $\text{Dim}(\mathcal{V})^N$. We will construct a projector from $|\psi\rangle \in \mathcal{H}$ to $|\psi_{\text{small}}\rangle \in \mathcal{H}_{\text{small}}$, where the dimension of $\mathcal{H}_{\text{small}}$ is less than the dimension of \mathcal{H} , this process is called course graining. It is helpful to consider the example of a 1-dimensional ising chain. To each lattice site, we can assign the vector space \mathbb{C}^2 , the Hilbert space for spin-half degrees of freedom. In an N -particle ising chain, the total Hilbert space of the chain with its group of linear operators is isomorphic to multilinear maps from N copies of \mathbb{C}^2 back to N copies of \mathbb{C}^2 .

Any vector in \mathcal{H} has a representation $|\psi\rangle = \sum_{i_1, \dots, i_N=1}^{\text{dim}(\mathcal{V})} \psi_{i_1, i_2, \dots, i_N} |i_1\rangle \otimes \dots \otimes |i_N\rangle$, where $\psi_{i_1, i_2, \dots, i_N}$ form the components of tensor.

Because

$$\mathcal{H} \cong \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \dots \otimes \mathcal{V}_j \otimes \mathcal{V}_{j+1} \otimes \dots \otimes \mathcal{V}_N$$

we can construct an operator $U = \mathbb{1} \otimes \mathbb{1} \otimes \dots \otimes w \otimes \dots \otimes \mathbb{1}$, which acts as the identity on all subspaces of \mathcal{H} except the support of w , $\text{supp}(w) = \mathcal{V}_j \otimes \mathcal{V}_{j+1}$, which we will call A for simplicity[7].

The trick here is finding the right operator, w , so that the minimal amount of information about $|\psi\rangle$ is lost while reducing the dimensionality of $\mathcal{H}_{\text{small}}$ as much as possible. In DMRG, we construct w by looking at the reduced density matrix of $|\psi\rangle$ in the subspace $\mathcal{V}_j \otimes \mathcal{V}_{j+1}$.

We are given a state $|\psi\rangle$ and construct the reduced density operator for some spacially connected[8] region A .

$$\rho^{[A]} = \text{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|),$$

where $\text{Tr}_{\bar{A}}$ is the partial trace over the closure of A . We can then diagonalize $\rho^{[A]}$ so that

$$\rho^{[A]} = \sum_{k=1}^{\text{dim}(\mathcal{V})^2} P_k |k\rangle\langle k|. \quad (1)$$

Suppose we arrange the way we label the states $|k\rangle$ so that $P_k \geq P_{k+1}$, as in we write eq. (1) in descending order. We can truncate $\rho^{[A]}$ by choosing some number, $m < (\dim(\mathcal{V}))^2$, of these vectors to keep in our new Hilbert space. We control the error in this truncation by choosing m subject to the condition that $1 - \sum_{k=1}^m P_k = \epsilon$. Then we let

$$w = \sum_{k=1}^m P_k |k\rangle \langle k|$$

Now we can see what happens when we apply U to $|\psi\rangle$.

$$\begin{aligned} U|\psi\rangle &= (\mathbb{1} \otimes \mathbb{1} \otimes \dots \otimes \sum_{k=1}^m P_k |k\rangle \langle k| \otimes \dots \otimes \mathbb{1}) |\psi\rangle \\ &= \sum_{k=1}^m \sum_{i_1, \dots, i_N=1}^{\dim(\mathcal{V})} \psi_{i_1, i_2, \dots, i_N} P_k C_{k, i_j, i_{j+1}} |i_1\rangle \otimes \dots \otimes |k\rangle \dots \\ &\quad \dots \otimes \dots \otimes |i_N\rangle \end{aligned}$$

This equation looks horrible, but it can also be written

$$U|\psi\rangle = \sum_{k=1}^m P_k |k\rangle \otimes |\phi_k\rangle$$

What we have done through this process is construct a *new* Hilbert space, \mathcal{H}' , with lower dimensionality. The degrees of freedom in the region A had total dimension $(\dim(\mathcal{V}))^2$, but we've course grained that space so that the information content is now represented in only m dimensions with minimal loss of information. Notice also that the new Hilbert space is factorizable; $\mathcal{H}' \cong \mathcal{V}_1 \otimes \dots \otimes \mathcal{A} \otimes \dots \otimes \mathcal{V}_N$, where \mathcal{A} is the Hilbert space containing the vectors $|k\rangle$.

If we were to cover the entire Hilbert space, \mathcal{H} , with R regions like A , [9] we could do this procedure to each region. A single iteration of that process is called a course graining step. For each course graining step, we must construct $\frac{\dim(\mathcal{H})}{\dim(A)}$ operators like U , one for each region. Then we will end up with a new Hilbert space $\mathcal{H}_{\text{small}}$ which we can guarantee will be factorizable $\mathcal{H}_{\text{small}} \cong \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_R$, where each \mathcal{A}_i carries the information content of the i th block. And for $|\psi_{\text{small}}\rangle \in \mathcal{H}_{\text{small}}$, we have that

$$|\psi_{\text{small}}\rangle = U_1 U_2 \dots U_R |\psi\rangle. \quad (2)$$

ENTANGLEMENT RENORMALIZATION

Notice a few key features of the DMRG method. The reduction in size of the Hilbert space by the course graining of a single block A is determined not only by the number ϵ , but also by the amount of entanglement between A

and \bar{A} . If there is maximal entanglement between A and \bar{A} , then every $P_k = \frac{1}{(\dim(\mathcal{V}))^2}$, and the renormalization becomes undefined since we can not order the eigenvalues in a unique way. Outside of this singular effect, we generally have that more entanglement means we keep a larger subspace, m is a function of the entanglement of the state and ϵ . [3]

To work around this, we can follow Vidal by introducing another set of operators that act on the state $|\psi\rangle$ before we perform the DMR. Rather than a single course graining step corresponding to the application of the operators U , we first apply a set of operators which are called disentanglers to $|\psi\rangle$. That is, for each of the R regions A_i , we apply an operator which can be represented as $D_i = \mathbb{1} \otimes \dots \otimes d_i \otimes \mathbb{1}$, where the support of d_i is the region A_i union its boundary in \bar{A} (which I will call the outer boundary). Additionally d_i is composed of unitaries whose support is one site on the boundary of A_i and the closest site on the outer boundary. The purpose of d is to reduce the number m in the DMRG step.

Again consider the example of a 1D ising chain with a block size of 2. A block A_i would then be two sites somewhere on the chain. We want to consider the block and its boundary, in this case, that corresponds to the block A_i and the sites immediately to its left and right. Then $D_i = \mathbb{1} \otimes d_{i+} \otimes d_{i-}$, where for notational simplicity, we have grouped together the closure of $\text{supp}(d)$ rather than writing the identity on each site, there is no loss of generality. d_{i-} is a unitary operator whose support is the two left-most sites and d_{i+} is a unitary operator whose support is the two right-most sites.

The simplest way to understand this process is through physical intuition. Unitary operators in quantum mechanics correspond to time evolution of a quantum state without a measurement. We can imagine taking a highly entangled ising chain and, for instance, applying just the right magnetic field locally at a pair of sites so that the particles become disentangled. This of course corresponds to changing state of quantum system, but in our theoretical treatment, we will know precisely what unitary operators were applied and thus there is no loss of information from this step. For a more explicit look at applying the MERA algorithm, check this reference. [2]

After proceeding with this disentanglement procedure for every site on the boundary of the course graining blocks, we proceed with the density matrix renormalization. This now constitutes a single step of the entanglement renormalization procedure, so that we replace eq. (2) with

$$|\psi_{\text{small}}\rangle = U_1 U_2 \dots U_R D_1 \dots D_R |\psi\rangle. \quad (3)$$

MULTISCALE ENTANGLEMENT RENORMALIZATION

We can now consider a system where we have repeatedly applied the Entanglement Renormalization prescription. For reasons which seem to contradict all English style and rules about parts of speech, this procedure is called (the) "Multiscale Entanglement Renormalization Ansatz." [10]

Again we can drop into the example of a 1D Ising chain. We are free to call the distance between lattice sites 1 unit. When computing expectation values of operators that act on the lattice, we say that we are computing properties at a length scale of 1 unit. If we choose a block size of 2, then each ER iteration combines 2 sites into a single site. We then say we are working on a length scale of 2. We will attempt to treat length scale as a sort of emergent dimension of our quantum many body system.

In this way we have a natural connection between the number of iterations of ER we have performed and the length scale which is our extra emergent dimension. If we call the length scale r , and the number of iterations m , then $r = 2^m$ or more generally, $r = a2^m$ for a lattice spacing of a in arbitrary units. Then we can define a discrete geometry by taking points to be lattice sites at various scales.

THE CONNECTION TO GEOMETRY FOR A PURE STATE

We then identify the size of a cell at any level with the entanglement entropy of the sites within that cell. That is to say that given some cell, we can look at the entanglement entropy of every site in that cell at each graining level m . As an upper bound to the entropy of a block, we can consider the entropy of all traced out sites along the causal cone of the block. Where the causal

cone is the set of all operators and sites that have been course grained into the block in question.

If we restrict ourselves to considering the ground state of a critical Hamiltonian, the scale invariance of the state means that for each MERA iteration, the disentangles will be the same, which is not generally true for an arbitrary state. In such a system, the entropy of a block in the bulk is proportional to the length of a minimal curve that stretches from the original lattice out the block in question. Following this paradigm, it can be shown that there is a discrete geometry generated by the quantum Transverse Field Ising Model which can be identified with anti-de Sitter space.

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 - [5] Operators with compact support. Intuitively, operators which act on a finite region of the lattice sites
 - [6] Again, notice the ambiguous part of speech
 - [7] There is no mathematical reason why we should choose A to span only two component vector spaces, but the generalization is straightforward
 - [8] The spacial connectedness of the region is not truly required in theory. We could equally choose any set of spin degrees of freedom and do the same process. However, in practice this would cause a loss of information about the close range entanglement within subsystems of A .
 - [9] As in, if we have an N particle ising chain, and course grain with regions of block size 2, considering reduced density operators for 2 qubit subsystems, we would have $R = \frac{N}{2}$ blocks that cover \mathcal{H} .
 - [10] It is lost on the author what the Ansatz is...