Parafermions

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The parafermion operators are a generalization of the free fermion operators used to solve the transverse field Ising spin chain. We will use the parafermion formalism to show that the Z_n spin chain Hamiltonian can be written in terms of free parafermion modes. Furthermore, we will show that parafermion edge zero modes exist when nearest-neighbour interactions are chiral.

INTRODUCTION

In the Ising model, each site on the lattice has only 2 possible states, and the Hamiltonian is constructed from Pauli matrices. In a spin chain where each spin has n possible states, the Paui matrices become $n \times n$ matrices. In particular, the eigenvalues of σ_z become ω^i, where i is an integer, and ω is a complex number that satisfies $\omega^n \equiv 1$. The σ_x becomes the "shift" operator τ, which takes an eigenstate of $\sigma \equiv \sigma_z$ from ω^i to ω^{i+1}. These properties can be written mathematically as:

$$\sigma^n = \tau^n \qquad \omega\tau\sigma = \sigma\tau$$
$$\sigma^\dagger = \sigma^{n-1} \qquad \tau^\dagger = \tau^{n-1}$$

The first identity comes from the fact that $\omega^n = 1$. The second comes from the definition of the "shift" operator, and the last two come from unitarity. Now, the general spin chain Hamiltonian with nearest-neighbor interactions is given by:

$$H = \sum_{j=1}^{L} f_j \tau_j + \sum_{j=1}^{L-1} J_j \sigma_j^\dagger \sigma_{j+1} \tag{1}$$

where f_j and J_j are spatially anisotropic interaction strengths.

THE PARAFERMION FORMALISM

Analogous to the Majorana fermion operators of the Ising model, there are 2 parafermion modes per site:

$$\chi_j = \left(\prod_{k=1}^{j-1} \tau_k\right) \sigma_j \qquad \tilde{\chi}_j = \omega^{\frac{n-1}{2}} \chi_j \tau_j \tag{2}$$

They look the same as the Ising fermions, but with σ and τ substituted for the respective Pauli matrices. In addition, sites are not filled or empty; there are n distinct fillings, as in figure 1. However, the second mode doesn't look quite like the second Majorana mode in the Ising model. This is due to the fact that $\omega \neq -1$ for the Z_n spin chain. From the properties of τ and σ, the parafermions

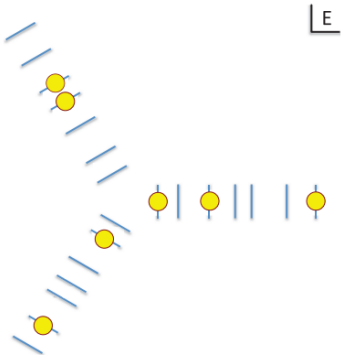

FIG. 1: Energy diagram for Z_3 parafermions. Each yellow dot represents the parafermion, and the filling is determined by which branch it is on.[1]

obey:

$$\chi_j^n = 1 \qquad \chi_j^\dagger = \chi_j^{n-1}$$
$$\chi_a \chi_b = \omega \chi_b \chi_a \text{ for } a < b$$

The constraint $a < b$ is necessary because $\frac{1}{\omega} \neq \omega$; therefore switching a and b would cause issues without this constraint.

Rewriting the Hamiltonian in terms of these parafermion modes,

$$H = \omega^{\frac{n-1}{2}} \sum_{j=1}^{L} f_j \tilde{\chi}_j \chi_j^\dagger + \omega^{\frac{n-1}{2}} \sum_{j=1}^{L-1} J_j \chi_{j+1} \tilde{\chi}_j^\dagger \tag{3}$$

To simplify the notation, define

$$h_j = \begin{cases} f_j \tilde{\chi}_j \chi_j^\dagger, & j \text{ odd} \\ J_j \chi_{j+1} \tilde{\chi}_j^\dagger, & j \text{ even} \end{cases} \tag{4}$$

From the parafermion operator relations, these operators obey relations:

$$h_j h_{j+1} = \omega h_{j+1} h_j \qquad h_j^n = \gamma_j$$

where $\gamma_j \equiv f_j^n$ for odd j and $\gamma_j \equiv J_j^n$ for even j. h_js commute with each other if they are more than 1 site apart. Then, the Hamiltonian is just

$$H = \omega^{\frac{n-1}{2}} \sum_{j=1}^{2L-1} h_j$$

HIGHER HAMILTONIANS

From the commutation relations for h_j, we can note that $h_{j-1}h_{j+1}$ commutes with h_j. This means that any operator of the form:

$$J^{(m)} = \sum_{b_m=b_{m-1}+2} 2L-1 \ldots \sum_{b_1=1} 2L-2m+1 h_{b_m} \ldots h_{b_1}$$

commutes with the Hamiltonian. For $m = 1$, this is just the Hamiltonian.

The higher Hamiltonians are operators corresponding to local conserved quantities; like the operators above, they commute with each other and the Hamiltonian. In a classical integrable model, the higher Hamiltonians are found by taking the logarithmic derivative of the transfer matrix and expanding it as a power series. Analogously, in the integrable quantum chain, the transfer matrix is replaced by a superposition of the non-local conserved quantities:

$$T(u) = 1 + \sum_{m=1}^{L} (-u)^m J^{(m)}$$

The relationship between the higher Hamiltonians and this "transfer matrix" is then:

$$-u\frac{d}{du}\ln T(u) = \sum_{m=1}^{\infty} H^{(m)} u^m$$

Matching powers of u^m in this expression, we can write down a recursion relation for generating the higher Hamiltonians in terms of the lower order Hamiltonians. Combined with the definition of $J^{(m)}$, this gives a closed-form expression for the higher Hamiltonians (in a chain of length L):

$$H^{(m)} = \sum_{c=1}^{L-1} \sum^{(m)} \frac{1-\omega^m}{1-\omega^{r_1}} \prod_{j}^{W} A_{r_{j+1},r_j} h_{c+j-1}^{r_j} \qquad (5)$$

where the sum over (m) is a sum over all r_j and W such that $\sum_{j=1}^{W} r_j = m$. The coefficients A are given by[1]:

$$A_{r,s} = \prod_{j=1}^{s-1} \frac{1-\omega^{r+j}}{1-\omega^j}$$

THE PARAFERMION SHIFT OPERATORS

The shift operators Ψ_j of the spin chain Hamiltonian transform energy eigenstates into different energy eigenstates. Thus, they must obey:

$$[H, \Psi_j] = \Delta_j \Psi_j$$

where Δ_j is the energy difference between the shifted and unshifted eigenstate for Ψ_j. This presents a problem, however, because commutators of parafermion operators with the Hamiltonian are not linear in the parafermion operators. If we define an operator that acts on operators $\mathcal{H}$ such that

$$\mathcal{H}X = \frac{[H, X]}{1-\omega}$$

, the "eigenoperators" of $\mathcal{H}$ are the shift operators, with energy shifts corresponding to the eigenvalues of $\mathcal{H}$. It can be shown[1] that there are only nL shift operators, where L is the length of the system. This makes sense, as each spin only has n unique eigenvalues, and there are L sites. To find the eigenoperators, we first apply $\mathcal{H}$ successively nL times to an arbitrary operator, and subtract out the bits that are proportional to the resultant lower order operators. This will create a basis of operators that can then be used to represent $\mathcal{H}$ as a matrix. The eigenvectors of this matrix are then the desired eigenoperators. Using the simplest parafermion $\eta_0 \equiv \chi_1$ as the starting operator and the parafermion commutation rules, we can generate the sequence:

$$\begin{aligned}
\eta_1 &= \mathcal{H}\eta_0 \\
&\vdots \\
\eta_{sn} &= \mathcal{H}\eta_{sn-1} - \gamma_{2m-1}\eta_{(s-1)n} \\
\eta_{sn+1} &= \mathcal{H}\eta_{sn} - \gamma_{2m}\eta_{(s-1)n+1} \\
\eta_{sn+l+1} &= \mathcal{H}\eta_{sn+l} \text{ for } l < n
\end{aligned}$$

Note that, because of these relationships, the result of applying $\mathcal{H}^n$ to some η_l will only contain η_r such that $r \bmod n = l \bmod n$. This means that, in the basis of η_ls, the matrix representation of $\mathcal{H}^n$ breaks into several independent blocks; for every $0 \le q < n$, there is a corresponding $L \times L$ matrix that mixes between all the different η_l with $l = sn + q$, $0 \le s \ge L$. Computing the (normalized) eigenvector of $\mathcal{H}^n$ with eigenvalue u_k, we obtain, for the lth block:

$$\phi_k^{(0)} = \frac{1}{N_k}\sum_{m=0}^{L-1} Q_{2m}(u_k)\eta_{sn} \qquad \phi_k^{(l)} = \frac{1}{N_k}\sum_{m=0}^{L-1} Q_{2m+1}(u_k)\eta_{sn+l}$$

where $Q_a(u)$ is the polynomial defined by

$$det\left(u^{\frac{1}{2}} - \mathcal{H}'\right) \prod_{b=a+1}^{2L-1} \gamma_b$$

$\mathcal{H}'$ is $\mathcal{H}$ with the a bottom rows and a rightmost columns deleted. $\mathcal{H}$ acts on these eigenvectors by $\mathcal{H}\phi_k^{(l)} = \phi_k^{(l+1)}$, except for $l = n-1$, where the action is $\mathcal{H}\phi_k^{(l)} = u_k\phi_k^{(0)}$. Now, defining $\epsilon_k = u_k^{\frac{1}{n}}$, the shift operators are given by

$$\Psi_{p,k} = \sum_{q=0}^{n-1} \left(\omega^p \epsilon_k\right)^{-q} \phi_k^{(q)} \epsilon_k^n \qquad (6)$$

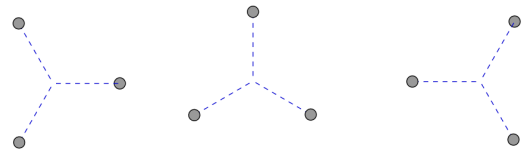

FIG. 2: Various coupling phases in the Z_3 chiral clock model. The ground state switches from ferromagnetic to antiferromagnetic order as the phase is varied[2]

where p is an integer. The commutator of these eigenvectors and the Hamiltonian is, from the definition of $\mathcal{H}$, $(1-\omega)\omega^p\epsilon_k$. Thus, relative to the ground state, the spectrum is given by the sum over all possible shifts:

$$E = \sum k = 1^L \omega^{p_k}\epsilon_k \tag{7}$$

PARAFERMION ZERO MODES

The creation operator for a zero mode of a system must satisfy 2 criteria. First, it must commute with the Hamiltonian (the energy is conserved so zero energy is added). It must also shift the particle number by 1; for fermions, this would correspond to the operator anticommuting with $(-1)^F$, where F is the fermion number. For parafermions, this becomes:

$$\omega^P \Psi = \omega \Psi \omega^P$$

where Ψ is the zero mode operator and ω^P is the Z_n charge. An edge zero mode must also be localized near an edge; the zero-mode operator's dependence on the state of parafermions at some site l away from the edge should be exponentially small.

The most general nearest-neighbour Hamiltonian for the Z_n spin chain in terms of parafermions is[2]:

$$\begin{aligned} H_n &= -\sum_{j=1}^{L}\sum_{m=1}^{n-1} \alpha_m f_j \omega^{m(m-n)/2} \chi_j^{n-m}\psi_j^m \\ &\quad - \sum_{j=1}^{L-1}\sum_{m=1}^{n-1} \hat{\alpha}_m J_j \omega^{m(m-n)/2} \psi_j^{n-m}\chi_{j+1}^m \\ &= \mathcal{F} + \mathcal{V} \end{aligned}$$

where $\mathcal{F}$ contains all shift terms, and $\mathcal{V}$ contains all interaction terms. Note that the couplings α are not necessarily real.

We can generate the zero-mode iteratively by noting that the parafermion at the edge, χ_1, commutes with $\mathcal{V}$, but not $\mathcal{F}$ (it is a zero mode if there are no shift terms). If we can write the commutator $[\mathcal{F}, \chi_1]$ as a commutator $[\mathcal{V}, X]$ for some operator X, then subtracting X from χ_1 will cancel out the remainder from $\mathcal{F}$ and add a new remainder. Iterating this:

$$\mathcal{F}X_n = \mathcal{V}X_{n+1}$$

Continuing this to infinity generates the iterative zero mode solution. It should be noted that the iterative method fails for α_m real[2].

For complex α_m, the iterative solution exists. Consider the operator $\mathcal{H}$ defined by $\mathcal{H}\nu \equiv [H, |\nu\rangle]$, where $|\nu\rangle$ is an ordered product of parafermion operators acting on the vacuum REFERENCE. A zero mode corresponds to an eigenvector of $\mathcal{H}$ with eigenvalue zero. This matrix can be written as

$$\begin{aligned} \mathcal{H} &= -\sum_{j=1}^{L} f_j \mathcal{H}_{2j-1} - \sum_{j=1}^{L-1} J_j \mathcal{H}_{2j} \\ &\equiv \mathcal{F}' + \mathcal{V}' \end{aligned}$$

where $\mathcal{H}_k$ acts on $|\nu\rangle$ by shifting the parafermion at site k down m, and shifting the parafermion at site $k+1$ up m. If site k is unoccupied, then $\mathcal{H}_k$ annihilates that state. Now we can apply the iterative method to find a state of $\mathcal{H}$ with zero eigenvalue, starting from $|\nu_0\rangle = \chi_1|0\rangle$ which commutes with $\mathcal{V}'$. However, $\mathcal{V}'$ has several zero eigenvalue eigenvectors. If $\mathcal{F}'|\nu_n\rangle$ overlaps with any of these eigenvectors, then $|\nu_{n+1}\rangle$ cannot be found from inverting $\mathcal{V}'$ and multiplying by $\mathcal{F}'|\nu_n\rangle$. Thus, the contribution of these zero eigenvalue eigenvectors to $|\nu_n\rangle$ must be subtracted out to get the exact zero mode. This zero mode, however, is not necessarily an edge mode. The χ_l parafermion is of order $\frac{f}{J}^{l-1}$ or greater in the expansion of an edge zero mode operator. Thus, only the parafermion operator furthest from the edge for each power of $\frac{f}{J}$ determines whether or not the solution is an edge mode. The furthest parafermion operator in $\mathcal{F}'|\nu_n\rangle$ is χ_{2n+2}, from the action of $\mathcal{H}_{2n+1}$. All terms acted on by $\mathcal{H}_{2n+1}$ contain χ_{2n+2} and thus $\mathcal{V}'$ is invertible for the leading term without corrections. Iterating just the leading term, the parafermion χ_{2n+2} is proportional to $\frac{f}{J}^{n-1}|\nu_0\rangle$, which satisfies the criteria for an edge mode.

[1] P. Fendley, J. Phys. A **47**, 075001 (2014) [arXiv:1310.6049 [cond-mat.stat-mech]].

[2] P. Fendley, J. Stat. Mech. **1211**, P11020 (2012) [arXiv:1209.0472 [cond-mat.str-el]].