

Lieb-Robinson Bounds and Locality in Quantum Systems

Mark Derdzinski¹

¹*Department of Physics, University of California at San Diego, La Jolla, CA 92093*

In quantum lattice systems with local interactions, the Lieb-Robinson bounds serve as an upper limit on the speed at which information can propagate. They can be naturally interpreted in terms of an “effective light cone”, where the information that leaks out to space-like separated regions is small. We highlight the content of the theorem originally due to Lieb and Robinson, and conclude by discussing how these ideas can be used to understand different phenomena, such as correlation functions and the development of topological order.

INTRODUCTION

Causality is one of the basic tenets of modern physics. In relativistic quantum theories, causality is manifest due to the existence of a light cone. However, for non-relativistic quantum systems, proving causality is non-trivial. In principle, Hamiltonians containing only local interactions can develop correlations over arbitrarily long distances in arbitrarily small times [1]. Indeed, for a Hamiltonian consisting of a sum of local, non-commuting terms, Nth order perturbations theory will correlate variables a distance N away after arbitrarily small times.

Fortunately, the situation is not so hopeless. In this paper we study one case of a theorem placing upper bounds on the correlation between two local operators with space-like separation, first proved by Lieb and Robinson [2] and later extended by Hastings [3, 4] and Nachtergaele and Sims [5]. We find that non-relativistic quantum mechanics gives rise to an “effective light-cone”, where the information leakage to space-time regions not connected by a light-cone is exponentially small. Lastly, we comment on some of the many results which can be derived from the Lieb-Robinson bounds, and see that correlation times and topological order are bounded by the theorem.

THE LIEB-ROBINSON BOUNDS

Consider a typical Hamiltonian, such as the transverse-field Ising model (TFIM):

$$H = -J \sum_{i=1}^{N-1} S_i^z S_{i+1}^z + B \sum_{i=1}^N S_i^x \quad (1)$$

Such a Hamiltonian lives on a finite size lattice and implicitly assumes local interactions, but we need to develop the language to make precise the notion of locality.

We must first define the notion of distance on the lattice. The lattice must be endowed with a metric that defines the notion of distance. In one dimension, a natural metric is $\text{dist}(i, j) = |i - j|$, (or, for periodic boundary conditions, $\text{dist}(i, j) = \min_n |i - j + nN|$, $n \in \mathbb{Z}$). Then we define the distance between two sets A and B and the

diameter of a set to be

$$\begin{aligned} \text{dist}(A, B) &= \min_{i \in A, j \in B} \text{dist}(i, j) \\ \text{diam}(A) &= \max_{i, j \in A} \text{dist}(i, j) \end{aligned}$$

We say that an operator O is “supported on the set A ” if O “acts on A ”, or more formally if we can write O as a tensor product of two operators $O = I_{\Lambda \setminus A} \otimes P$, where $I_{\Lambda \setminus A}$ is the identity operator on sites not in A .

Lastly, we need the notion of “operator norm”. We define the norm $\|O\|$ to be

$$\|O\| = \max_{\phi, |\phi|=1} |O\phi|$$

so that for Hermitian operators, the norm is equal to the absolute value of the largest eigenvalue.

Now we can consider general Hamiltonians of the form $H = \sum_Z H_Z$, where H_Z is supported on the set Z . A local Hamiltonian is one where $\|H_Z\|$ decays rapidly with the diameter of sets Z . In general, one wants to choose a metric such that the Hamiltonian is local, but the set of sites Λ has a large diameter. For example, the TFIM Hamiltonian (1) with the metric above is a sum of terms with diameter 0 (the magnetic field term) or 1 (the Ising interaction), so that $\|H_Z\| = 0$ for $\text{diam}(Z) > 1$.

We are now equipped to discuss the Lieb-Robinson bounds, which describe the time-evolution of operators in a local Hamiltonian. The bounds were first proven by Lieb and Robinson in [2], and then generalized in [3–5]. While we are focused on quantum systems on a finite lattice, the theorem can be proved for general lattices of arbitrary dimension [6]. We state the Lieb-Robinson bounds as in [7], where a proof can be found.

Theorem 1. *Suppose for all sites i , the following holds*

$$\sum_X \|H_X\| |X| \exp[\mu \text{diam}(X)] \leq s < \infty \quad (2)$$

for some positive constants μ and s . Let A_X, B_Y be operators supported on sets X, Y respectively. Then, if $\text{dist}(X, Y) > 0$,

$$\begin{aligned} \|[A_X(t), B_Y]\| &\leq \\ 2\|A_X\| \|B_Y\| |X| \exp[-\mu \text{dist}(X, Y)] &\left(e^{2s|t|} - 1 \right) \end{aligned} \quad (3)$$

To explore the physical meaning of the theorem, suppose A_X is some operator and $B_l(X)$ the ball of radius l about the set X (*i.e.* the set of sites i such that $\text{dist}(i, j) \leq l$). We can define an operator $A_X^l(t)$ supported on the set of sites $\Lambda \setminus B_l(X)$ by

$$A_X^l(t) = \int U A_X(t) U^\dagger dU$$

where the integral is over unitaries supported on $\Lambda \setminus B_l(X)$. Now since $U A_X(t) U^\dagger = A_X(t) + U[A_X(t), U^\dagger]$,

$$\|A_X^l(t) - A_X(t)\| \leq \int \|[A_X, U]\| dU$$

We can use the Lieb-Robinson bounds (3) to bound the RHS and show that $A_X^l(t)$ is exponentially close to $A_X(t)$ so long as l is sufficiently small compared to $2st/\mu$. Thus, an “effective light-cone” arises — we can approximate an operator $A_X(t)$ by one supported on a set $B_l(X)$ to exponential accuracy, so that the information flow out of the light-cone is small.

The Lieb-Robinson bounds can be reformulated into a more convenient form. Supposing the assumptions (2) hold, there is a constant v_{LR} depending only on s, μ such that for $t \leq \text{dist}(X, Y)/v_{LR}$, one has

$$\|[A_X(t), B_Y]\| \leq \frac{v_{LR}}{l} g(l) |X| \|A_X\| \|B_Y\| \quad (4)$$

where $l = \text{dist}(X, Y)$ and $g(l)$ decays exponentially in l . For example, choosing $v_{LR} = 4s/\mu$ yields (4) directly from (3). The quantity v_{LR} is usually referred to in literature as the Lieb-Robinson velocity, and establishes an upper limit on the rate at which information can propagate in quantum systems, in the sense that the “leakage” of information outside of the light-cone established by v_{LR} vanishes exponentially.

CORRELATION AND TOPOLOGICAL ORDER

While the bounds are a statement about the dynamical properties of quantum systems as a function of time, they can be used to prove interesting results in the static properties of such systems as well as the dynamic properties. For example, using our previous results, one can prove a bound on the decay of correlations in the ground state of Hamiltonians with spectral gaps [7]:

Theorem 1. *For a quantum lattice system with a unique ground state and a spectral gap ΔE , and any operators A_X, B_Y supported on sets X, Y , we have*

$$|\langle \psi_0, A_X B_Y \psi_0 \rangle - \langle \psi_0, A_X \psi_0 \rangle \langle \psi_0, B_Y \psi_0 \rangle| \leq C \left(\exp\left(\frac{-l\Delta E}{2v_{LR}}\right) + \min(|X|, |Y|)g(l) \right) \|O_A\| \|O_B\| \quad (5)$$

for some constant C , where $l = \text{dist}(X, Y)$. Thus the development of long-range correlations is bounded by the information which propagates no faster than v_{LR} .

The Lieb-Robinson bounds also establish limits on the development and destruction of “topological order” in quantum systems. While the one-dimensional TFIM has either 1 ground state (in the paramagnetic phase) or 2 ground states (in the ferromagnetic phases), other systems, such as Kitaev’s toric code model [8] exhibit a ground state degeneracy that depends on the topology of the lattice. The property of ground state degeneracy dependent on topology is characteristic of Hamiltonians with topological order.

Explicitly, we say a system has (l, ϵ) topological order if for any operator O supported on a set of diameter at most l , the corresponding matrix M with matrix elements $M_{ab} = \langle \psi_0^a | O \psi_0^b \rangle$ (where ψ_0^a are the ground states) is within ϵ of the identity matrix. In other words, there exists a complex number z such that $\|M - zI\| \leq \epsilon$. Note that topological order is actually defined on sets of states which the operator O acts on, and one can talk about the topological ordering of any set of orthogonal states ψ_0^a whether or not they are the ground states of some Hamiltonian.

The Lieb-Robinson bounds limit the behavior of topological order under time-evolution [9]. If a state ϕ has topological order (l, ϵ) , there exists (at least one) partner state ϕ' . One can show that if we evolve the states forward in time to obtain

$$\psi = \exp(-iHt)\phi, \quad \psi' = \exp(-iHt)\phi'$$

that ψ and ψ' retain some memory of the topological order. Specifically, the topological order (l, ϵ) relaxes to $(l - m, \epsilon + g(m)l^d * \text{const.})$ for any $m \leq v_{LR}t$ [7]. The length scale is smaller and the error grows larger as we evolve the states forward in time, but in a way that is bounded. Thus, topological order cannot be completely destroyed (or produced) in an arbitrarily short time.

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