

Entanglement Renormalization with local symmetry

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I discuss a specific implementation of entanglement renormalization which preserves local symmetry which can be used in conjunction with MERA to obtain ground states of entangled systems.

INTRODUCTION

This term paper is a brief introduction to the idea of entanglement renormalization and the multi-scale entanglement ansatz (MERA). I discuss a new proposal by Tagliacozzo and Vidal to carry out a particular MERA which preserves local symmetries. Throughout, I will assume a moderate level of familiarity with Kitaev's toric code model[3] and the basics of lattice gauge theory.

Entanglement Renormalization (ER) is a real space RG method which renormalizes the entanglement of the lattice[4]. This is useful because highly entangled systems are very computationally costly to simulate, whereas less entangled systems are easier. In practice, this is carried out by unitary transformations called disentglers. In an attempt to avoid technical details, these may simply be thought of as black boxes which reduce the entanglement between a block of spins and its neighbors. During ER, disentglers u are combined with isometries w , which map multiple spins into a single effective spin, to coarse grain an entangled lattice system. An important property of these operators is that if they are symmetric under a particular symmetry, the state obtained by acting with these operators will retain the symmetry. One step of coarse graining is denoted by an operator $W : \mathcal{L} \rightarrow \mathcal{L}'$.

By repeated application of W , a lattice may be coarse grained to a small lattice denoted \mathcal{L}^{top} with an effective Hamiltonian H^{top} which can be diagonalized to find $|\Psi_{GS}^{top}\rangle$. On the other hand, if the particular forms of W_1, W_2, W_3, \dots are known, the coarse graining may be reversed and the original ground state $|\Psi_{GS}\rangle$ may be recovered. The essence of MERA is to use the set of W_1, W_2, \dots as a variational approach to solve for the ground state by manipulating the coefficients of the set of u and w to minimize the energy $\langle \Psi | H | \Psi \rangle$. Having provided a lightning speed review of ER and MERA, we now move on to the specifics of the proposal in [1]. The goal is to coarse grain the toric code model.

THE STRATEGY

The key idea in [1] is that if the original Hamiltonian has symmetry, be it global or local, this should be incorporated into the ER procedure for two reasons. One, it is naturally desirable that upon coarse graining, our new model retains the symmetries present in the original

model. It turns out that this condition may not be satisfied if we do not explicitly enforce it. Another reason is that exploiting symmetry allows for significant reduction in computational time due to the nature of symmetric matrices.

First we consider a system with a global symmetry R . We desire two properties of the operator W ; first, it should map local operators in \mathcal{L} to local operators in \mathcal{L}' . Furthermore, it should preserve the symmetry or, in other words, the coarse graining transformation should commute with the symmetry operation, up to some subtlety about how the symmetry operation acts on sites in \mathcal{L}' .

Now for the main content of the paper, we consider the presence of a gauge symmetry. Two possible approaches to coarse graining such a model are to a) ignore the symmetry and just coarse grain away or b) use the duality between lattice gauge models and spin systems to perform the coarse graining in the spin system, then translate back. However both of these approaches are not ideal for various reasons.

Therefore, the approach we take is as follows. We introduce a specific coarse graining implementation W which exactly preserves the local \mathbb{Z}_2 symmetry of the toric code. This transformation W is composed of two parts, an exact transformation W_{exact} and a numerical process W_{num} . To describe the effect of these transformations, I need to introduce some terminology. A spin is *constrained* if it is part of some star operator (in the TC Hamiltonian), and *free* if no star operator acts on it. Then, W_{exact} is a transformation which acts on constrained spins, and W_{num} acts on free spins. The gauge symmetry is manifested in the constrained spins, so by acting on them only with an exact transformation, we preserve the symmetry. The combination of preservation of locality and preservation of local symmetry is what distinguishes this approach from others.

In fact, the action of W_{exact} is to free the constrained spins. This is done through the application of CNOT gates, which are a well known operation from quantum information. Understanding the action of CNOT gates is more or less necessary to understand how this coarse graining transformation works, yet I consider it to be 'auxiliary' knowledge and therefore relegate it to the appendix. I now describe the two intermediate processes of the CG transformation.

$$W_{exact}$$

W_{exact} is a systematic application of CNOT gates intended to ‘free’ certain constrained spins. Its effect is that of 8 initially constrained spins, 3 are freed, 3 are disentangled (in fact removed, because they get projected out), and 2 are left constrained. The constrained spins again live on links of a larger square lattice, and so the procedure may be iterated, with some subtlety. The transformation itself is shown diagrammatically in Figure 2. We have already seen (upon consulting the appendix) that CNOT gates can be used to focus a star operator onto a single qubit, which must be in a pure state $|+\rangle$ and subsequently gets projected out. Then we have leftover constrained spins on the links of the plaquettes, and free spins in the center of the plaquettes. In my experience, the transformation is best understood by studying the diagram. As a side note, W_{exact} may be interpreted as *locally* applying the duality transformation from gauge system to Ising model on individual plaquettes.

$$W_{num}$$

W_{num} takes the free spins created by W_{exact} and coarse grains them into a single effective spin. This is intuitively the same idea as in Kadanoff’s block spin RG in which we separate the spin interactions into ‘fast’ and ‘slow’ terms and trace out the fast degrees of freedom at each step. W_{num} may be thought of as a map

$$w : (\mathbb{C}_2)^{\otimes 3} \rightarrow \mathbb{C}_\chi \quad (1)$$

where the RHS is the χ dimensional vector space of an effective free spin. As alluded to before, this is the part of the transformation which may muddle up the local symmetry if we are not careful. This step introduces some numerical errors, so it is important that it acts only minimally on the constrained spins, which is where the symmetry acts. W_{num} is designed to commute with all of the star operators which are leftover after W_{exact} , and so it leaves the local symmetry intact. As promised, the transformation W exactly preserved the local symmetry.

Effective Hamiltonian

During each of W_{exact} and W_{num} , the Hamiltonian is transformed into an effective hamiltonian given by $H' = W^\dagger H W$. After the first step, four star operators acting in \mathcal{L} are transformed into a single star operator acting in \mathcal{L}' . Four plaquette operators are transformed into three single site σ^z operators and a single seven-spin plaquette operator. The magnetic field term acting on single sites is transformed into a coupling interaction of the form

$-\hbar_x \sigma^x \sigma^x$ between intra-plaquette spins and $-\hbar_x \sigma^x \sigma^x \sigma^x$ between inter plaquette spins (each of the pauli matrices acts on a different spin). After the W_{num} transformation, the Hamiltonian acquires new terms which represent the operation of σ^z and σ^x on the coarse grained free spin given by

$$\Sigma^z \equiv w(\sigma_1^z + \sigma_2^z + \sigma_3^z)w^\dagger \quad (2)$$

$$\Sigma^x \equiv w(\sigma_1^x \sigma_2^x + \sigma_1^x \sigma_3^x + \sigma_2^x + \sigma_3^x)w^\dagger \quad (3)$$

ITERATION OF W

There are some subtleties involved in iterating the procedure discussed above. When the first W is applied, the lattice is composed entirely of constrained spins and is absent of free spins. After W is applied, there appears a free spin at the center of each plaquette, so we need to slightly modify the subsequent coarse grainings. Originally, W_{num} took 3 free spins and mapped them to one effective spin, described by

$$(\mathbb{C}_2)^{\otimes 3} \rightarrow \mathbb{C}_\chi \quad (4)$$

Recall that every application of W_{exact} creates three new free spins. The subsequent W_{num} , therefore, takes two types of free spins. This new map can be described as

$$(\mathbb{C}_2)^{\otimes 3} \otimes (\mathbb{C}_\chi)^{\otimes 4} \rightarrow \mathbb{C}_{\chi'} \quad (5)$$

All factors of W after the first are therefore taking old coarse grained spins and newly free spins and creating a new effective spin with yet a new vector space of dimensionality $\chi', \chi'', \chi''', \text{etc.}$ χ is called the refinement parameter, and it determines the accuracy of the transformation, since this part is an approximation after all. Note that the lattice after applying W has the same pattern as \mathcal{L} so with this subtlety in mind, we may continue to iterate the transformation.

ENDING POINT

After iterating W some number of times, we will eventually be left with a 2 by 2 lattice. This is the penultimate step in the ER scheme. This 2 by 2 lattice will be mapped onto a set of 3 spins which contain topological information about the state we started with (recall the toric code has fourfold degenerate ground states which may be classified into topological sectors). It can be shown that the product of σ^x (σ^z) along the non-contractible cuts (loops) are mapped onto σ^x (σ^z) measured on the two free spins under repeated application of W . The final factor of W_{num} , denoted as W_{num}^* coarse grains the remaining free spins into a single free spin which lives in the vector space \mathbb{C}_{χ^*} . The final 3 spin system will be composed of all free spins; due to the boundary conditions

the star operators act trivially on each spin. This completes the description of the coarse graining procedure; we have reduced a vector space $\mathbb{C}^{\otimes 2L^2}$ to $\mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_{\chi^*}$

FINAL REMARKS

In this paper, I described a specific coarse graining procedure. You may be wondering what is the point of doing this. The answer is that the idea of MERA is to find a ground state of the final lattice \mathcal{L}^* and undo all the applications of W in order to recover the ground state of the original model. The variational parameters to be optimized are the coefficients of the isometric and disentangling tensors, and there exist optimization processes by which this may be performed. The approach described in this paper compares favorably to a number of other similarly minded approaches, and are generalizable to more complex models where those other approaches may be invalid. Therefore, this approach may be worth consideration by people who are working with tensor network algorithms.

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 [2] G. Vidal, arXiv:0912.1651 (2009).
 [3] A. Kitaev, Annals of Physics **303**, 2 (2003), ISSN 0003-4916, URL <http://www.sciencedirect.com/science/article/pii/S0003491602000180>.
 [4] G. Vidal, Phys. Rev. Lett. **99**, 220405 (2007), URL <http://link.aps.org/doi/10.1103/PhysRevLett.99.220405>.

FIGURES

Here I copy several figures from reference [1] which illustrate the coarse graining procedure.

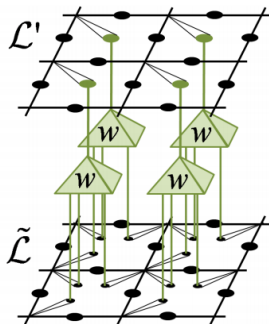


FIG. 1: This figure illustrates how three free spins are coarse grained into a single spin which sits at the center of the plaquette

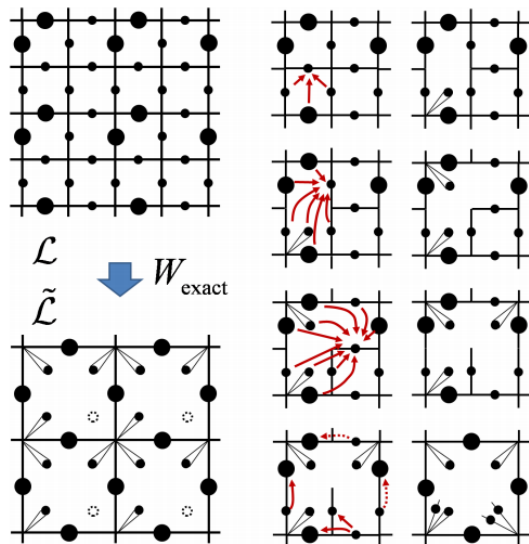


FIG. 2: This diagram illustrates the process W_{exact} . It consists of applying CNOT gates in the proper manner to turn three constrained spins into free ones. Three spins are left constrained. Two spins become completely disentangled and are forced to be eigenstates of σ^x , and are projected out.

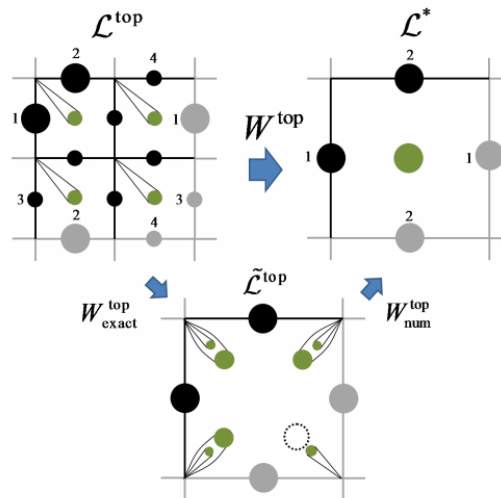


FIG. 3: This figure shows the last step which takes us from a two by two lattice to our final three spins

Appendix: Qubits, CNOTS, and the Toric Code

In this section I describe CNOT gates and show how they may be used to accomplish the tasks involved in this coarse graining procedure. First, a CNOT gate is a unitary transformation which takes as its argument two qubits: a control qubit and a target qubit. It acts in the following way

$$U_{CNOT} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma^x \quad (6)$$

$$= \mathbb{1} \otimes |+\rangle\langle +| + \sigma^z \otimes |-\rangle\langle -| \quad (7)$$

From these definitions, one may establish that the operations on the control and target qubit transform in the following way when conjugated by a CNOT gate:

$$\mathbb{1} \otimes \sigma^z \leftrightarrow \sigma^z \otimes \sigma^z \quad (8)$$

$$\sigma^z \otimes \mathbb{1} \leftrightarrow \sigma^z \otimes \mathbb{1} \quad (9)$$

$$\mathbb{1} \otimes \sigma^x \leftrightarrow \mathbb{1} \otimes \sigma^x \quad (10)$$

$$\sigma^x \otimes \mathbb{1} \leftrightarrow \sigma^x \otimes \sigma^x \quad (11)$$

The important thing for us is how these CNOT gates affect the plaquette and star operators which appear in the toric code hamiltonian. We shall see how the terms in the Hamiltonian change by looking at several poorly drawn diagrams. First, we introduce a diagrammatic way to write the toric code hamiltonian, which basically consists of just drawing a lattice. It is implied that there is a plaquette operator acting on every plaquette, and a star operator acting at every junction. A part of the regular toric code hamiltonian looks like Figure 4.

Now consider acting with a CNOT gate with control and target qubits A and B, respectively. The CNOT gate is indicated by an arrow drawn from control to target (Fig 5). Which terms will be affected by this operation? Clearly the joint plaquette of qubits A and B as well as the joint vertex will change. The plaquette term has the form

$$-J_m \sigma_A^z \sigma_B^z \sigma_C^z \sigma_D^z \quad (12)$$

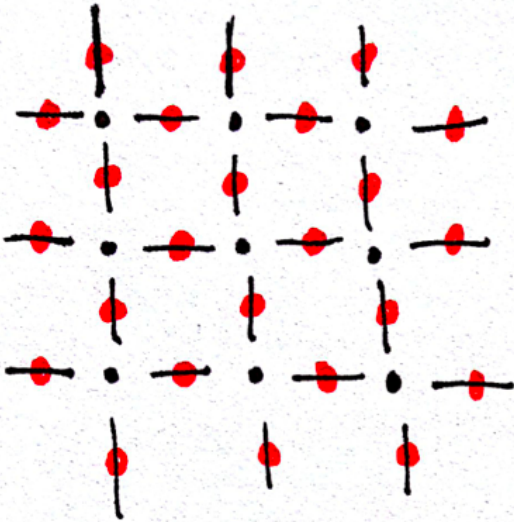


FIG. 4: The toric code Hamiltonian can be represented diagrammatically by drawing the lattice, with implied plaquette (star) operators acting on every (plaquette) (vertex). I have drawn the spins living on the links of the lattice as red dots. The black dots are points where star operators act.

From (8) we see that the $\sigma_A^z \sigma_B^z$ interaction is changed into a $\mathbb{1}_A \sigma_B^z$ so this plaquette term becomes $-J_m \sigma_B^z \sigma_C^z \sigma_D^z$. Similarly, consider the star operator acting at the joint vertex of A and B, it has the form

$$-J_e \sigma_A^x \sigma_B^x \sigma_G^x \sigma_H^x \quad (13)$$

From (11), the $\sigma_A^x \sigma_B^x$ term changes to $\sigma_A^x \mathbb{1}_B$, and so this term becomes $-J_e \sigma_A^x \sigma_G^x \sigma_H^x$.

On the other hand, the BGEF plaquette term can be thought of as $\mathbb{1}_A \sigma_B^z \sigma_G^z \sigma_E^z \sigma_F^z$, and under conjugation by CNOT the $\mathbb{1}_A \sigma_B^z$ term becomes, $\sigma_A^z \sigma_B^z$, so

$$P_{BGEF} \rightarrow \sigma_A^z \sigma_B^z \sigma_G^z \sigma_E^z \sigma_F^z \quad (14)$$

We see that plaquette ABCD effectively shrinks and plaquette BGEF grows. In addition, the star operator acting at the joint vertex of A and B no longer acts on qubit B after CNOT conjugation. With some thought, one can see that this transformation may be represented by modifying diagram 4. I invite the reader to convince themselves that Figure 6 represents the new interaction terms which we have just derived. Therefore, we say that the action of a CNOT is to *reconnect* the target qubit. By repeated application of CNOT gates, it is possible to focus plaquette or star operators to a single site. An example is given in Figure 7.

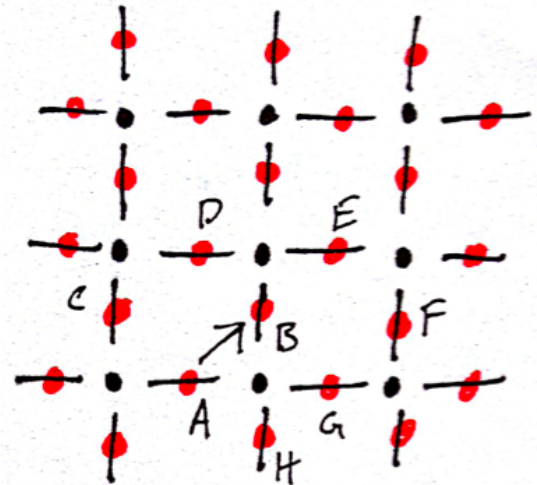


FIG. 5: A CNOT gate is indicated by drawing an arrow from control qubit to target qubit.

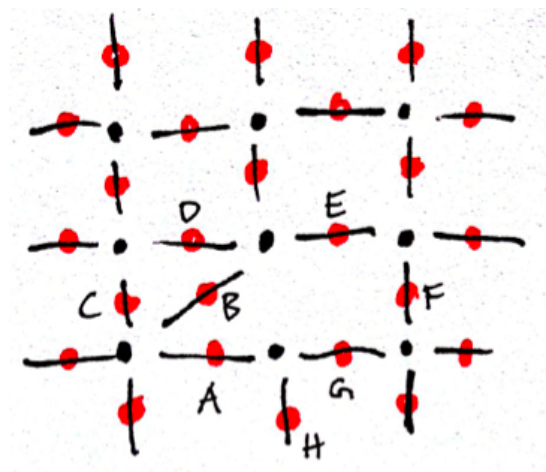


FIG. 6: This diagram represents the transformed Hamiltonian after the action of the CNOT gate from A to B

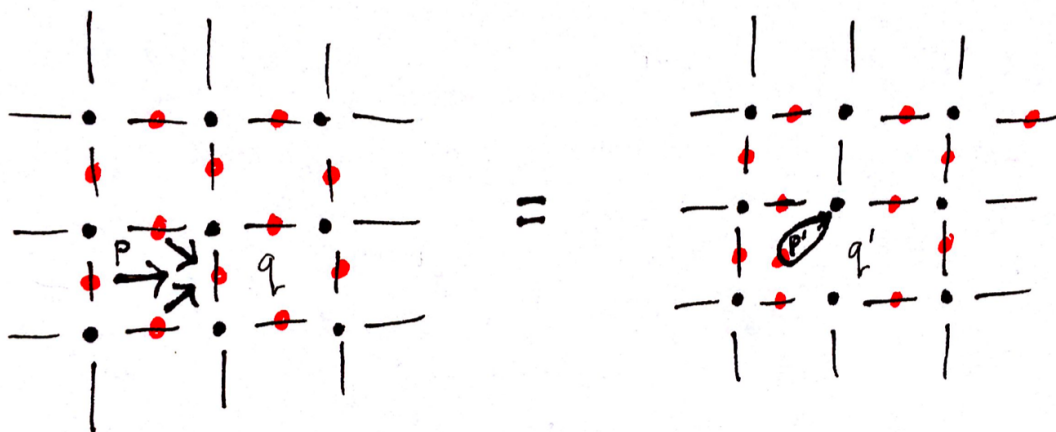


FIG. 7: This diagram shows how it is possible to focus the action of a plaquette operator to a single site. The plaquette p is effectively reduced to plaquette p' , which acts only on the site in the middle