The Title of Your Paper

Miles Albert[1]

[1]*Department of Physics, University of California at San Diego, La Jolla, CA 92093*

A fermion vortex system in non-compact two dimensional space is studied. The vortex is made up of a gauge field and a scalar field, both of which interact with the massless fermion field. $|n|$ independent zero modes are found, where n is the winding number for the vortex. The fermions acquire a mass through their interaction with the scalar field.

INTRODUCTION

By placing fermions in the background field of an infinite 2-dimensional vortex we can get some interesting results. The vortex is made up of an abelian gauge field and a charged scalar field, and the fermion field is a massless charged Dirac that interacts with both the gauge field and the scalar field. The fermions will acquire a mass and have their charge quantized only though their interaction with the scalar field. In an n-vortex, there will be $|n|$ linearly independent zero modes. Charge conjugation will not play its usual role of taking positive energy solutions to negative energy solutions and preserving zero modes, but we can define a particle conjugation which does this.

In the next section details the vortex system and interaction lagrangian. The following section discusses the zeromode solutions for the fermion field. The final section gives some concluding remarks.

VORTEX SYSTEM EQUATIONS

The background n-vortex field is defined by the gauge field $\vec{A}$ and scalar field ϕ in the following way:

$$\begin{aligned} \phi(\vec{r}) &= \mathrm{e}^{in\theta} f(\vec{r}), \\ g\vec{A}(\vec{r}) &= \varepsilon^{ij}\hat{r}^j A(r), \\ \vec{r} &= (r\cos(\theta), r\sin(\theta)) \end{aligned} \tag{1}$$

The vortex is spherically symmetric because a rotation followed by a gauge transformation leaves it invariant. The Lagrangian which couples the fermion field to these fields is given by

$$\mathcal{L} = \bar{\psi}(\gamma^\mu[i\partial_\mu - eA_\mu])\psi - \frac{1}{2}ig\phi\bar{\psi}\Gamma\psi^c + \frac{1}{2}ig^*\phi^*\bar{\psi}^c\bar{\Gamma}\psi. \tag{2}$$

Here e is the charge of the fermion ψ, and ψ^c is the charge conjugate spinor. There are two possible spinor scalar interactions, $\Gamma = I$ and $\Gamma = \gamma_5$. However, a γ_5 transformation leaves the lagrangian invariant, allowing us to use two component spinors and making both interactions give the same results [1]. The Dirac equation for the lagrangian is then

$$\begin{aligned} i\partial_t\psi &= \vec{\alpha}\cdot(\vec{p} - e\vec{A})\psi - g\phi\sigma^2\psi^* \\ \vec{\alpha} &= (\sigma^1, \sigma^2) \end{aligned} \tag{3}$$

We may separate variables by

$$\psi = \mathrm{e}^{-iEt}\psi^+(\vec{r}) + \mathrm{e}^{iEt}\psi^-(\vec{r}) \tag{4}$$

so that (3) becomes

$$\begin{aligned} E\psi^+ &= \vec{\alpha}\cdot(\vec{p} - e\vec{A})\psi^+ - g\phi\sigma^2\psi^{-*}, \\ -E\psi^- &= \vec{\alpha}\cdot(\vec{p} - e\vec{A})\psi^- - g\phi\sigma^2\psi^{+*}. \end{aligned} \tag{5}$$

The transformation which takes solutions of eigenvalue E to $-E$ is $\psi^\pm \to \sigma^3\psi^\pm$, which we call particle conjugation. At zero energy we may just replace the left hand side of (3) by 0, and the solutions may be taken to be eigenmodes of particle conjugation.

In the vacuum sector, $r \to \infty$, assume the asymptotic forms $f(r) \to f_\infty$ and $A(r) \to 0$. Here the lagrangian reduces to

$$\mathcal{L} = \bar{\psi}(\gamma^\mu i\partial_\mu)\psi - \frac{1}{2}igf_\infty\mathrm{e}^{in\theta}\bar{\psi}\psi^c + \frac{1}{2}ig^*f_\infty\mathrm{e}^{-in\theta}\bar{\psi}^c\psi. \tag{6}$$

which reduces simply to the lagrangian for a fermion of mass $\mu = gf_\infty$. Therefore, fermions not bound to the vortex acquire a mass, and we expect a continuum of states with $|E| \geq \mu$, since the fermions are free far from the vortex.

ZEROMODE SOLUTIONS

The Dirac equation in the vortex sector may be solved analytically for $E = 0$ using the definition for the spinor

$$\psi \equiv \begin{pmatrix} \mathrm{e}^{1/2\int_0^r \mathrm{d}\rho A(\rho)}\psi_\mathrm{U} \\ \mathrm{e}^{-1/2\int_0^r \mathrm{d}\rho A(\rho)}\psi_\mathrm{L} \end{pmatrix} \tag{7}$$

Finding the General Solution for the Spinors: The definition (7) eliminates the gauge potential from (3) and decouples the components of the spinors.

$$\begin{aligned} e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta \psi_{\rm U}\right) + gfe^{in\theta}\psi_{\rm U}^* = 0 \\ e^{-i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta \psi_{\rm L}\right) - gfe^{in\theta}\psi_{\rm L}^* = 0 \end{aligned} \tag{8}$$

Another ansatz separates out the angular dependance

$$\begin{aligned} \psi_{\rm U} = U_{\rm U}(r)e^{im\theta} + V_{\rm U}(r)e^{i(n-1-m)\theta} \\ \psi_{\rm L} = U_{\rm L}(r)e^{-im\theta} - V_{\rm L}(r)e^{i(n+1+m)\theta} \end{aligned} \tag{9}$$

Where m is an integer not equal to $(\pm n - 1)/2$, in which case we must use a different ansatz. Now we have

$$\begin{aligned} \left(\partial_r - \frac{m}{r}\right) U_{\rm U} + gfV_{\rm U}^* = 0 \\ \left(\partial_r - \frac{n-1-m}{r}\right) V_{\rm U} + gfU_{\rm U}^* = 0 \end{aligned} \tag{10}$$

The L subscript equations are obtained by $n \to -n$ on the equations with the U subscript, so we may proceed with the subscript removed. These equations are not linear due to the complex conjugation, so let us define the real functions u_m, v_m that satisfy

$$\begin{aligned} \left(\partial_r - \frac{m}{r}\right) u_m + gfv_m = 0 \\ \left(\partial_r - \frac{n-1-m}{r}\right) v_m + gfu_m = 0 \end{aligned} \tag{11}$$

The solutions to the complex equations are

$$U = a_m u_m(r), \qquad V = a_m^* v_m(r)$$

At the origin, $f \to f_0 r^{|n|}$, so the two solutions for u_m and v_m have the form

$$\begin{aligned} u_m \to r^m, \quad r^{|n|+n-m} \\ v_m \to r^{|n|+1+m}, \quad r^{n-1-m} \end{aligned} \tag{12}$$

At infinity the solution will be $e^{\pm\mu r}$ for a zeromode, of which only one is acceptable. Therefore both solutions at the origin must be well behaved for matching conditions to be satisfied. Acceptable values for m occur when n is positive, so either $\psi_{\rm U}$ or $\psi_{\rm L}$ always vanishes. For positive n, we have $|n| - 1 \geq m \geq 0$ and $\psi_{\rm L}$ vanishes, similarly for negative n, but $\psi_{\rm U}$ vanishes. When $m = (\pm n - 1)/2$, the u_m and v_m are not solutions to the same equations, but the solution for $\psi_{\rm U}$ and $\psi_{\rm L}$ can be put into the same form [1]. The u_m and v_m follow a recurrence relation allowing us to write the latter half in terms of the former:

$$\begin{aligned} u_m = \varepsilon_{n-1-m} v_{n-1-m}, \quad v_m = \varepsilon_{n-1-m} u_m \\ \varepsilon_{n-1-m}\varepsilon_m = 1, \qquad \varepsilon_m \text{ real} \end{aligned} \tag{13}$$

The general solution for n positive and odd is

$$\begin{aligned} \psi_{\rm U} = e^{(i/2)(n-1)\theta} \sum_{m=0}^{n/2-3/2} [b_m u_m(r) e^{-i(n/2-1/2-m)\theta} \\ + b_m^* v_m(r) e^{i(n/2-1/2-m)\theta}] \\ + e^{(i/2)(n-1)\theta} e_{(n-1)/2} u_{(n-1)/2}(r) \end{aligned} \tag{14}$$

The solution for n even is the same except that the upper limit on the sum is $n/2 - 1$ and the term after the sum is removed. The solution for negative n is very similar, different only in that it is $\psi_{\rm L}$ that is nonvanishing and in some signs in the angular dependance. In each case, the independent coefficients are the complex b_m, so there are $|n|$ independent zeromodes. The solutions are not angular momentum eigenstates despite arising from a spherically symmetric background. This is due to the nonlinearity of the Dirac equation, which relates the spinor to its complex conjugate. Therefore we may not superpose the complex solutions, which stops us from constructing angular momentum eigenstates by superposition.

CONCLUSION

We have found $|n|$ bound fermion zeromodes for the vortex, and shown that far from the vortex the fermions behave as free particles with mass μ. Therefore we expect a continuous distribution of states for $|E| \geq \mu$ (remembering that ψ has terms with positive and negative E in (4)). We haven't explored the possibility of bound states with $0 < E < \mu$. There will still be only one normalizable solution at infinity, so two linearly independent well behaved solutions are still required in the vortex region. While we haven't required that the gauge field and scalar field solve the non-linear Dirac equations, rather treating them as background fields, their general form was enough for our purposes. Since the topological flux of the vortex is conserved, the zeromodes we found can't become unbound through some change in the vortex.

REFERENCES

[1] R. Jackiw and P. Rossi, Nucl. Phys. B **190**, 681 (1981).