University of California at San Diego – Department of Physics – Prof. John McGreevy

Whence QFT? (239a) Spring 2014
Assignment 6 – Solutions

Posted June 4 2014 **Due later**

All problems are optional in the following sense: if you are sure that you know the ideas involved so well that it would be a waste of your time to do the problem, don't do it, or merely sketch the answer. By this point in your education you don't need to rely on me to determine what you know and don't know.

1. **Warmup.** Compute the scaling dimension of the operator $\mathcal{O}_k \equiv: e^{ikX}:$ in the free boson theory whose stress tensor is $T(z) = -\frac{1}{\alpha'} : \partial X \partial X(z) :$, using the $T\mathcal{O}_k$ OPE.

   The solutions to many of these problems are here.

2. **Linear dilaton CFT.**

   The *linear dilaton theory* is a 2d CFT made from a free boson in the presence of a 'background charge.' This means that the boson X has some funny coupling to the worldsheet gravity, which can be described by a linear dilaton term in the action $S_\Phi = \int d^2\sigma \Phi(X) R^{(2)}$, $\Phi(X) = QX$, Q is a constant. On a flat worldsheet, the quantization of X proceeds as before. In that case, the only difference from the ordinary free boson is that the stress tensor (which is sensitive to how the theory is coupled to gravity) has the form

   $$T_Q = -\frac{1}{\alpha'} : \partial X \partial X : + V \partial^2 X$$

   (where $V \sim Q$).

   [Optional: relate V to Q.]

   (a) Verify that T_Q has the right OPE with itself to be the stress tensor for a CFT. Compute the Virasoro central charge for the linear dilaton theory.

   (b) Compute the scaling dimension of the operator $: e^{ikX} :$ in the linear dilaton theory.

3. **The stress tensor is not a conformal primary if $c \neq 0$.** [1]

   (a) For any 2d CFT, use the general form of the TT OPE to show that the transformation of T under an infinitesimal conformal transformation $z \mapsto z + \xi(z)$ is

   $$\delta_\xi T(w) = (\xi \partial + 2\partial\xi) T(w) + \frac{c}{12} \partial^3 \xi. \quad (1)$$

[1] I got this problem from Robbert Dijkgraaf.

(b) Consider the *finite* conformal transformation $z \mapsto f(z)$. Show that (1) is the infinitesimal version of the transformation law

$$T(z) \mapsto (\partial f)^2 T\left(f(z)\right) + \frac{c}{12}\{f, z\}$$

where

$$\{f, z\} \equiv \frac{\partial f \partial^3 f - \frac{3}{2}(\partial^2 f)^2}{(\partial f)^2}$$

is called a *Schwarzian derivative.*

[Optional: verify that this extra term does the right thing when composing two maps $z \to f(z) \to g(f(z))$.]

(c) Given that the conformal map from the cylinder to the plane is $z = e^{-iw}$, show that (b) means that

$$T_{\text{cyl}}(w)(dw)^2 = \left(T_{\text{plane}}(z) - \frac{c}{24}\right)(dz)^2.$$

Use this relation to show that the Hamiltonian on the cylinder

$$H = \int \frac{d\sigma}{2\pi} T_{\tau\tau}$$

is

$$H = L_0 + \tilde{L}_0 - \frac{c + \bar{c}}{24}.$$

Comment: After all this complication, the result has a very simple physical interpretation: when putting a CFT on a cylinder, the scale invariance is spontaneously broken by the fact that the cylinder has a *radius*, *i.e.* the cylinder introduces a (worldsheet) length scale into the problem. The term in the energy extensive in the radius of the cylinder but not the length (and proportional to c) is actually experimentally observable sometimes.

4. **OPE and mode commutator algebra.** [Bonus tedium] This is a continuation of the problem on HW 5 about the $\mathfrak{SU}(2)$ radius.

   Defining modes of the current (as usual for a dimension 1 operator) by

   $$J^a(z) = \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1}$$

   show (from the current-current OPE) that

   $$[J^a_m, J^b_n] = i\epsilon^{abc} J^c_{m+n} + mk\delta^{ab}\delta_{m+n}$$

   with $k = 1$, which is an algebra called Affine $SU(2)$ at level $k = 1$. Note that the $m = 0$ modes satisfy the ordinary $SU(2)$ lie algebra.

5. **Constraints from Unitarity.** Show that in a unitary CFT, $c > 0$, and $h \geq 0$ for all primaries. Hint: consider $\langle\phi|[L_n, L_{-n}]|\phi\rangle$.

6. **Orbifolds.**

Suppose you have in your possession a 2d CFT with the correct central charge to provide (part of) a critical string background. Suppose further that this CFT has a symmetry group Γ (for simplicity let's assume it's discrete, finite and abelian). For example, say the CFT is a sigma model, and the target space has a discrete isometry, which could be a subgroup of a continuous isometry.

Let's try to make a new CFT by gauging this symmetry; in particular, this will mean projecting onto states invariant under Γ.

(a) By considering the torus amplitude of this theory, show that a modular transformation ($\tau \to -1/\tau$) relates the projection onto Γ-invariant states to a sum over new sectors of states called *twisted sectors*. These are sectors of strings which are only closed modulo the action of Γ:

$$x(\sigma + 2\pi) = \gamma x(\sigma)$$

for some $\gamma \in \Gamma$.

The resulting procedure of projecting onto invariant states and summing over twisted sectors in a modular-invariant way is called the *orbifold* construction.

(b) Consider the simple example where our initial CFT is four free bosons of infinite radius; group them into complex pairs $z^{i=1,2}$, and consider the group $\Gamma = \mathbb{Z}_N$ generated by

$$\gamma : \left(\begin{pmatrix} z^1 \\ z^2 \end{pmatrix}\right) \to \left(\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}\right) \left(\begin{pmatrix} z^1 \\ z^2 \end{pmatrix}\right), \quad \omega^N = 1.$$

For the case of $N = 2$, compute the spectrum of marginal operators. Note that γ does not act freely; the geometric space you get by identifying points under gamma is therefore scary to mathematicians; it is called an A_{N-1} singularity. Strings propagation on this space on the other hand is completely fine.

(c) [BONUS] Note that the orbifolded theory has a new global discrete symmetry, called the *quantum symmetry* (not my fault), which acts with a phase α^k on the sector twisted by ω^k. This symmetry constrains the interactions of the orbifold theory. What can you say about the theory you get by orbifolding by the quantum symmetry?