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Whence QFT? (239a) Spring 2014
Assignment 1 – Solutions

All problems are optional in the following sense: if you are sure that you know the ideas involved so well that it would be a waste of your time to do the problem, don't do it, or merely sketch the answer. By this point in your education you don't need to rely on me to determine what you know and don't know.

Note that although the deadline I've written above is not so soon, these problems are mainly illustrations of points from Chapter 1 (QFT from springs), and reminders about simple aspects of quantum field theory. You should get these done now, because we're going to do more stuff!

1. **Heisenberg picture**

   Here we will try to understand in what sense the field momentum of a free scalar field is $\pi \sim \dot{\phi}$, and we will explain the factor of $\mathbf{i}\omega$ by which π and ϕ differ.

   I usually think in what is called *Schrödinger picture*, where we evolve the states in time

   $$|\psi(t)\rangle = \mathbf{U}(t)|\psi(0)\rangle = e^{-\mathbf{i}\mathbf{H}t/\hbar}|\psi(0)\rangle$$

   and leave the operators alone. It is sometimes useful to define time-dependent operators by implementing the change of basis associated with $\mathbf{U}$ on the operators:

   $$\mathbf{A}(t) \equiv \mathbf{U}(t)\mathbf{A}\mathbf{U}(t)^\dagger = e^{+\mathbf{i}\mathbf{H}t/\hbar}\mathbf{A}e^{-\mathbf{i}\mathbf{H}t/\hbar}.$$

   First consider a simple harmonic oscillator,

   $$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{x}^2 = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)$$

   with

   $$\mathbf{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\mathbf{a} + \mathbf{a}^\dagger\right) = \sqrt{\frac{\hbar}{2m\omega}}2\mathrm{Re}\left(\mathbf{a}\right).$$

   $$\mathbf{p} = -\mathbf{i}\sqrt{\frac{\hbar\omega}{2m}}\left(\mathbf{a} - \mathbf{a}^\dagger\right) = \sqrt{\frac{m\hbar\omega}{2}}2\mathrm{Im}\left(\mathbf{a}\right).$$

(a) Using the algebra satisfied by $\mathbf{H}$ and $\mathbf{a}$, show that

$$\mathbf{x}(t) \equiv e^{+\mathbf{i}\mathbf{H}t}\mathbf{x}e^{-\mathbf{i}\mathbf{H}t} = \sqrt{\frac{\hbar}{2m\omega}}2\mathrm{Re}\left(e^{-\mathbf{i}\omega t}\mathbf{a}\right).$$

(b) Using the expression above, show that

$$\mathbf{p}(t) \equiv e^{+\mathbf{i}\mathbf{H}t}\mathbf{p}e^{-\mathbf{i}\mathbf{H}t} = m\partial_t\mathbf{x}(t)$$

in agreement with what you would want from the Lagrangian formulation and from classical mechanics.

The above was pretty simple, I hope. Now we consider a scalar quantum field theory, in say $d + 1 = 1 + 1$ dimensions:

$$\mathbf{H} = \int d^dx \left(\frac{\boldsymbol{\pi}(x)^2}{2\mu} + \frac{1}{2}\mu v_s^2\left(\vec{\nabla}\phi \cdot \vec{\nabla}\phi\right) + m^2\phi^2\right) = \sum_k \hbar\omega_k\left(\mathbf{a}_k^\dagger\mathbf{a}_k + \frac{1}{2}\right).$$

$$\begin{aligned}
\boldsymbol{\phi}(x) &= \sum_k \sqrt{\frac{\hbar}{2\mu\omega_k}}\left(e^{\mathbf{i}\vec{k}\cdot\vec{x}}\mathbf{a}_k + e^{-\mathbf{i}\vec{k}\cdot\vec{x}}\mathbf{a}_k^\dagger\right), \\
\boldsymbol{\pi}(x) &= \frac{1}{\mathbf{i}}\sum_k \sqrt{\frac{\hbar\mu\omega_k}{2}}\left(e^{\mathbf{i}\vec{k}\cdot\vec{x}}\mathbf{a}_k - e^{-\mathbf{i}\vec{k}\cdot\vec{x}}\mathbf{a}_k^\dagger\right),
\end{aligned}$$

(c) Find ω_k. (Note that I've added a mass term, relative to the model we studied in lecture.)

(d) Do a Legendre transformation to construct the action, $S[\phi] = \int dt d^dx \mathcal{L}(\phi, \dot{\phi})$.

(e) Show that

$$\boldsymbol{\phi}(t,x) \equiv e^{+\mathbf{i}\mathbf{H}t}\boldsymbol{\phi}(x)e^{-\mathbf{i}\mathbf{H}t} = \sum_k \sqrt{\frac{\hbar}{2\mu\omega_k}}\left(e^{\mathbf{i}\vec{k}\cdot\vec{x}-\mathbf{i}\omega_k t}\mathbf{a}_k + e^{-\mathbf{i}\vec{k}\cdot\vec{x}+\mathbf{i}\omega_k t}\mathbf{a}_k^\dagger\right)$$

(f) Using the previous result, show that

$$\boldsymbol{\pi}(t,x) \equiv e^{+\mathbf{i}\mathbf{H}t}\boldsymbol{\phi}(x)e^{-\mathbf{i}\mathbf{H}t} = \mu\partial_t\boldsymbol{\phi}(t,x)$$

so that all is right with the world.

2. **Complex scalar field and antiparticles**

So far we've discussed scalar field theory with one *real* scalar field. The particles created by this field are their own antiparticles.

To understand this statement, consider a scalar field theory in $d+1$ dimensions with *two* real fields ϕ_1, ϕ_2. Organize them into one complex field $\Phi \equiv \phi_1 + \mathbf{i}\phi_2$, with $\Phi^\star = \phi_1 - \mathbf{i}\phi_2$.

$$S[\Phi, \Phi^\star] = \int d^dx dt \left(\frac{1}{2}\mu\partial_t\Phi\partial_t\Phi^\star - \frac{1}{2}\mu v^2\vec{\nabla}\Phi \cdot \vec{\nabla}\Phi^\star - V(\Phi^\star\Phi)\right).$$

(a) Show that

$$S[\Phi] = \int \left(\sum_{i=1,2} \left(\frac{1}{2}\mu \left(\partial_t \phi_i \right)^2 - \frac{1}{2}\mu v^2 \vec{\nabla}\phi_i \cdot \vec{\nabla}\phi_i \right) - V \left(\phi_1^2 + \phi_2^2 \right) \right) .$$

That is, if $V = 0$, it is just the sum of two copies of the action of the theory we considered previously.

(b) Show by doing the Legendre transformation that the associated hamiltonian is

$$\mathbf{H} = \int d^d x \left(\frac{1}{\mu}\Pi\Pi^\star + \mu v^2 \vec{\nabla}\Phi \cdot \vec{\nabla}\Phi^\star + V \left(\Phi\Phi^\star \right) \right)$$

where the canonical momenta are

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \frac{1}{2}\mu \dot{\Phi}^\star, \Pi^\star = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^\star} = \frac{1}{2}\mu \dot{\Phi}$$

with $S = \int dt d^d x \mathcal{L}$.

(c) This theory has a continuous symmetry under which $\Phi \to e^{i\alpha}\Phi, \Phi^\star \to e^{-i\alpha}\Phi^\star$ with α a real constant. Show that the action S does not change if I make this replacement. [1]

(d) The existence of a continuous symmetry means a conserved charge – a hermitian operator which commutes with the Hamiltonian, which generates the symmetry (this is the Emmy "Quantum" Nöther theorem). Show that

$$\mathbf{q} \equiv \int d^d x \; \mathbf{i} \left(\Phi^\star\Pi^\star - \Pi\Phi \right)$$

generates this transformation, in the sense that

$$\delta\Phi = \mathbf{i}\alpha\Phi = -\mathbf{i}\alpha[\mathbf{q}, \Phi], \delta\Phi^\star = -\mathbf{i}\alpha\Phi^\star = -\mathbf{i}\alpha[\mathbf{q}, \Phi^\star].$$

Show that $[\mathbf{q}, \mathbf{H}] = 0$.

(e) For the case where $V(\Phi\Phi^\star) = m^2\Phi\Phi^\star$ the hamiltonian is quadratic. Diagonalize it in terms of *two sets* of creation operators and annihilation operators. You should find something of the form

$$\Phi = \sqrt{\frac{\hbar}{2\mu}} \sum_k \frac{1}{\sqrt{\omega_k}} \left(e^{\mathbf{i}kx}\mathbf{a}_k + e^{-\mathbf{i}kx}\mathbf{b}_k^\dagger \right)$$

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[1] This is called a U(1) symmetry: it is a unitary rotation (hence 'U') on a one-dimensional (hence '(1)') complex vector. Notice that on the real components ϕ_1, ϕ_2 it acts as a two-dimensional rotation:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} .$$

The name for this group is SO(2). So U(1) is the same as SO(2).

(f) Write the canonical commutators

$$[\Phi(x), \Pi(x')] = i\hbar\delta(x - x'), \quad [\Phi(x), \Pi^\star(x')] = 0$$

(and the hermitian conjugate expressions) in terms of $\mathbf{a}$ and $\mathbf{b}$.

(g) Rewrite $\mathbf{q}$ in terms of the mode operators.

(h) Evaluate the charge of each type of particle created by $\mathbf{a}_k^\dagger$ and $\mathbf{b}_k^\dagger$ (*i.e.* find $[\mathbf{q}, \mathbf{a}^\dagger]$).

I claim that the particle created by $\mathbf{a}^\dagger$ is the antiparticle of that created by $\mathbf{b}^\dagger$ in the sense that they have opposite quantum numbers. This means that we can add terms to the hamiltonian by which they can annihilate each other without breaking any symmetries.

3. **Goldstone boson**

Here is a simple example of the Goldstone phenomenon. Consider again the complex scalar field from problem 2.

Suppose the potential is
$$V(\Phi^\star\Phi^\star) = g\left(\Phi^\star\Phi - v^2\right)^2$$

where g, v are constants. The important features of V are that (1) it is only a function of $|\Phi|^2 = \Phi\Phi^\star$, so that it preserves the particle-number symmetry generated by $\mathbf{q}$ which was the hero of problem 2, and (2) the minimum of $V(x)$ away from $x = 0$.

Treat the system classically. Write the action $S[\Phi, \Phi^\star]$ in polar coordinates in field space:
$$\Phi(x, t) = \rho e^{i\theta}$$

where ρ, θ are functions of space and time.

(a) Consider constant field configurations, and show that minimizing the potential fixes ρ but not θ.

The potential is $g(\rho^2 - v^2)^2$, independent of θ So $\theta \to \theta + \alpha$ is a symmetry.

(b) Compute the mass^2 of the ρ field about its minimum, $m_\rho^2 = \frac{1}{2}\partial_\rho^2 V|_{\rho=v}$.

$4gv^2$.

(c) Now ignore the deviations of ρ from its minimum (it's heavy), but continue to treat θ as a field. Plug the resulting expression

$$\Phi = ve^{i\theta(x,t)}$$

into the action. Show that θ is a massless scalar field.

$$S[\Phi = ve^{i\theta}] = \int d^d x dt \left(\frac{1}{2}\mu v^2\left(\dot{\theta}^2 - \vec{\nabla}\theta \cdot \vec{\nabla}\theta\right)\right)$$

There's no mass term because that would violate the $\theta \to \theta + \alpha$ symmetry.

(d) How does the $U(1)$ symmetry generated by $\mathbf{q}$ act on θ?

Since $\mathbf{q} : \Phi \to e^{i\alpha}\Phi = e^{i\alpha}\rho e^{i\theta}$, we can infer that $\mathbf{q} : \theta \to \theta + \alpha$. A more ugly calculation using commutators will agree.

4. **Gaussian identity.**

Show that for a gaussian quantum system

$$\langle e^{\mathbf{i}K\mathbf{q}} \rangle = e^{-A(K)\langle \mathbf{q}^2 \rangle}$$

and determine $A(K)$. Here $\langle ... \rangle \equiv \langle 0|...|0\rangle$. Here by 'gaussian' I mean that $\mathbf{H}$ contains only quadratic and linear terms in both $\mathbf{q}$ and its conjugate variable $\mathbf{p}$ (but for the formula to be exactly correct as stated you must assume $\mathbf{H}$ contains only terms quadratic in $\mathbf{q}$ and $\mathbf{p}$; for further entertainment fix the formula for the case with linear terms in $\mathbf{H}$).

I recommend using the path integral representation.

This result is useful for the following problem and in many other places.

Versions of this problem appear in Peskin problem 11.1a) and in Green-Schwarz-Witten volume 1 page 429.

One can do it by algebra (as in GSW), using Campbell-Baker-Hausdorff to put the annihilation operators on the right and the creation operators on the left:

$$e^{\alpha(\mathbf{a}+\mathbf{a}^\dagger)} = e^{\alpha^2/2}e^{\alpha \mathbf{a}^\dagger}e^{\alpha \mathbf{a}} \ .$$

The vacuum expectation value of the RHS is $e^{\alpha^2/2}$.

But it is, I think, more illuminating to do it by path integral. The path integral representation is

$$\langle e^{\mathbf{i}K\mathbf{q}} \rangle = \frac{1}{Z} \int \prod_i dq_i \ e^{-q_i D_{ij} q_j} e^{\mathbf{i}K q_0}$$

with $Z = \int \prod_i dq_i \ e^{-q_i D_{ij} q_j}$. Here i, j are discrete time labels, and D_{ij} is the matrix which discretizes the action. Repeated indices are summed. Two steps: First, we can absorb the insertion into the action:

$$e^{-q_i D_{ij} q_j} e^{\mathbf{i}K q_0} = e^{-q_i D_{ij} q_j + \mathbf{i}K q_0} = e^{-\tilde{q}_i D_{ij} \tilde{q}_j} e^{J_i D_{ij}^{-1} J_j}$$

with $J_j = -\mathbf{i}K\delta_{j,0}$, and D^{-1} is the inverse of the kinetic matrix, i.e. the propagator:

$$\langle q_i q_j \rangle = D_{ij}^{-1}.$$

So, plugging in the value of J, we arrive at the answer:

$$\langle e^{\mathbf{i}K\mathbf{q}} \rangle = e^{-K^2 D_{00}^{-1}} = e^{-K^2 \langle q_0 q_0 \rangle} = e^{-K^2 \langle \mathbf{q}^2 \rangle}.$$

5. **Zero-phonon process.**

We wish to understand the probability for a photon to hit (our crude model of) a crystalline solid without exciting any vibrational excitations.

Fermi's golden rule says that the probability for a transition from one state of the lattice $|L_i\rangle$ to another $|L_f\rangle$ is proportional to

$$W(L_i \to L_f) = |\langle L_f|\mathbf{H_L}|L_i\rangle|^2.$$

Here $\mathbf{H_L}$ is the hamiltonian describing the interaction between the photon and an atom in the lattice. For the first parts of the problem, use the following form (to be justified in the last part of the problem):

$$\mathbf{H_L} = Ae^{iK\mathbf{x}} + h.c. \tag{1}$$

here $\mathbf{x}$ is the (center of mass) position operator of the atom in question; K is a constant (the photon wavenumber), and (for the purposes of the first parts of the problem) A is a constant. $+h.c.$ means 'plus the hermitian conjugate of the preceding stuff'.

(a) Recalling that $\mathbf{x}$ (up to an additive constant) is part of a collection of coupled harmonic oscillators:

$$\mathbf{x} = nx + \mathbf{q}_n$$

evaluate the "vacuum persistence amplitude" $\langle 0|\mathbf{H_L}|0\rangle$. You will find the result of problem 4 useful.

(b) From the previous calculation, you will find an expression that requires you to sum over wavenumbers. Show that in one spatial dimension, the probability for a zero-phonon transition is of the form

$$P_{\text{Mössbauer}} \propto e^{-\Gamma \ln L}$$

where L is the length of the chain and Γ is a function of other variables. Show that this infrared divergence is missing for the analogous model of crystalline solids with more than one spatial dimension. (Cultural remark: these amplitudes are called 'Debye-Waller factors').

(c) [more optional] Convince yourself that a coupling $\mathbf{H_L}$ of the form (1) arises from the minimal coupling of the electromagnetic field to the constituent charges of the atom, after accounting for the transition made by the radiation field when the photon is absorbed by the atom. 'Minimal coupling' means replacing the momentum operator of the atom $\mathbf{p}$, with the gauge-invariant combination $\mathbf{p} \to \mathbf{p} + \mathbf{A}$. You will also need to recall the form of the quantized electromagnetic field in terms creation and annihilation operators for a photon of definite momentum K.

The key ingredient is

$$\mathbf{x}_n = na + \mathbf{q}_n = na + \sum_k \mathcal{N}_k \left(e^{ikna}\mathbf{a}_k + e^{-ikna}\mathbf{a}_k^\dagger\right),$$

where a is the lattice spacing and $\mathcal{N}_k = \sqrt{\frac{\hbar}{2mN\omega_k}}$. So

$$P_{\text{Mössbauer}} \propto |\langle 0|e^{iK(na+\mathbf{q}_n)}|0\rangle|^2 \ .$$

Which means

$$P_{\text{Mössbauer}} \propto e^{-K^2 \sum_k \frac{\hbar}{2m\omega_k}} \sim e^{-K^2 \frac{\hbar}{2m} \frac{1}{Na} \ln(Na)} \ .$$

In the last step we've approximated the sum as an integral. Here Na is the system size, and the (IR) divergence is a 1d artifact, related to the Mermin-Wagner theorem which forbids the existence of solids in one dimension.