Emergence of Gauge Symmetry in a Lattice Spin System

Minyoung You[1]

[1]*Department of Physics, University of California at San Diego, La Jolla, CA 92093*

A symmetry not present in the Hamiltonian of a 2-D lattice spin system can emerge at low energies.

INTRODUCTION

Following [1], we will study a model consisting of a lattice of spins, taking it to the low energy limit in two steps. In the first, intermediate step, we will see that a lattice gauge theory emerges; in the second step, which takes us to the ground states, we will encounter anyons, particles which change phase when moving around one another.

THE MODEL

We start with a lattice on a torus with spins lying on the edges (figure 1), with the Hamiltonian

$$H = -a\sum_s A_s - b\sum_p B_p \tag{1}$$

where

$$A_s \equiv \prod_{j\in \text{star}(s)} \sigma_j^x \qquad B_p \equiv \prod_{j\in \text{boundary}(p)} \sigma_j^z \tag{2}$$

and a and b are constants.

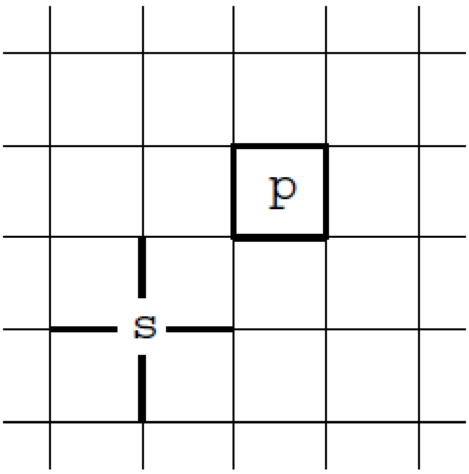

FIG. 1: Lattice with spins lying on the edges. (Taken from [1].)

star(s) and boundary(p) are shown in figure 1. Each of the operators A_s and B_p is a product of four spin operators at different locations. Since any given plaquette can have either zero or two (i.e. an even number of) edges in common with any star and vice versa, al the terms in the Hamiltonian (A_s and B_p for all s and p) commute with each other, and thus can be simultaneously diagonalized.

EMERGENCE OF LATTICE GAUGE THEORY

Let us assume that a is much larger than b. At high energies, any configuration of spins is possible. The Hilbert space consists of all the combinations of spin states at each edge.

At somewhat low energies – low compared to the coefficient a but still high enough to allow excitations of order b – only states that minimize the first term of the Hamiltonian survive; i.e. the Hilbert space becomes:

$$\mathcal{H} = \Big\{ |\psi\rangle : A_s|\psi\rangle = |\psi\rangle \text{ for all } s \Big\}. \tag{3}$$

This Hilbert space is not local, in that it is not a tensor product of local Hilbert spaces.

The Hamiltonian, then, reduces to

$$H = -an + b\sum_p B_p \tag{4}$$

where n is the number of lattice sites s.

If we imagine an initial configuration in which all spins are 'down' in the x-direction, states $|\psi\rangle$ satisfying the condition $A_s|\psi\rangle = |\psi\rangle$ are closed strings of 'up' spins. It has to be closed, because if it has endpoints, $A_s|\psi\rangle = -|\psi\rangle$ at those points. You can see in figure 2 that for each vertex s on the closed string, star(s) contains an even number of up spins and down spins (two of each), so that $A_s = 1$, whereas at the endpoints of the open string, the 'star' of those points contain only one up spin.

Since σ^z flips the spins in the x-direction, the operation B_p amounts to flipping the spins around the plaquette p. As can be seen from figure 3, the result of such an operation on a closed string results in a closed string; we are still in the same Hilbert space, and the dominant (first) term of the Hamiltonian remains at its minimum. Hence a $\mathbb{Z}_2$ gauge symmetry emerges [2]. This is a local symmetry since we can operate with B_p at select locations.

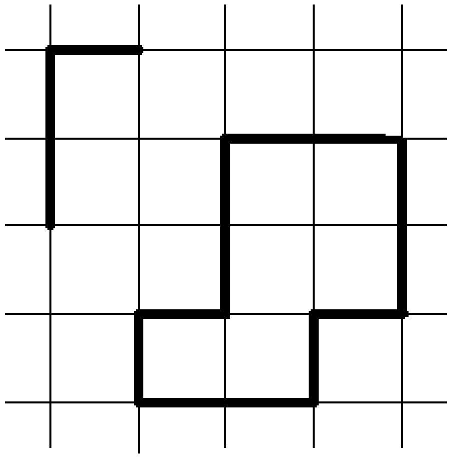

FIG. 2: An open string (upper left) and a closed string (right). Bold lines are 'up' spins.

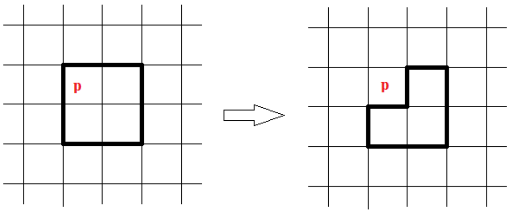

FIG. 3: Operating with B_p leaves the energy unchanged. (Taken from [1].)

ABSENCE OF MAGNETIZATION

This system can have no magnetization. We can see this in the following way [2]:

$$\langle\psi|\sigma_j^z|\psi\rangle = \langle\psi|A_sA_s^{-1}\sigma_j^zA_sA_s^{-1}|\psi\rangle = \langle\psi|-\sigma_j^z|\psi\rangle \quad (5)$$

for $j \in \text{star}(s)$, since $A_s|\psi\rangle = |\psi\rangle$ (we have used the facts that σ^x and σ^z anti-commute and σ^x is its own inverse). Hence

$$\langle\sigma^z\rangle = 0. \quad (6)$$

THE GROUND STATES

As we go to even lower energies, states not satisfying the second constraint, $B_p|\psi\rangle = |\psi\rangle$ also get thrown away, and only the true ground states remain. The Hilbert space further reduces to:

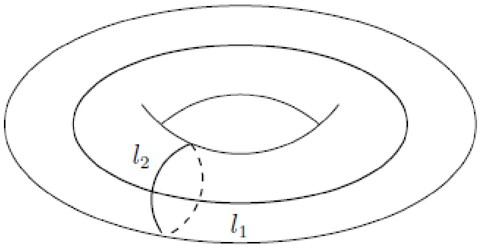

FIG. 4: Closed loops around the torus. (Taken from [3].)

$$\mathcal{H} = \left\{|\xi\rangle : A_s|\xi\rangle = |\xi\rangle, \qquad B_p|\xi\rangle = |\xi\rangle \text{ for all } s \text{ and } p\right\} \quad (7)$$

Since σ^z flips the spins in the x-direction, satisfying $B_p|\psi\rangle = |\psi\rangle$ means the state $|\psi\rangle$ has to be invariant under flipping. Hence the ground states are uniform superposition of closed strings: the flipping operation will deform some strings into other strings, but since the coefficients of each string is the same the operation will result in the same state. Schematically,

$$|\psi\rangle = \frac{1}{\sqrt{N_C}}\sum_{\{C\}}|C\rangle \quad (8)$$

where $\{C\}$ runs over spin configurations which consist of collections of close strings.

Since we are on the torus, there can be four kinds of distinct closed loops: there are two ways in which the strings can wrap around the torus (see figure 4), and the number of up spins (in the z-direction) along those loops can be even or odd (whether it is even or odd is conserved with respect to the operations A_s and B_p). Hence there is a four-fold degeneracy for the ground state.

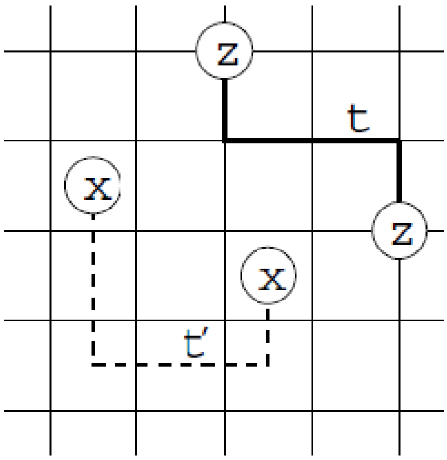

FIG. 5: Open strings with particles at the endpoints. (Taken from [1].)

PARTICLES: ANYONS

Elementary excitations, against the backdrop of ground states, occur when the smallest number of constraints are violated. If we have an open string, the endpoints, where the constraints are violated, can be called particles. Just as electric field lines can end on electrons, the flux lines can end on these particles. There can be two types of particles, z-type particles which live on the vertices, and x-type particles which live on the plaquettes (see figure 5). These states are, respectively, $|\psi^z(t)\rangle = S^z(t)|\xi\rangle$ and $|\psi^x(t')\rangle = S^x(t')|\xi\rangle$, where

$$S^z(t) = \prod_{j\in t} \sigma_j^z \qquad S^x(t') = \prod_{j\in t'} \sigma_j^x \tag{9}$$

with t being a string of edges, and t' a set of edges that the string t' intersects.

Let us consider what happens when we move one particle around another. Let the initial configuration consist of two x-type particles and two z-type particles whose strings (q and t respectively) do not intersect: $|\Psi_{\text{initial}}\rangle = S^z(t)S^x(q)|\xi\rangle$. If we move an x-type particle around a z-type particle along a loop c (as in figure 6),

$$|\Psi_{\text{final}}\rangle = S^x(c)S^z(t)S^x(q)|\xi\rangle = -S^z(t)S^x(q)|\xi\rangle = -|\Psi_{\text{initial}}\rangle \tag{10}$$

where we have used the fact that $S^x(c)S^z(t) = -S^z(t)S^x(c)$ since c and t have one edge in common, and that $S^x(c)|\xi\rangle = |\xi\rangle$ since c is a closed string. We see that the state acquires a factor of -1, or a phase of π. Such particles which acquire a phase when moving around one another are called anyons.

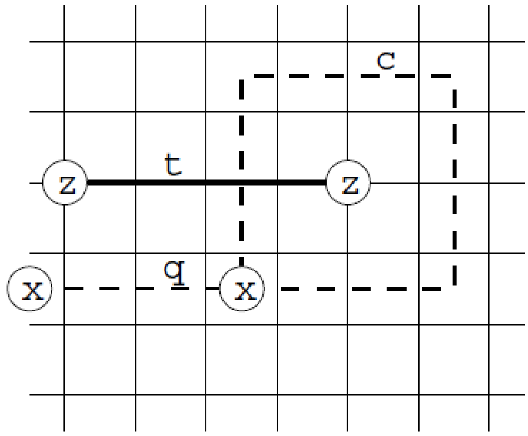

FIG. 6: Moving one particle around another. (Taken from [1].)

Acknowledgements

I thank John McGreevy for helpful discussion and guidance.

[1] A. Y. .Kitaev, “Fault tolerant quantum computation by anyons,” Annals Phys. **303**, 2 (2003) [quant-ph/9707021].
[2] J. B. Kogut, Rev. Mod. Phys. **51**, 659 (1979).
[3] A. Y. .Kitaev, “Topological phases and quantum computation,” [cond-mat.mes-hall].